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On the Calogero–Degasperis–Fokas equation in (2 + 1) dimensions

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Abstract

In this paper we study a (2 + 1)-dimensional integrable Calogero–Degasperis–Fokas equation derivable by using a method proposed by Calogero. A catalogue of classical symmetry reductions are given. These reductions to partial differential equations in (1+1) admit symmetries which lead to further reductions, i.e., to second-order ordinary differential equations. These ODEs provide several classes of solutions; all of them are expressible in terms of known functions, some of them are expressible in terms of the second and third Painleve trascendents. The corresponding solutions of the (2 + 1)-dimensional equation, involve up to three arbitrary smooth functions. Consequently, the solutions exhibit a rich variety of qualitative behaviour. Indeed by making appropriate choices for the arbitrary functions, we exhibit solitary waves, coherent structures and bound states.

1. Introduction

In this paper, we discuss the generalized (2 + 1)-dimensional integrable generalization of the Calogero–Degasperis–Fokas (CDF) equation

$$u_t + \frac{1}{4} u_{xxz} - \frac{1}{2} \frac{u_x u_{xz}}{u} - \frac{1}{4} \frac{u_{xx} u_z}{u} + \frac{1}{2} \frac{u_x^2 u_z}{u^2} - \frac{1}{8} u_x \partial_x^{-1} \left(\frac{u_x^2}{u^2} \right)_z + \frac{1}{4} a^2 u^2 u_z + \frac{1}{8} a^2 u_x \partial_x^{-1} (u^2)_z + \frac{1}{2} abu_z + \frac{1}{4} abu_x + \frac{1}{4} b^2 \frac{u_z}{u^2} + \frac{1}{8} b^2 u_x \partial_x^{-1} \left(\frac{1}{u^2} \right)_z = 0, \quad (1)$$

where $\partial_x^{-1} u = \int u \, dx$. This equation has been derived by Toda and Yu [1].

A wide class of differential equations with interesting properties are integrable by the inverse spectral transformation method. One of these equations is the CDF equation in (1 + 1) dimensions. The CDF equations is a (1 + 1)-dimensional nonlinear equation having the form

$$u_t + \frac{1}{4} u_{xxx} - \frac{3}{4} \frac{u_x u_{xx}}{u} + \frac{3}{8} \frac{u_x^3}{u^2} + \frac{3}{8} \frac{u_x}{u^2} (au^2 + b)^2 = 0, \quad (2)$$

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where a and b are arbitrary constants. Eq. (2) was introduced by Calogero and Degasperis [2] investigating equations solvable by a matrix variant of the inverse transformation, and independently by Fokas [3] investigating KdV-type equations with certain Lie–Backlund symmetries. Exact multi-soliton solutions to (2) was obtained from its bilinear form (when $a>0$) [4]. The CDF equation was studied also by other authors [5,6]. In Ref. [7], an extended Dym equation was generated by the purely binormal motion of an inextensible curve of constant curvature. This extended Dym equation is readily established by a reciprocal link to the CDF equation. Besides, it is well known that CDF equation can be reduced to: the Calogero–Korteweg–de Vries (CKdV) equation [8] when $a=0$ and $b=\pm 1$, the Chen equation [9] when $a=-b=1$, after the transformation $u=\exp(kw)$ and $w\rightarrow \pm iw$, the Schwartian KdV (SKdV) equation when $a=b=0$ and $u=\phi_x$ which is a potential transformation of u. The $(2+1)$ (SKdV) has been considered in Ref. [10].

It is well known that the similarity solutions of integrable nonlinear partial differential equations (PDEs) give rise to Painlevé trascendents [11–14]. This connection between Painlevé equations and soliton-type equations has led to the Ablowitz, Ramani and Segur (ARS) conjecture [15]. Namely, it was exemplified that ODEs obtained as reductions of the well-known soliton equations yield ODEs with the Painlevé property (PP). A modern survey about PP can be found in Refs. [16,17]. Moreover, similarity reductions of the best-known soliton equations lead to second-order Painlevé equations [11,18]. In Ref. [10] classes of solutions of the Schwarzian Korteweg–de Vries equation in $(2+1)$ dimensions has been derived, all of them expressible in terms of known functions such as the second and third Painlevé trascendents.

