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# Hausdorff measures, capacities and compact composition operators

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**Abstract** It is shown that there exist analytic self-maps $\varphi$ of the unit disc $\mathbb{D}$ inducing compact composition operators on the Hardy space $\mathcal{H}^p$, $1 \leq p < \infty$ such that the Hausdorff dimension of the set $E_\varphi = \{e^{i\theta} \in \partial\mathbb{D} : \ |\varphi(e^{i\theta})| = 1\}$ is one; sharpening a classical result due to Schwartz. Moreover, the same holds in the weighted Dirichlet spaces $\mathcal{D}_\alpha$ with $0 < \alpha < 1$. As a consequence, we deduce that there exist symbols $\varphi$ inducing compact composition operators on $\mathcal{D}_\alpha$ such that the $\alpha$-capacity of $E_\varphi$ is positive, which is no longer true for those just inducing Hilbert-Schmidt composition operators on $\mathcal{D}_\alpha$.

**Keywords** Composition operator · Weighted Dirichlet spaces · Hausdorff measures

**Mathematics Subject Classification (2000)** Primary 30C85 · 47B38

## 1 Introduction

Let $\mathbb{D}$ denote the open unit disk of the complex plane and $\partial\mathbb{D}$ its boundary. The Hardy space $\mathcal{H}^p$, $1 \leq p < \infty$, consists of holomorphic functions $f$ on $\mathbb{D}$ for which

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the norm

$$\|f\|_p = \left( \sup_{0 \le r < 1} \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p}$$

is finite. If $p = \infty$, $\mathcal{H}^\infty$ is the space of holomorphic functions $f$ on $\mathbb{D}$ such that

$$\|f\|_\infty = \sup_{\mathbb{D}} |f(z)| < \infty.$$

Fatou's Theorem asserts that any Hardy function $f$ has radial limit at $e^{i\theta} \in \partial\mathbb{D}$ except on a set Lebesgue measure zero (see [4], for instance). Throughout this work, $f(e^{i\theta})$ will denote the radial limit of $f$ at $e^{i\theta}$, i. e., $f(e^{i\theta}) = \lim_{r \to 1^-} f(re^{i\theta})$.

If $\varphi$ is an analytic function on $\mathbb{D}$ which takes $\mathbb{D}$ into itself, Littlewood Subordination Principle [8] ensures that the composition operator induced by $\varphi$

$$C_\varphi f = f \circ \varphi, \qquad (f \in \mathcal{H}^p)$$

is bounded on $\mathcal{H}^p$, $1 \le p \le \infty$.

On the other hand, compactness of composition operators have attracted the attention of many experts in the area for decades. If $p = \infty$, $C_\varphi$ is compact on $\mathcal{H}^\infty$ if and only if $\|\varphi\|_\infty < 1$. When $1 \le p < \infty$, compact composition operators on $\mathcal{H}^p$ were characterized in terms of asymptotic properties of distribution values of the inducing symbols in [14] (see also [13]). For a comprehensive treatment of related problems concerning composition operators on spaces of analytic functions we refer to Cowen and MacCluer's book [3].

In this work, we focus on the relationship between the compactness of composition operators and the size of the boundary set

$$E_\varphi = \{e^{i\theta} \in \partial\mathbb{D} : \ |\varphi(e^{i\theta})| = 1\}.$$

Observe that, since $\varphi \in \mathcal{H}^\infty$, $\varphi(e^{i\theta})$ is defined a. e. on $\partial\mathbb{D}$ and therefore, the set $E_\varphi$ consists of those points $e^{i\theta} \in \partial\mathbb{D}$ such that $\varphi(e^{i\theta})$ is defined and $\varphi(e^{i\theta}) \in \partial\mathbb{D}$. Obviously, if $C_\varphi$ is compact on $\mathcal{H}^\infty$, the set $E_\varphi$ is empty. The first result relating compactness of $C_\varphi$ to the size of $E_\varphi$ was shown by Schwartz [12] in the sixties. He proved that if $C_\varphi$ is compact on $\mathcal{H}^p$, then the Lebesgue measure of $E_\varphi$ is zero. We ask how sharp is Schwartz's result in terms of Hausdorff dimension. Actually, it is the best one could expect as our main result states:

**Main Theorem** *There exists a compact composition operator $C_\varphi$ on $\mathcal{H}^p$ for all $1 \le p < \infty$ such that the Hausdorff dimension of $E_\varphi$ is one.*

So, our main theorem could be read as follows: there exist maps inducing compact composition operators on $\mathcal{H}^p$, $1 \le p < \infty$, approaching to the boundary $\partial\mathbb{D}$ in a huge set!

