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A counterexample to Wood's conjecture

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Abstract

We prove that if the one-point compactification of a locally compact, noncompact Hausdorff space L is the topological space called pseudoarc, then $\mathcal{C}_0(L, \mathbb{C})$ is almost transitive. We also obtain two necessary conditions on a metrizable locally compact Hausdorff space L for $\mathcal{C}_0(L)$ being almost transitive.

Keywords: Almost transitivity; Pseudoarc; $C(K)$ spaces

1. Introduction

Let X be a Banach space. We denote by $\mathcal{G}(X)$ the group of all surjective linear isometries on X. It is said that X is *transitive* if for every $x, y \in S_X$ there exists $T \in \mathcal{G}(X)$ such that $Tx = y$. It is said that X is *almost transitive* if for every $x, y \in S_X$ and $\varepsilon > 0$ there exists $T \in \mathcal{G}(X)$ such that $\|Tx - y\| < \varepsilon$.

If L is a locally compact Hausdorff space, $\mathcal{C}_0(L, \mathbb{K})$ will be the space of continuous functions from L into $\mathbb{K}$ which vanishes at infinity. We consider it as a Banach space over $\mathbb{K}$ with the supremum norm; if we treat results that do not depend on the scalar field, we shall simply write $\mathcal{C}_0(L)$. $\hat{L}$ is the one-point compactification of L, and we assume $L \subset \hat{L}$. If L is not compact, the only element of $\hat{L} \setminus L$ is represented by the symbol ∞. If $f \in \mathcal{C}_0(L)$, $\hat{f}$ is its only continuous extension to $\hat{L}$. Likewise, if $\sigma : L \to L$ is a homeomorphism, we

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denote by $\hat{\sigma}$ the extension of σ to $\hat{L}$ given by $\hat{\sigma}(\infty) = \infty$. It is easy to see $\hat{\sigma}$ is also a homeomorphism.

Wood's conjecture, which first appeared in [14], is the following statement:

Conjecture 1.1. *If L is a locally compact Hausdorff space with more than one point, then $\mathcal{C}_0(L)$ is not almost transitive.*

Henceforth, L will be a locally compact Hausdorff space with more than one point. It is known that the conjecture is true in the real case [7], and the next proposition summarizes some of the most important results on the general case.

Proposition 1.2. *Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. The following statements are equivalent*:

(1) *There exists L such that $\mathcal{C}_0(L, \mathbb{K})$ is almost transitive.*
(2) *There exists L such that $\mathcal{C}_0(L, \mathbb{K})$ is almost transitive and $\hat{L}$ is metrizable.*
(3) *There exists L such that $\mathcal{C}_0(L, \mathbb{K})$ is transitive.*

The proof of $1 \Leftrightarrow 2$ is in [3], and that of $1 \Leftrightarrow 3$ is in [7].

Following the terminology of [4] and [5], which is inspired in [7], we say that $\mathcal{C}_0(L)$ is *almost positive transitive* if given $\varepsilon > 0$ and $f, g \in \mathcal{C}_0(L)$ with $\|f\| = \|g\| = 1$ and $f(L) \cup g(L) \subset \mathbb{R}^+$, there exists $T \in \mathcal{G}(\mathcal{C}_0(L))$ such that $\|Tf - g\| < \varepsilon$. We say that $\mathcal{C}_0(L)$ admits *almost polar decompositions* if for every $f \in \mathcal{C}_0(L)$ and $\varepsilon > 0$, there exists $T \in \mathcal{G}(\mathcal{C}_0(L))$ such that $\|T(|f|) - f\| < \varepsilon$, where $|f|(t) = |f(t)|$.

Of course, $\mathcal{C}_0(L)$ is almost transitive if and only if it is almost positive transitive and admits almost polar decompositions. We also should take into account that if $\mathcal{C}_0(L)$ is almost positive transitive, the isometry T mentioned in the definition can be chosen in the form $Th = h \circ \sigma$, where $\sigma : L \to L$ is a homeomorphism. This fact is an easy consequence of Banach–Stone theorem.

In both [7] and [14] it is observed that if $\mathcal{C}_0(L)$ is almost transitive then L cannot be compact and $\hat{L}$ must be connected. These results remain valid for the weaker property of almost positive transitivity.