Similarity reductions of partial differential equations (PDEs) to ODEs of Painlevé type are very important theoretically. In this sense it is of great interest to consider the similarity reductions of the CDF equation in $(2+1)$ dimensions to ODEs. In Ref. [19], the similarity reductions of the CDF modified KdV equation in $(1+1)$ dimensions have been obtained. In Ref. [20], the Painlevé property was defined for PDEs and the connection between the PP and the occurrence of the Lax pair and Bäcklund transformation was demonstrated for the KdV equation [21], as well as for some other well-known equations.

Lou [22] proposed a $(1+1)$-dimensional integrable model with space–time exchange symmetry under the meaning that the model can be changed to a form with the Painlevé property. A list of $(1+1)$-dimensional integrable equations and their symmetries has been done in Ref. [23].

The study of higher-dimensional integrable systems is one of the main themes in integrability theory. Several models in the context of $(2+1)$-dimensional equations, i.e., equations with two spatial and one temporal variables, which are integrable have been developed by Toda and Yu [24]. These equations has been recently derived by using a method proposed by Calogero. That is, by modifying one of the operators of the Lax pair for $(1+1)$-dimension. In this way from the CDF equation they obtain Eq. (1). Although this equation arises in a non-local form it can be written as follows:

$$\begin{aligned}
&8\,\frac{u_{tx}}{u_x}-8\,\frac{u_tu_{xx}}{u_x^2}+2\,\frac{u_{xxxz}}{u_x}-2\,\frac{u_{xx}u_{xxz}}{u_x^2}-4\,\frac{u_{xxz}}{u}\\
&\quad+4\,\frac{u_xu_{xz}}{u^2}-2\,\frac{u_zu_{xxx}}{uu_x}-2\,\frac{u_{xx}u_{xz}}{uu_x}+2\,\frac{u_zu_{xx}^2}{uu_x^2}+6\,\frac{u_zu_{xx}}{u^2}\\
&\quad+2\,\frac{u_xu_{xz}}{u^2}-6\,\frac{u_zu_x^2}{u^3}+6uu_za^2+2a^2\,\frac{u^2u_{xz}}{u_x}-2a^2\,\frac{u^2u_zu_{xx}}{u_x^2}\\
&\quad+4ab\,\frac{u_{xz}}{u_x}-4ab\,\frac{u_zu_{xx}}{u_x^2}+2b^2\,\frac{u_{xz}}{u^2u_x}-2b^2\,\frac{u_zu_{xx}}{u^2u_x^2}-6b^2\,\frac{u_z}{u^3}=0.
\end{aligned}\tag{3}$$

Although there exist different tools to investigate the properties of the integrable $(2+1)$-dimensional equations we choose the Lie symmetry analysis. The invariance properties of some of the physically important nonlinear evolution equations such as Kadomtsev–Petviashvili equation (KP) and Davey–Stewartson equation (DS) have been studied through Lie symmetry analysis [25,26]. In most of the cases the corresponding Lie algebra has the Kac–Moody–Virasoro-type subalgebra, but some of the integrable $(2+1)$-dimensional equations do not admit Virasoro-type subalgebra. Examples of such equations are a breaking soliton equation introduced by Bogovalenski, a $(2+1)$-dimensional generalization of the nonlinear Schrödinger equation [27] and the Schwartian–Korteweg–de-Vries equation [10].

The classical method for finding symmetry reductions of PDEs is the Lie group method of infinitesimals transformations. Using this method we bring out the unexplored invariance properties and similarity reduced $(1+1)$ PDEs of the above Eq. (3). First we obtain a point transformation group which leaves system (3) invariant. In order to find all invariant solutions with respect to s-dimensional subalgebras, it is sufficient to construct invariant solutions for the optimal system of order s. The set of invariant solutions obtained in this way is called an *optimal system of invariant solutions*.