Section 2 is devoted to proving our main result. The proof will be accomplished by constructing a simply connected domain $\Omega$ contained in $\mathbb{D}$ such that the Riemann map that takes $\mathbb{D}$ onto $\Omega$ induces a compact composition operator on $\mathcal{H}^p$ with the required behavior.

In section 3 we introduce the weighted Dirichlet spaces $\mathcal{D}_\alpha$, with $\alpha > -1$. We show that the composition operator $C_\varphi$ in our main result is actually compact on $\mathcal{D}_\alpha$ for any $\alpha > 0$. In particular we deduce that the $\alpha$-capacity of $E_\varphi$ is positive. Nevertheless, this is no longer true for Hilbert-Schmidt composition operators on $\mathcal{D}_\alpha$ whenever $\alpha \ge 0$, as shown in [5] and [6].

## 2 Proof of Main Result

This section is devoted to proving Main Theorem. The proof will be accomplished by constructing a simply connected domain $\Omega$ contained in $\mathbb{D}$ such that the Riemann map taking $\mathbb{D}$ onto $\Omega$ induces a compact composition operator on $\mathcal{H}^2$ and furnishes the required conditions. From here, the statement of our Main Theorem will follow just recalling the fact that if $C_\varphi$ is compact on $\mathcal{H}^p$ for some $1 \leq p < \infty$, then it is compact on $\mathcal{H}^p$ for all $1 \leq p < \infty$ (see [15, Theorem 6]).

Before proceeding with the construction of the domain $\Omega$, we recall the concepts of the $\alpha$-Hausdorff dimensional measure and the Hausdorff dimension of a set, for the sake of completeness.

### 2.1 $\alpha$-Hausdorff dimensional measure

Let $0 < \alpha \leq 1$ be any real number and $E$ a Borel set contained in $\partial\mathbb{D}$. For any $\varepsilon > 0$ we define

$$\Lambda_\alpha^\varepsilon(E) = \inf\left\{\sum_{k=1}^{\infty}(\text{diam } B_k)^\alpha \, : \, E \subset \bigcup_{k=1}^{\infty} B_k, \ \text{diam } B_k \leq \varepsilon\right\}$$

where diam $B_k$ denotes the diameter of the set $B_k$, that is,

$$\text{diam } B_k = \sup\{|x - y| \, : \, x, y \in B_k\}.$$

The $\alpha$-dimensional Hausdorff measure of $E$ is defined by

$$\Lambda_\alpha(E) = \lim_{\varepsilon \to 0^+} \Lambda_\alpha^\varepsilon(E).$$

Observe that $\alpha = 1$ corresponds to the Lebesgue measure of $E$, which will be denoted by $|E|$ in what follows. In addition, note that if $\Lambda_{\alpha_1}(E) = 0$ then $\Lambda_{\alpha_2}(E) = 0$ for any $\alpha_2 > \alpha_1$. Recall that the Hausdorff dimension of $E$ is defined by

$$\text{d}(E) = \inf\{\alpha : \Lambda_\alpha(E) = 0\}.$$

It holds that $\Lambda_\alpha(E) = \infty$ whenever $0 < \alpha < \text{d}(E)$, and $\Lambda_\alpha(E) = 0$ for $\alpha > \text{d}(E)$; while $0 \leq \Lambda_{\text{d}(E)}(E) \leq \infty$. For more about Hausdorff measures, we refer the reader to Pommerenke's book [11].

The following definition will be also useful in the construction of the domain $\Omega$ in the proof of our main theorem.

**Definition 2.1** Let $\varepsilon$ and $\delta$ be positive numbers. A bump of height $\varepsilon$ supported on the interval $[-\delta, \delta]$ is the graph of the real function

$$f(x) = \begin{cases} \varepsilon \exp\left(\frac{x^2}{x^2-\delta^2}\right) & -\delta < x < \delta \\ 0 & x \in \mathbb{R} \setminus (-\delta, \delta). \end{cases}$$

*A word about notation.* In the sequel, we will denote $a \simeq b$ whenever there exists two universal constants $c$ and $C$ such that $c\, a \leq b \leq C\, a$.