A generalization of Wood's conjecture was raised in [1]: Is every almost transitive JB^*-triple a Hilbert space? As in Proposition 1.2, here the word "almost" can be dropped from the question without changing the answer. This was proved also in [1].

Recently, Lusky [11] proved that if a separable complex Banach space X is (isometrically) the predual of an L_1-space then it is contractively complemented in a C^*-algebra. In particular, such X will be a JB^*-triple and therefore every separable complex almost transitive L_1-predual is a non-Hilbertian JB^*-triple; examples of this are the complex Gurarij space [8] and the complex M-space constructed in [3], thus both examples answer in the negative the aforementioned question in [1].

2. Necessary conditions and continua theory

Concerning the basic facts about continua, the author followed the book [12], although there are many specialized texts on the subject.

A *continuum* (in the plural, *continua*) is a compact connected topological space. If K is a continuum, it is said that K is *indecomposable* if there do not exist A, B proper subcontinua of K such that $K = A \cup B$. Equivalently, every proper subcontinuum of K has empty interior in K. Finally, it is said that a continuum K is *hereditarily indecomposable* if every subcontinuum of K is indecomposable (including K itself).

The intersection of a countable decreasing family of continua is a continuum. If K is a continuum, for every $x \in K$ we define the *composant* of x as $\bigcup\{C \subset K\colon C$ is a proper subcontinuum of K and $x \in C\}$. It is well known that if K is an indecomposable metrizable continuum, then every composant is dense in K and the set of all the composants forms an uncountable partition of K.

The following lemma will be useful in what follows, yet it is not strictly necessary.

Lemma 2.1. [12] *If K is a Hausdorff continuum, $x \in K$ and there exists a continuum $D \subsetneq K$ with $\operatorname{int}(D) \neq \emptyset$ and $x \in \operatorname{ext}(D)$, then there exists a continuum $C \subsetneq K$ with $x \in \operatorname{int}(C)$.*

Proof. If $K \setminus D$ is connected, it is enough to take $C = \overline{K \setminus D}$, then $C \neq K$ since $\operatorname{int}(D) \neq \emptyset$, and $x \in K \setminus D \subset \operatorname{int}(\overline{K \setminus D}) = \operatorname{int}(C)$.

If $K \setminus D$ is not connected, then $K \setminus D = U \cup V$ with U and V open disjoint sets and $x \in U$. Consider $C = D \cup U$, we have that $x \in \operatorname{int}(C)$. Also, $C = K \setminus V$, so C is compact. If C is not connected, there exist A and B compact disjoint nonempty sets with $A \cup B = D \cup U$. Therefore, $A \cap D$ and $B \cap D$ are closed sets which partition D, and as D is connected one of them must be empty. Suppose for instance that $A \cap D = \emptyset$, then $A = U \cap (K \setminus B)$ and therefore A is a compact open proper subset of K, which yields to a contradiction. □

Lemma 2.2. *Let $\mathcal{T}$ be a metric space, F a closed proper subset of $\mathcal{T}$ with nonempty interior, $A \subset \operatorname{int}(F)$ and $B \subset \operatorname{ext}(F)$ nonempty sets such that $\min\{d(A, \operatorname{ext}(F)), d(B, \operatorname{int}(F))\} > 0$. In this situation, given $\varepsilon \in (0, 1)$ there exists a continuous function $f : \mathcal{T} \to [0, 1]$ with $f(A) = 0$, $f^{-1}([0, \varepsilon]) = F$ and $f(B) = 1$.*

Proof. Take

$$f(x) = \min\left\{1, \max\left\{0, \varepsilon + \frac{d(x, \operatorname{int}(F)) - d(x, \operatorname{ext}(F))}{\min\{d(A, \operatorname{ext}(F)), d(B, \operatorname{int}(F))\}}\right\}\right\}.$$

It is straightforward to check that f has the required properties. □

Next we present the theorem which was the guide to the counterexample. The necessary conditions that we obtain for a space to be almost positive transitive are quite restrictive.