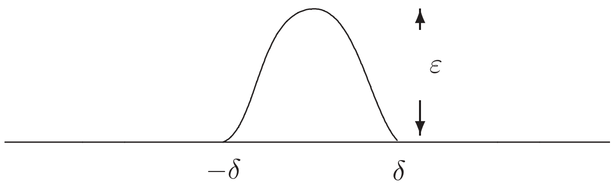


**Fig. 1** Bump of height $\varepsilon$ supported on $[-\delta, \delta]$

*Proof of Main Theorem.* First, we point out that the construction of the simply connected domain $\Omega$ such that a Riemann map $\varphi$ taking $\mathbb{D}$ onto $\Omega$ satisfies the required condition will be carried out by constructing a Jordan curve contained in $\overline{\mathbb{D}}$ which will correspond to the boundary $\partial\Omega$.

In order to do that, let $\{\delta_n\}_{n\geq 0}$ and $\{\varepsilon_n\}_{n\geq 0}$ be two decreasing sequences of positive numbers in $[0, 1]$ such that $\lim_n \delta_n = 0$ and $\lim_n \varepsilon_n = 0$. Throughout the proof we will impose additional conditions on $\delta_n$ and $\varepsilon_n$ so that the theorem holds.

Firstly, we construct a Cantor set $E$ contained in the unit interval $[0, 1]$ with variable ratio of dissection depending on the sequence $\{\delta_n\}_{n\geq 0}$. For that purpose, let $I_0$ denote the unit interval $[0, 1]$. Let $E_n$ be the $n$-th approximation of the Cantor set $E$ which consists of $2^n$ open intervals obtained as follows: if $E_{n-1} = \bigcup_{k=1}^{2^{n-1}} I_{n-1}^k$, the intervals in $E_n$ are obtained from $I_{n-1}^k$, $1 \leq k \leq 2^{n-1}$, by removing the middle third closed interval $J_n^k$ of length $\delta_n |I_{n-1}^k|$. Thus, the Cantor set $E$ is defined by

$$E = \bigcap_{n\geq 0} E_n.$$

Note that, by construction,

$$E_{n-1} \setminus E_n = \bigcup_{k=1}^{2^{n-1}} J_n^k.$$

In addition, we observe that the length of the two intervals obtained at the stage $n$ from $I_{n-1}^k$, for $1 \leq k \leq 2^{n-1}$, is the same. Thus, in what follows, we will denote by $|I_n^k|$ such a length. It is clear that

$$|I_n^k| = |I_{n-1}^k|(1 - \delta_n)/2, \qquad 1 \leq k \leq 2^{n-1}.$$

Now, for $1 \leq k \leq 2^{n-1}$, let $\Gamma_n^k$ be a bump supported on the interval $J_n^k$ of height $\varepsilon_n\, |I_n^k|$ (see Figure 2).

Set $\mathcal{R} = \bigcup_n \bigcup_k \Gamma_n^k$ (see Figure 3) and consider the natural identification of the unit circle $\partial\mathbb{D}$ with $[0, 1)$.

Let $\Omega$ be the simply connected domain contained in $\mathbb{D}$ whose boundary $\partial\Omega$ is the curve $\mathcal{R}$. Let $\varphi$ be a Riemann map that takes $\mathbb{D}$ onto $\Omega$ satisfying $\varphi(0) = 0$. We will show that, under certain restrictions on the sequences $\{\varepsilon_n\}$ and $\{\delta_n\}$, the map $\varphi$ has the desired behavior.

*Claim 1* If $\delta_n = 1/\log(n + 2)$, then the Cantor set $E$ has Lebesgue measure zero and Hausdorff dimension one.

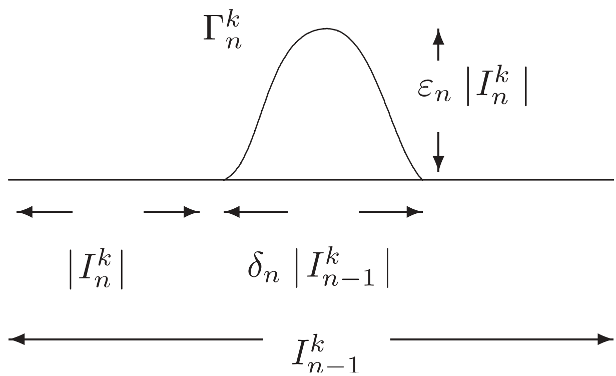


**Fig. 2**

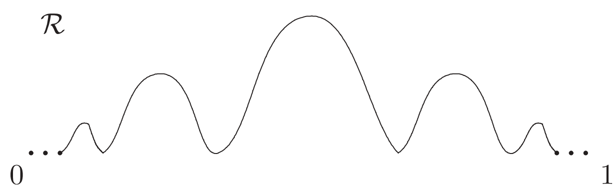


**Fig. 3** Curve $\mathcal{R}$

*Proof of Claim 1.* If $E_n = \bigcup_{k=1}^{2^n} I_n^k$, by standard arguments to estimate the Hausdorff dimension of a Cantor set (see [11, Chapter 10] or [2], for instance), it is enough to ensure that for any $1 \le k \le 2^n$

$$\lim_{n\to\infty} 2^n \, |I_n^k| = 0 \tag{1}$$

along with

$$\lim_{n\to\infty} 2^n \, |I_n^k|^\alpha = \infty \qquad (0 < \alpha < 1). \tag{2}$$

Condition (1) implies that the Lebesgue measure of $E$ is zero, whereas from condition (2) follows that $\Lambda_\alpha(E) = \infty$ for any $0 < \alpha < 1$, and therefore the Hausdorff dimension of $E$ is one.

To check that both conditions (1) and (2) hold whenever $\delta_n = 1/\log(n+2)$, we observe that, by construction, all the intervals obtained at the same level have the same length. So far, for any $1 \le k \le 2^n$ we can assert that the length of $I_n^k$ is $|I_{n-1}^1|(1-\delta_n)/2$. Thus, iterating we deduce that

$$\lim_{n\to\infty} 2^n \, |I_n^k| = \prod_{j=1}^{\infty}(1-\delta_j),$$

and

$$\lim_{n\to\infty} 2^n \, |I_n^k|^\alpha = \lim_{n\to\infty} 2^{(1-\alpha)n} \prod_{j=1}^{n}(1-\delta_j)^\alpha.$$

Therefore, it is enough to take into account that $\prod_j(1-\delta_j) \simeq \exp(-\sum_j \delta_j)$ to get the desired conditions on the Cantor set $E$, and therefore, the statement of the Claim 1.

Observe that the conditions required on $E$ are actually those ones that the set $E_\varphi$ must satisfy. Although $E_\varphi = \varphi^{-1}(E)$ by construction, it is not possible to guarantee straightforward that $E_\varphi$ satisfies such conditions. It will require a little work.

Firstly, the fact that the Lebesgue measure of $E_\varphi$ is zero follows since $E$ has also Lebesgue measure zero and, by construction, the boundary $\partial\Omega$ is a rectificable curve (see Riesz's Theorem in [11, Chapter 6]).

Secondly, to show that the Hausdorff dimension of $E_\varphi$ is one, we will show that, under certain restrictions for $\varepsilon_n$, the curve $\partial\Omega$ is smooth. Recall that a curve is smooth if there is a parametrization $w$ of the curve of class $\mathcal{C}^1$ (see [11, Chapter 3]). In particular, this implies that $\partial\Omega$ is asymptotically smooth, that is

$$\max_{w\in\partial\Omega(a,b)} \frac{|a-w|+|w-b|}{|a-b|} \to 1 \qquad \text{as } a, b \in \partial\Omega,\ |a-b| \to 0,$$

where $\partial\Omega(a,b)$ denotes the smaller arc of $\partial\Omega$ between $a$ and $b$ (see [11, Chapter 11] for more about asymptotically smooth curves). Therefore, $\varphi : \mathbb{D} \to \Omega$ is $\alpha$-Hölder continuous for any $0 < \alpha < 1$ (see [11, Chapter 11, Exercise 11.2.1], for instance). Thus, the $\alpha$-Hausdorff measure is preserved for any $0 < \alpha < 1$, and therefore we may deduce that $E_\varphi$ has Hausdorff dimension one since so does $E$. We claim the following:

*Claim 2* If the sequence $\{\varepsilon_n/\delta_n\}$ tends to zero, then $\partial\Omega$ is smooth.

*Proof of Claim 2.* First, note that $\partial\Omega = E \bigcup \left(\bigcup_{n,k} \Gamma_n^k\right)$ and each of the bumps $\Gamma_n^k$ is a $\mathcal{C}^\infty$ curve. Even more, if the construction of the Cantor set stops at the stage $n_0$, and we consider $\Omega_{n_0} = \mathbb{D} \setminus \bigcup_{n=1}^{n_0} \bigcup_k \Gamma_n^k$, the domain $\Omega_{n_0}$ is bounded by a $\mathcal{C}^\infty$ curve. So, roughly speaking, to show that $\partial\Omega$ is smooth, we only need to consider the points in the Cantor set $E$ which are limit of an infinite sequence of decreasing bumps. Note that each of such bumps belongs to a different generation $n$.