Theorem 2.3. *If $\hat{L}$ is metrizable and $\mathcal{C}_0(L)$ is almost positive transitive, then*:

(1) *Given C and K subcontinua of $\hat{L}$ with $C \subsetneq K \subset \hat{L}$ and $\infty \notin \operatorname{fr}^K C$, we have $\operatorname{int}^K C = \emptyset$. In particular, every subcontinuum of L is hereditarily indecomposable.*

(2) *Given G_1, G_2 open subsets of L with $\infty \notin \overline{G_1 \cup G_2}$ and $H \subset G_2$ with $d(H, \hat{L} \setminus G_2) > 0$, there exists $\sigma : L \to L$ homeomorphism such that $H \subset \sigma(G_1) \subset G_2$. In particular, L is almost homogeneous (i.e., given $x, y \in L$ and $\varepsilon > 0$ there exists $\tau : L \to L$ homeomorphism with $d(\tau(x), y) < \varepsilon$).*

Proof. (1) Let C and K be continua of $\hat{L}$ with $C \subsetneq K \subset \hat{L}$ and $\infty \notin \mathrm{fr}^K C$, and suppose that $\mathrm{int}^K C \neq \emptyset$. Due to Lemma 2.1, we can suppose that $\infty \notin K$ or $\infty \in \mathrm{int}^K C$. In both cases, Lemma 2.2 allows us to obtain a continuous surjective function $F : K \to [0, 1]$ such that $F^{-1}([0, 3/5]) = C$ and F has a continuous extension f to $\hat{L}$ such that $f|_L \in \mathcal{C}_0(L)$ with $\|f|_L\| = 1$.

Besides, let $G : [0, 1] \to [0, 1]$ be a continuous function defined by $G(1) = 1$, $G(1/2) = 2/5$, $G(1/3) = 4/5$, $G(0) = 0$ and G is linear in the intervals $(0, 1/3)$, $(1/3, 1/2)$ and $(1/2, 1)$. We can suppose $\max\{d(x, \infty)\colon x \in \hat{L}\} = 1$. Let $g : \hat{L} \to [0, 1]$ be defined by $g(x) = G(d(x, \infty))$. We have $g|_L \in \mathcal{C}_0(L)$ and $\|g|_L\| = 1$.

By the almost positive transitivity of $\mathcal{C}_0(L)$, there exists $\sigma : L \to L$ homeomorphism such that $|f(t) - g(\sigma(t))| < 1/5$ for every $t \in L$. Choose $t_1 \in K$ such that $f(t_1) = 1$, then $g(\sigma(t_1)) > 4/5$ and this implies $d(\sigma(t_1), \infty) > 1/2$. Choose $t_2 \in K$ such that $f(t_2) = 1/5$, then $g(\sigma(t_2)) < 2/5$ and this implies $d(\sigma(t_2), \infty) < 1/3$. As $\hat{\sigma}(K)$ is connected and $\{\sigma(t_1), \sigma(t_2)\} \subset \hat{\sigma}(K)$, there exists $t_3 \in K$ such that $d(\sigma(t_3), \infty) = 1/2$. Then $g(\sigma(t_3)) = 2/5$, which implies $t_3 \in C$. As $\hat{\sigma}(C)$ is connected and $\{\sigma(t_2), \sigma(t_3)\} \subset \hat{\sigma}(C)$, there exists $t_4 \in C$ such that $d(\sigma(t_4), \infty) = 1/3$. This implies $g(\sigma(t_4)) = 4/5$ and $f(t_4) > 3/5$, which leads to a contradiction.

(2) By using Lemma 2.2 twice, we can construct continuous surjective functions $f, g : \hat{L} \to [0, 1]$ such that $f^{-1}([0, 2/3]) = \hat{L} \setminus G_1$, $f(\infty) = 0$, $g^{-1}([0, 1/3]) = \hat{L} \setminus G_2$, $g(\infty) = 0$ and $g(H) = 1$. Let $\sigma : L \to L$ be a homeomorphism such that $|f(t) - g(\sigma(t))| < 1/3$ for every $t \in L$. If $t \in G_1$ then $f(t) > 2/3$ and $g(\sigma(t)) > 1/3$, therefore $\sigma(t) \in G_2$. If $t \in H$ then $g(t) = 1$ and $f(\sigma^{-1}(t)) > 2/3$, which implies $\sigma^{-1}(t) \in G_1$, i.e., $t \in \sigma(G_1)$. □

The most famous example of hereditarily indecomposable continuum is a subset of $\mathbb{R}^2$ called *pseudoarc*. Actually, the construction can be done in a great family of metric spaces, but we shall work in the plane to give a small support to the intuition. We introduce first some concepts we shall need.