Let $n$ be fixed. For any $1 \le k \le 2^n$, the bump $\Gamma_n^k$, of height $\varepsilon_n |I_n^k|$, is supported on the interval $J_n^k$, whose length is $\delta_n |I_{n-1}^k|$. Therefore, its slope $m_n^k$ is bounded by

$$m_n^k \le \frac{\varepsilon_n\ |I_n^k|}{\delta_n\ |I_{n-1}^k|} \le \frac{\varepsilon_n}{\delta_n}$$

(see Figure 4). It is clear that if $\lim_n \varepsilon_n/\delta_n = 0$, then $\partial\Omega$ is smooth; which completes the proof of Claim 2.

According to the above, we have constructed a domain $\Omega \subset \mathbb{D}$ such that a Riemann map $\varphi$ taking $\mathbb{D}$ onto $\Omega$ satisfies that the set $E_\varphi$ has Lebesgue measure zero but Hausdorff dimension one. The task is now to show that such a $\varphi$ induces a compact composition operator $C_\varphi$ on $\mathcal{H}^2$. The key point for that is the Angular Derivative Criterion (see [9, Theorem 5.3]), which states that for univalent

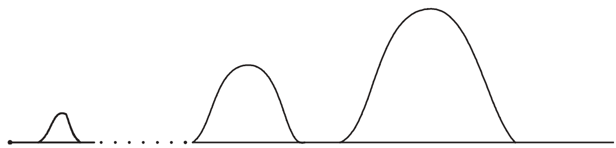

**Fig. 4**

symbols $\varphi$, compactness of $C_\varphi$ is equivalent to the fact that the angular derivative of $\varphi$ at any point $e^{i\theta} \in E_\varphi$ is infinite (see also [3, Corollary 3.21]).

Let $e^{i\theta_0} \in E_\varphi$ be fixed. By construction, $\varphi(e^{i\theta_0})$ belongs to the Cantor set $E$. Recall that the angular derivative at $e^{i\theta_0}$ is defined by the angular limit

$$\varphi'(e^{i\theta_0}) = \angle \lim_{z \to e^{i\theta_0}} \frac{\varphi(z) - \varphi(e^{i\theta_0})}{z - e^{i\theta_0}} \qquad (z \in \mathbb{D}). \tag{3}$$

That is, $z \to e^{i\theta_0}$ through any non tangential approach region at $e^{i\theta_0}$. For more about angular derivatives and composition operators see, for example, [3, Chapter 2]. In our case, by construction, to prove that $\varphi'(e^{i\theta_0}) = \infty$, it is enough to show that

$$\lim_n \frac{|I_n^k|}{|\varphi^{-1}(I_n^k)|} = \infty, \tag{4}$$

for any $1 \le k \le 2^n$.

Let us fix $1 \le k \le 2^n$. To estimate $|\varphi^{-1}(I_n^k)|$, first we consider the relation between the Lebesgue measure of a set $A \subset \partial\mathbb{D}$ and the harmonic measure of a $A$ in the unit disc $\mathbb{D}$ at the origin $\omega(0, A, \mathbb{D})$, given by $|A| = 2\pi\, \omega(0, A, \mathbb{D})$. Now, the invariance of the harmonic measure under conformal mappings yields that

$$\omega(0, \varphi^{-1}(I_n^k), \mathbb{D}) = \omega(0, I_n^k, \Omega),$$

and therefore the limit in (4) diverges if and only if

$$\lim_n \frac{|I_n^k|}{\omega(0, I_n^k, \Omega)} = \infty. \tag{5}$$

Assume, for the moment, that the following claim is already proved.

*Claim 3* Let $\varepsilon$ and $\delta$ be positive numbers. Let $\Gamma$ be a bump supported on the interval $J = [-\delta, \delta]$ of height $\varepsilon(1-\delta)$. Let $\mathbb{R}^2_+ = \{(x, y) : \ y > 0\}$ and $\widetilde{\Omega} \subset \mathbb{R}^2_+$ be the domain $\widetilde{\Omega} = \mathbb{R}^2_+ \setminus \Gamma$. If $I$ is the interval $(\delta, 1)$, then

$$\omega(i, I, \widetilde{\Omega}) \le (1 - C_0\, \varepsilon\, \delta)|I|, \tag{6}$$

where $C_0$ is a universal constant (see Figure 5).

Then, with Claim 3 at hand and iterating formula (6) normalized at each stage, it follows that

$$\omega(0, I_n^k, \Omega) \le \left( \prod_{j=1}^{n} (1 - C_0 \varepsilon_j \, \delta_j) \right) |I_n^k|.$$

Therefore, it follows that the limit in (5) diverges if the series

$$\sum_{j=1}^{\infty} \varepsilon_j \, \delta_j = \infty. \tag{7}$$

Now, since $\delta_n = 1/\log(n+2)$ by Claim 1, it is enough to chose $\varepsilon_n = 1/(n+2)$ so that the conditions imposed by the Claim 2 and equation (7) are satisfied. Thus, the proof of Main Theorem will be completed whenever we prove Claim 3.

*Proof of Claim 3.* Let $\Phi$ be a Riemann mapping from $\mathbb{R}^2_+$ onto $\widetilde{\Omega}$ with $\Phi(i) = i$. We assert that there exists a universal constant $C_1$ such that

$$|\Phi'(x)| \ge 1 + C_1 \varepsilon \, \delta \tag{8}$$

for all $x \in \Phi^{-1}(I)$. If this is the case, observe that

$$|I| = \int_{\Phi^{-1}(I)} |\Phi'(x)| \, dx \ge (1 + C_1 \varepsilon \, \delta) |\Phi^{-1}(I)|,$$

and therefore, inequality (6) follows since $|\Phi^{-1}(I)| \simeq \omega(i, I, \widetilde{\Omega})$. So, to get Claim 3 proved, it is enough to show that equation (8) holds.

For this purpose, recall that the Hilbert transform of a function $g \in L^2(\mathbb{R})$ is given by

$$(\mathcal{H}g)(x) = \lim_{\eta \to 0^+} \frac{1}{\pi} \int_{|x-t|>\eta} \frac{g(t)}{(x-t)} \, dt,$$

and it represents the boundary values of the harmonic conjugate of the harmonic extension of $g$ to $\mathbb{R}^2_+$. For more on this topic, we refer to Garnett's book [7].

Since $i \log \Phi' = -\arg \Phi' + i \log |\Phi'|$ represents an analytic function in $\mathbb{R}^2_+$, we can write $\log |\Phi'(x)|$ as the Hilbert transform of the function $g(x) = -\arg \Phi'(x)$, for $x \in \mathbb{R}$. Note that $g(x)$ is odd and vanishes outside an interval which will be denoted by $J_1 = (-\delta_1, \delta_1)$.

Let $x \in \Phi^{-1}(I)$ be fixed. Then,

$$\begin{aligned} \log |\Phi'(x)| &= \lim_{\eta \to 0^+} \frac{1}{\pi} \int_{\substack{|t| < \delta_1 \\ |x-t| > \eta}} \frac{g(t)}{x-t} \, dt \\ &\ge \lim_{\eta \to 0^+} \frac{1}{\pi} \int_{\substack{0 < t < \delta_1 \\ |x-t| > \eta}} g(t) \left( \frac{1}{x-t} - \frac{1}{x+t} \right) dt. \end{aligned} \tag{9}$$

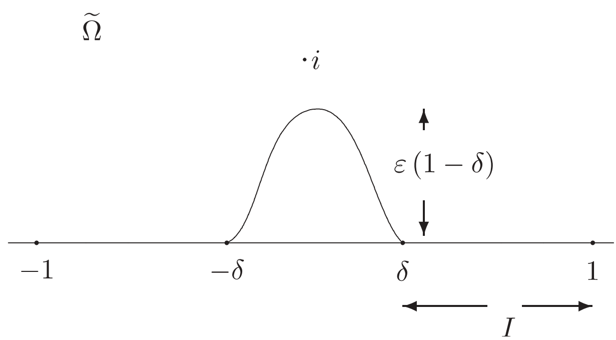


**Fig. 5**

Now, we observe that $g(t) \geq 0$ if $t \in [0, \delta_1]$. Moreover, there exists an open interval $J_2 \subset [0, \delta_1]$ such that $g(t) \simeq \varepsilon/\delta$ for any $t \in J_2$. Therefore, it follows that (9) is bigger than or equal to

$$\lim_{\eta \to 0^+} \frac{1}{\pi} \int_{\substack{J_2 \\ |x-t|>\eta}} g(t)\, \frac{t}{x^2 - t^2}\, dt \gtrsim \frac{\varepsilon}{\delta}\, |J_2|^2. \tag{10}$$