A *chain* is an n-uple $D = (d_1, d_2, \ldots, d_n)$ of open bounded sets such that $d_i \cap d_j \neq \emptyset$ if and only if $|i - j| \leqslant 1$. Every d_i is called *link*, and we denote $\#D = n$, $\mathrm{mesh}(D) = \max\{\mathrm{diam}(d_i)\colon i \in \{1, 2, \ldots, \#D\}\}$ and $D^* = \bigcup_{i=1}^{\#D} d_i$. If $a \in d_1$ and $b \in d_{\#D}$, we say that D is a chain from a to b. If we work with a sequence of chains $(D_n)_{n\in\mathbb{N}}$, the links of the chain D_q are denoted $d(q)_1, d(q)_2, \ldots, d(q)_{\#D_q}$.

Given two chains D and E, it is said that D is *contained* in E if every link of D is included in a link of E. It is said that E is a *consolidation* of D if D is contained in E and every link of E is union of links of D. It is said that D is *crooked* in E if D is contained in E and for every e_h, e_k links of E with $|h - k| > 2$ and d_i, d_j links of D with $i < j$ and such that $d_i \subset e_h$ and $d_j \subset e_k$, there exist r, s with $i < r < s < j$ such that d_r is contained in e_{k-1} or in e_{k+1} and d_s is contained in e_{h-1} or in e_{h+1}.

Given $r, s \in \mathbb{N}$, a map $N : \{1, \dots, r\} \to \{1, \dots, s\}$ is a *pattern* if $|N(i+1) - N(i)| \leqslant 1$ for every $i \in \{1, \dots, r-1\}$. It is said that a chain D *follows the pattern* N in a chain E if $r = \#D$, $s = \#E$ and for every $i \in \{1, \dots, r\}$ we have $d_i \subset e_{N(i)}$.

Let us construct now the pseudoarc. Let $a, b \in \mathbb{R}^2$ be different points, and $(D_n)_{n\in\mathbb{N}}$ a sequence of chains from a to b with $(\mathrm{mesh}(D_n))_{n\in\mathbb{N}}$ convergent to zero, and such that for every $n \in \mathbb{N}$ the links of D_n are connected, the chain D_{n+1} is crooked in D_n, and every link of D_{n+1} has its closure included in a link of D_n. Our space is the continuum $P = \bigcap_{n=1}^{\infty} D_n^*$.

In [2] it is proved that all the spaces which follow the construction above are homeomorphic, and that P is hereditarily indecomposable and homogeneous. The author of that article also proves there the four following results.

Theorem 2.4. *If D, E and F are chains such that E contains D and E is crooked in F, then D is crooked in F.*

By the previous theorem, if $(a_n)_{n\in\mathbb{N}}$ is an increasing sequence of natural numbers then $P = \bigcap_{n=1}^{\infty} D_{a_n}^*$ and the chain $D_{a_{n+1}}$ is crooked in D_{a_n}, being D_n as above. This fact will be used without further reference to it.

Theorem 2.5. *If D, E and F are chains such that F is a consolidation of E and D is crooked in E, then D is crooked in F.*

Theorem 2.6. *Let $N : \{1, \dots, r\} \to \{1, \dots, s\}$ be a pattern with $N(1) = 1$ and $N(r) = s$, and $(D_n)_{n\in\mathbb{N}}$ a sequence of chains from the point a to the point b, such that $\#D_1 = s$ and for every $n \in \mathbb{N}$ the chain D_{n+1} is crooked in D_n, every link of D_{n+1} has its closure included in a link of D_n, and* $\mathrm{mesh}(D_n) \leqslant 1/n$. *Then there exist $j \in \mathbb{N}$ and a chain E from a to b such that E is a consolidation of D_j and follows the pattern N in D_1.*