Using the conformal invariance of the harmonic measure, we may conclude from (9) and (10) that

$$\log |\Phi'(x)| \gtrsim \frac{\varepsilon}{\delta}\, \left(\omega(i, \Phi(J_2), \Omega)\right)^2 . \tag{11}$$

The last ingredient in the proof comes from the observation that, for any set $E \subset J_1 = (-\delta_1, \delta_1)$ holds the following inequality

$$\omega(i, \Phi(E), \Omega) \geq \omega(i, \Phi(E)^*, \mathbb{R}^2_+), \tag{12}$$

where $\Phi(E)^*$ denotes the projection of $\Phi(E)$ on to $\mathbb{R}$. Note that, actually, $\Phi(E)$ is a piece of the bump $\Gamma$ (see Figure 6).

Once again the conformal invariance of the harmonic measure along with the fact that $\Phi(J_2)^*$ are just the points $x \in (0, \delta)$ for which the slope of the bump $\Gamma$ is comparable to $\varepsilon/\delta$, yields that

$$\omega(i, \Phi(J_2)^*, \mathbb{R}^2_+) \simeq |\Phi(J_2)^*| \simeq \delta. \tag{13}$$

Then, combining (12) and (13) in (11), it follows

$$\log |\Phi'(x)| \gtrsim \varepsilon\, \delta$$

for any $x \in \Phi^{-1}(I)$. From here, equation (8) follows, which completes the Claim 3. Therefore, the proof of our Main Theorem is now complete.

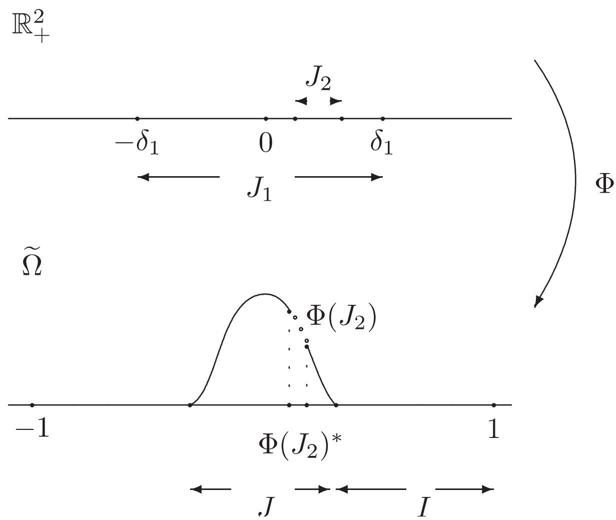


**Fig. 6**

## 3 Compact composition operators on $\mathcal{D}_\alpha$

In this section, we overtake some previous results which allow us, along with our main result, to analyze the size of $E_\varphi$ whenever $C_\varphi$ is compact on the weighted Dirichlet spaces $\mathcal{D}_\alpha$.

Recall that for $\alpha > -1$, the weighted Dirichlet space $\mathcal{D}_\alpha$ consists of analytic functions $f$ on $\mathbb{D}$ for which the norm

$$\|f\|_\alpha = \left(|f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2(1-|z|^2)^\alpha \, dA(z)\right)^{1/2}$$

is finite. Here, $dA(z) = \frac{1}{\pi}\, dx\, dy$ denotes the normalized Lebesgue area measure on $\mathbb{D}$. Particular instances of $\alpha$ yield well known Hilbert spaces of analytic functions. Indeed, $\alpha = 0$ corresponds to the Dirichlet space $\mathcal{D}$. If $\alpha = 1$ the norm obtained is equivalent to the usual one in the Hardy space $\mathcal{H}^2$. Analogously, for $\alpha = 2$, we get the Bergman space $\mathcal{A}^2$. Note that if $\alpha_1 < \alpha_2$, then $\mathcal{D}_{\alpha_1}$ is strictly contained in $\mathcal{D}_{\alpha_2}$.

If $\alpha > 1$, it was shown in [9] that there exists an inner function $\varphi$ inducing a compact $C_\varphi$ on $\mathcal{D}_\alpha$ (see Example 3.6). So, in such a case, $E_\varphi = \partial\mathbb{D}$, and therefore its Lebesgue measure is positive. Obviously, the Hausdorff dimension of $E_\varphi$ is one.

When $-1 < \alpha < 0$ and $C_\varphi$ is compact on $\mathcal{D}_\alpha$, the set $E_\varphi$ is empty (see [3, Theorem 4.5]) and therefore, there is nothing to discuss.