Theorem 2.7. *Let a_i, b_i, c_i $(i = 1, 2)$ be points of P such that a_i and c_i belong to the same composant of P and b_i belongs to a different composant. Then there exists a homeomorphism $H : P \to P$ which maps a_1 to a_2, b_1 to b_2 and c_1 to c_2.*

Corollary 2.8. *If a_i, b_i $(i = 1, 2)$ are points of P, for every $\varepsilon > 0$ there exists a homeomorphism $H : P \to P$ such that $H(a_1) = a_2$ and $\|H(b_1) - b_2\| < \varepsilon$.*

Proof. Just consider the previous theorem and that every composant is dense in P. □

If $\mathcal{T}$ is a completely regular topological space, we use $\dim \mathcal{T}$ to denote its covering dimension (see [6], for instance). It is known that if $A \subset \mathbb{R}^n$ has empty interior then $\dim A \leqslant n - 1$. Therefore, $\dim P \leqslant 1$ (actually, it is equal to 1, but we do not need this fact).

The interested reader can find a thorough survey of the pseudoarc in [10].

3. The counterexample

The following statement is easily deduced from the results in [13, pp. 42, 44 and 61–64].

Theorem 3.1. *Let $\mathcal{T}$ be a normal topological space with* $\dim \mathcal{T} \leqslant 1$. *If $F \subset \mathcal{T}$ is closed and $f : F \to S_{\mathbb{C}}$ is continuous, there exists $\bar{f} : \mathcal{T} \to S_{\mathbb{C}}$ continuous extension of f.*

The next theorem appears as an observation, in the case of $\hat{L}$ being metrizable, in [5, p. 315]. The proof given here is slightly different and we include it for completeness.

Theorem 3.2. *If L is a locally compact Hausdorff space with* $\dim \hat{L} \leqslant 1$, *then $\mathcal{C}_0(L,\mathbb{C})$ admits almost polar decompositions.*

Proof. Take $g \in S_{\mathcal{C}_0(L,\mathbb{C})}$ and $\varepsilon \in (0,2)$. The set $K = \{t \in L\colon\ |g(t)| \geqslant \varepsilon/2\}$ is compact and nonempty. Consider $f : K \to S_{\mathbb{C}}$ given by $f(t) = \frac{g(t)}{|g(t)|}$, by Theorem 3.1 there exists $\bar{f} : \hat{L} \to S_{\mathbb{C}}$ continuous extension of f. We define $T : \mathcal{C}_0(L,\mathbb{C}) \to \mathcal{C}_0(L,\mathbb{C})$ by $Th(t) = \bar{f}(t)h(t)$, T is a surjective linear isometry and we have that if $t \in K$ then $|T(|g|)(t) - g(t)| = |f(t)|g(t)| - g(t)| = 0$, and if $t \notin K$ then $|T(|g|)(t) - g(t)| \leqslant 2|g(t)| \leqslant \varepsilon$. Therefore, $\|T(|g|) - g\| \leqslant \varepsilon$. □

In particular, if $\hat{L}$ is the pseudoarc then $\mathcal{C}_0(L,\mathbb{C})$ admits almost polar decompositions.

Lemma 3.3. *Let $a, b \in \mathbb{R}^2$ be different points, and $(D_n)_{n\in\mathbb{N}}$ a sequence of chains from a to b such that $(\mathrm{mesh}(D_n))_{n\in\mathbb{N}}$ is convergent to zero and for every $n \in \mathbb{N}$ the chain D_{n+1} is contained in D_n. Given $i \in \mathbb{N}$ and a continuous function $g : \overline{D_1^*} \to [0,1]$ with $g(a) = 0$ and $g(b) > 1 - 1/\#D_i$, there exist $q \in \mathbb{N}$ and $N : \{1,\dots,\#D_q\} \to \{1,\dots,\#D_i\}$ which verify:*

(1) *N is a pattern with $N(1) = 1$ and $N(\#D_q) = \#D_i$.*
(2) *For every $r \in \{1,2,\dots,\#D_q\}$, $d(q)_r \subset g^{-1}(\frac{N(r)-2}{\#D_i}, \frac{N(r)}{\#D_i}]$.*

Proof. Let $\delta > 0$ be such that if $\|x - y\| < \delta$ then $|g(x) - g(y)| < 1/\#D_i$. Take $q \in \mathbb{N}$ with $\mathrm{mesh}(D_q) < \delta$, for each $r \in \{1,\dots,\#D_q\}$ there exist $\alpha \in [0,1]$ and $k \in \{1,\dots,\#D_i\}$ verifying $d(q)_r \subset g^{-1}[\alpha, \alpha + \frac{1}{\#D_i}] \subset g^{-1}(\frac{k-2}{\#D_i}, \frac{k}{\#D_i}]$.

Let us construct N inductively. Define $N(1) = 1$. As $a \in d(q)_1$, we can assure that $d(q)_1 \subset g^{-1}(\frac{-1}{\#D_i}, \frac{1}{\#D_i}]$. Let $r \in \{1,\dots,\#D_q - 1\}$ be such that $N(1), N(2),\dots,N(r)$ have already been defined, verifying that $N : \{1,\dots,r\} \to \{1,\dots,\#D_i\}$ is a pattern and $d(q)_r \subset g^{-1}(\frac{N(r)-2}{\#D_i}, \frac{N(r)}{\#D_i}]$. Choose $k \in \{1,\dots,\#D_i\}$ such that $d(q)_{r+1} \subset g^{-1}(\frac{k-2}{\#D_i}, \frac{k}{\#D_i}]$. As $d(q)_r \cap d(q)_{r+1} \neq \emptyset$, necessarily $|N(r) - k| \leqslant 1$. Define $N(r+1) = k$, it is clear that $N : \{1,\dots,r+1\} \to \{1,\dots,\#D_i\}$ is a pattern.

Finally, $b \in d(q)_{\#D_q} \subset g^{-1}(\frac{N(\#D_q)-2}{\#D_i}, \frac{N(\#D_q)}{\#D_i}]$ implies $N(\#D_q) = \#D_i$. □

Theorem 3.4. *Let P be a pseudoarc from the point a to the point b. Given $\varepsilon > 0$, there exists $j : P \to [0,1]$ such that for every continuous surjective function $f : P \to [0,1]$ and every $x \in f^{-1}(0)$ there exists $\varphi : P \to P$ homeomorphism such that $\varphi(x) = a$ and $|j(\varphi(t)) - f(t)| < \varepsilon$ for each $t \in P$.*

Proof. Let $(D_n)_{n\in\mathbb{N}}$ be a sequence of chains from the point a to the point b with $P = \bigcap_{n\in\mathbb{N}} D_n^*$ and such that for every $n \in \mathbb{N}$ the links of D_n are connected, $\mathrm{mesh}(D_n) < 1/n$, the chain D_{n+1} is crooked in D_n, and every link of D_{n+1} has its closure included in a link of D_n. Also, we can suppose without loss of generality that for every $n \in \mathbb{N}$ we have $d(n+1)_1 \subset d(n)_1$, $d(n+1)_{\#D_{n+1}} \subset d(n)_{\#D_n}$, $d(n)_1$ is the only link of D_n which includes $\{a\}$ and $d(n)_{\#D_n}$ is the only link of D_n which includes $\{b\}$.

Choose $i \in \mathbb{N}$ such that $\#D_i > 2/\varepsilon$. Take any function $j : D_i^* \to [0,1]$ verifying $j(d(i)_k) \subset (\frac{k-2}{\#D_i}, \frac{k}{\#D_i}]$ for every $k \in \{1, 2, \ldots, \#D_i\}$ (it is easy to construct such a function).

Let $f : P \to [0,1]$ be a continuous surjective function and $x \in f^{-1}(0)$, by virtue of Corollary 2.8 there exists a homeomorphism $H : P \to P$ with $H(a) = x$ and $H(b) \in f^{-1}(1 - 1/\#D_i, 1]$. Let $g : \overline{D_1^*} \to [0,1]$ be any continuous extension of $f \circ H$, g verifies $g(a) = 0$ and $g(b) > 1 - 1/\#D_i$.