On the other hand, since the map $\varphi$ constructed in the proof of main result is univalent, it follows that $C_\varphi$ is bounded on the Dirichlet space $\mathcal{D}$. Moreover, the angular derivative of $\varphi$ fails to exist at each point of the boundary $\partial\mathbb{D}$, so upon applying the Angular Derivative Criterion (see [9, Theorem 5.3]), it follows that $C_\varphi$ is actually compact on $\mathcal{D}_\alpha$ for any $\alpha > 0$. Then, we may conclude.

**Corollary 3.1** *There exists a compact composition operator $C_\varphi$ on $\mathcal{D}_\alpha$, $\alpha > 0$, such that the Hausdorff dimension of $E_\varphi$ is one.*

*Remark 3.2* We note that the map $\varphi$ in the proof of Main Theorem does not induce a compact composition operator on the Dirichlet space $\mathcal{D}$, that is, $\alpha = 0$. To check this, it is enough to show that for any Carleson disk $S(\xi, \delta) = \{z \in \mathbb{D} : |z - \xi| < \delta\}$ centered in $\xi \in \partial\mathbb{D}$ of radius $\delta$, $0 < \delta < 1$, the limit $\lim_{\delta \to 0} A\left(S(\xi, \delta) \cap \Omega\right)/\delta^2$ is not zero. For the characterization of compact composition operators on the Dirichlet space see [9, Proposition 5.9], for instance.

On the other hand, if $0 \le \alpha < 1$, there is a close relation between functions in $\mathcal{D}_\alpha$ and the concept of $\alpha$-capacity of a set. Recall that, for $0 \le \alpha < 1$ and any measure $\mu$ of bounded support $S_\mu$, the $\alpha$-potential of $\mu$ is defined by

$$u_\alpha^\mu(x) = \begin{cases} \displaystyle\int \log \frac{1}{|x-y|}\, d\mu(y), & \alpha = 0; \\[2ex] \displaystyle\int \frac{d\mu(y)}{|x-y|^\alpha}, & 0 < \alpha < 1. \end{cases}$$

If $I_\alpha(\mu)$ denotes the energy integral of $\mu$,

$$I_\alpha(\mu) = \int u_\alpha^\mu \, d\,\mu(x),$$

then the $\alpha$-capacity of a bounded Borel set $E$ is defined by

$$C_\alpha(E) = \{\inf I_\alpha(\mu)\}^{-1}$$

where the infimum is taken over all positive measures $\mu$ with total mass 1 and support $S_\mu$ contained in $E$. When $\alpha = 0$, the $\alpha$-capacity is also called logarithmic capacity. Observe that there exist Borel sets $E$ of Lebesgue measure zero and $\alpha$-capacity positive for any $0 \le \alpha < 1$. In addition, if $C_{\alpha_1}(E) = 0$ then $C_{\alpha_2}(E) = 0$ for any $\alpha_2 > \alpha_1$. For more about capacities, we refer to Carleson's book [2].

When functions in the weighted Dirichlet spaces $\mathcal{D}_\alpha$ are considered, the following relation holds: if $f \in \mathcal{D}_\alpha$, then the radial limits $f(e^{i\theta}) = \lim_{r \to 1^-} f(re^{i\theta})$ exist except on a set of $\alpha$-capacity zero (see [1] and [16]).

In addition, when $\alpha > 0$, there is a connection between the $\alpha$-capacity and the $\alpha$-dimensional Hausdorff measure. In fact, a Meyers' result on Bessel capacities [10] implies that $C_\alpha(E) = 0$ whenever $\Lambda_\alpha(E) < \infty$. Moreover, if $C_\alpha(E) = 0$ then $\mathcal{H}_\beta(E) = 0$ for any $\beta > \alpha$ (see also [17]). Thus, as a consequence, we may deduce the following

**Corollary 3.3** *There exists a compact composition operator $C_\varphi$ on $\mathcal{D}_\alpha$, $0 \le \alpha < 1$, such that the $\alpha$-capacity of $E_\varphi$ is positive.*

The statement in Corollary 3.3 for $\alpha = 0$ corresponds to Theorem 3.1 in [5], where the authors provided an example of a compact composition operator $C_\varphi$ on $\mathcal{D}$ such that the logarithmic capacity of $E_\varphi$ is positive.

Finally, for $0 < \alpha < 1$, Corollary 3.3 completes a previous result proved in [6] which asserts that the $\alpha$-capacity of $E_\varphi$ is zero whenever $C_\varphi$ is Hilbert-Schmidt on $\mathcal{D}_\alpha$.

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