Let N and q be as in Lemma 3.3. By Theorem 2.6, there exist $t_1 \in \mathbb{N}$, $t_1 \geqslant i$, and a chain C_1 from a to b which is a consolidation of D_{t_1} and follows the pattern N in D_i. By Theorem 2.5, D_{t_1+1} is crooked in C_1. Taking into account the construction of the pseudoarc we have made and the properties of C_1, we can deduce $d(t_1+1)_1 \subset c(1)_1$ and $d(t_1+1)_{\#D_{t_1+1}} \subset c(1)_{\#C_1}$. Thus, there exists a pattern N' which D_{t_1+1} follows in C_1 and such that $N'(1) = 1$, $N'(\#D_{t_1+1}) = \#C_1 = \#D_q$. By Theorem 2.6, there exist $s_1 \in \mathbb{N}$, $s_1 \geqslant q$, and a chain B_2 which is a consolidation of D_{s_1} and follows the pattern N' in D_q. Carrying on inductively, we get to the following situation:

$$\begin{array}{ccccccccc} g & \xrightarrow{N} & B_1\ (= D_q) & \hookleftarrow & B_2\ (\succcurlyeq D_{s_1}) & \hookleftarrow & B_3\ (= D_{s_1+1}) & \hookleftarrow & \cdots \\ & \Downarrow & & \Uparrow & & \Downarrow & & \Uparrow & \\ D_i & \hookleftarrow & C_1\ (\succcurlyeq D_{t_1}) & \hookleftarrow & C_2\ (= D_{t_1+1}) & \hookleftarrow & C_3\ (\succcurlyeq D_{t_2}) & \hookleftarrow & \cdots \end{array}$$

where the vertical arrows indicate the direction in which patterns are induced, an expression like $B_2 \hookleftarrow D_{s_1+1}$ indicates that the chain D_{s_1+1} is crooked in B_2, and one such as $C_3(\succcurlyeq D_{t_2})$ means that C_3 is a consolidation of D_{t_2}.

As well as the facts explicitly stated in the figure, we should note that $\#B_n = \#C_n$ for every $n \in \mathbb{N}$, $P = \bigcap_{n\in\mathbb{N}} B_n^* = \bigcap_{n\in\mathbb{N}} C_n^*$ and $(\mathrm{mesh}(B_n))_{n\in\mathbb{N}}$, $(\mathrm{mesh}(C_n))_{n\in\mathbb{N}}$ are sequences convergent to zero.

Now we shall construct a homeomorphism $\psi : P \to P$. Given $x \in P$, there exist a sequence of natural numbers $(r_1, r_2, \ldots)$ such that $\{x\} = \bigcap_{n\in\mathbb{N}} b(n)_{r_n}$. We define the image of x by ψ by means of $\{\psi(x)\} = \bigcap_{n\in\mathbb{N}} c(n)_{r_n}$. It is straightforward to see that ψ is well defined and bijective. To see that ψ is continuous, let $x_0 \in P$ and V be an open subset of P which contains $\{\psi(x_0)\}$. Let $(r_1, r_2, \ldots)$ be a sequence of natural numbers such that $\{x_0\} = \bigcap_{n\in\mathbb{N}} b(n)_{r_n}$. There exists $n \in \mathbb{N}$ such that for every r with $\psi(x_0) \in c(n)_r$ we have $c(n)_r \cap P \subset V$. Take $U = b(n)_{r_n} \cap P$, $x_0 \in U$ and for each $x \in U$, we have $\psi(x) \in c(n)_{r_n} \cap P$, therefore $\psi(U) \subset V$.

Besides, let us see that ψ verifies, relative to g, the required inequality. Let $t \in P$. Take $(r_1, r_2, \ldots)$ sequence of natural numbers such that $\{t\} = \bigcap_{n\in\mathbb{N}} b(n)_{r_n}$, and thus $\{\psi(t)\} = \bigcap_{n\in\mathbb{N}} c(n)_{r_n}$. In particular, $\psi(t) \in c(1)_{r_1}$ for some $r_1 \in \{1, \ldots, \#C_1\} = \{1, \ldots, \#D_q\}$. By the previous lemma, $b(1)_{r_1} = d(q)_{r_1} \subset g^{-1}(\frac{N(r_1)-2}{\#D_i}, \frac{N(r_1)}{\#D_i}]$. So $g(t) \in g(b(1)_{r_1}) \subset (\frac{N(r_1)-2}{\#D_i}, \frac{N(r_1)}{\#D_i}]$ and, on the other hand, as C_1 follows the pattern N in D_i, we have

$c(1)_{r_1} \subset d(i)_{N(r_1)}$, which implies $j(\psi(t)) \in j(c(1)_{r_1}) \subset (\frac{N(r_1)-2}{\#D_i}, \frac{N(r_1)}{\#D_i}]$. We deduce that $|j(\psi(t)) - g(t)| < \frac{2}{\#D_i} < \varepsilon$.

It is also clear that $\psi(a) = a$. Finally, take $\varphi = \psi \circ H^{-1}$. □

The map j that appears in the previous theorem could be constructed being continuous, but it is not necessary.

Corollary 3.5. *If L is a locally compact, noncompact Hausdorff space such that $\hat{L}$ is the pseudoarc then $\mathcal{C}_0(L, \mathbb{C})$ is almost transitive.*

Proof. We have already seen that $\mathcal{C}_0(L, \mathbb{C})$ admits almost polar decompositions, next we shall prove that it is almost positive transitive. Take $\varepsilon > 0$ and $f, g \in \mathcal{C}_0(L, \mathbb{C})$ with $\|f\| = \|g\| = 1$ and $f(L) \cup g(L) \subset \mathbb{R}^+$. By the previous theorem, there exist a map $j : \hat{L} \to [0, 1]$ and two homeomorphisms $\varphi_f, \varphi_g : \hat{L} \to \hat{L}$ such that $\varphi_f(\infty) = \varphi_g(\infty)$ and for every $t \in \hat{L}$, we have $|j(\varphi_f(t)) - f(t)| < \varepsilon/2$ and $|j(\varphi_g(t)) - g(t)| < \varepsilon/2$. Let us observe that $\sigma : L \to L$ given by $\sigma(t) = \varphi_f^{-1}(\varphi_g(t))$ is a well-defined homeomorphism. Let $T : \mathcal{C}_0(L, \mathbb{C}) \to \mathcal{C}_0(L, \mathbb{C})$ be the surjective linear isometry given by $Th = h \circ \sigma$, for every $t \in L$, we have $|Tf(t) - g(t)| = |f(\varphi_f^{-1}(\varphi_g(t))) - g(t)| \leqslant |f(\varphi_f^{-1}(\varphi_g(t))) - j(\varphi_f(\varphi_f^{-1}(\varphi_g(t))))| + |j(\varphi_g(t)) - g(t)| < \varepsilon$. □

4. Final remarks

The author has recently known that the same counterexample to Wood's conjecture has been independently given by Kawamura [9], however his proof and the path leading to the results are substantially different to the ones followed here. As a consequence, the necessary conditions stated in Theorem 2.3 do not appear in [9].

We have obtained essentially one counterexample, since by the homogeneity of the pseudoarc, if L and L' are locally compact, noncompact Hausdorff spaces such that $\hat{L}$ and $\hat{L}'$ are the pseudoarc then L is homeomorphic to L'.

Anyway, we can easily deduce the existence of another counterexample from the results already mentioned. By Proposition 1.2, there exists a locally compact Hausdorff space L with more than one point and such that $\mathcal{C}_0(L, \mathbb{C})$ is transitive. Moreover, in [7] it is also proved that such L cannot be first countable; therefore, it is not metrizable.

As Theorem 2.3 gives us some restrictions on L, it is not too crazy to ask for a topological characterization of the locally compact Hausdorff spaces L such that $\hat{L}$ is metrizable and $\mathcal{C}_0(L, \mathbb{C})$ is almost transitive.

Perhaps a good starting point to look for another metrizable counterexample would be the pseudocircle, which is a topological space closely related to the pseudoarc (for example, every proper subcontinuum of the pseudocircle is homeomorphic to the pseudoarc).

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