By using the classical Lie method we obtain reductions of the $(1+1)$-dimensional PDEs, to obtain ODEs and by further reductions to second-order integrable ODEs. The solutions of all of these ODEs are expressible in terms of known functions, some of them can be expressed in terms of the second and third Painlevé trascendents. We also derive exact solutions for the $(2+1)$-dimensional integrable generalization of the CDF equation. Some of these solutions are soliton solutions, localized on a curve and that decay exponentially apart from that curve.

2. Lie symmetries

To apply the classical method to the $(2+1)$-dimensional PDE (3), we consider the one-parameter Lie group of infinitesimal transformations in (x, t, z, u) given by

$$\begin{aligned} x^* &= x + \varepsilon\xi(x, z, t, u) + \mathcal{O}(\varepsilon^2),\\ z^* &= z + \varepsilon\zeta(x, z, t, u) + \mathcal{O}(\varepsilon^2),\\ t^* &= t + \varepsilon\tau(x, z, t, u) + \mathcal{O}(\varepsilon^2),\\ u^* &= u + \varepsilon\eta(x, z, t, u) + \mathcal{O}(\varepsilon^2), \end{aligned} \tag{4}$$

where ε is the group parameter. Then one requires that this transformation leaves invariant the set of solutions of (3). This yields to an overdetermined, linear system of equations for the infinitesimals $\xi(x, z, t, u)$, $\zeta(x, z, t, u)$, $\tau(x, z, t, u)$ and $\eta(x, z, t, u)$. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$\mathbf{v} = \xi \frac{\partial}{\partial x} + \zeta \frac{\partial}{\partial z} + \tau \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u}. \tag{5}$$

Having determined the infinitesimals, the symmetry variables are found by solving the invariant surface condition

$$\Phi_1 \equiv \xi \frac{\partial u}{\partial x} + \zeta \frac{\partial u}{\partial z} + \tau \frac{\partial u}{\partial t} - \eta = 0. \tag{6}$$

Applying the classical method to PDE (3) the corresponding Lie symmetry algebra depends on the constants a and b and we can distinguish the following cases:

(i) If $a \neq 0$, $b \neq 0$

$$\mathbf{v}_1 = \frac{\partial}{\partial t}, \quad \mathbf{v}_2 = \frac{\partial}{\partial z}, \quad \mathbf{v}_3 = t \frac{\partial}{\partial t} + z \frac{\partial}{\partial z}, \quad \mathbf{v}_f = f(t) \frac{\partial}{\partial x}, \tag{7}$$

where $f(t)$ is an arbitrary function of t.

(ii) If $a \neq 0$, $b = 0$, we get generators (7) and

$$\mathbf{v}_4^1 = x \frac{\partial}{\partial x} - 2z \frac{\partial}{\partial z} - u \frac{\partial}{\partial u}.$$

(iii) If $a = 0$, $b \neq 0$, we get generators (7) and

$$\mathbf{v}_4^2 = x \frac{\partial}{\partial x} - 2z \frac{\partial}{\partial z} + u \frac{\partial}{\partial u}.$$

(iv) If $a=0$, $b=0$, we get generators (7) and

$$\mathbf{v}_4^3 = x\frac{\partial}{\partial x} - 2z\frac{\partial}{\partial z}, \quad \mathbf{v}_k = k(z)u\frac{\partial}{\partial u},$$

where $k(z)$ is an arbitrary function of z.

We, respectively, denote by $\mathscr{L}_1$, $\mathscr{L}_2$, $\mathscr{L}_3$ and $\mathscr{L}_4$, the corresponding Lie algebras.

In order to obtain all the invariant solutions with respect to one-dimensional subalgebras, we construct the one-dimensional optimal system of subalgebras.

(i) If $a\neq 0$, $b\neq 0$, the corresponding generators of the optimal system of subalgebras for $\mathscr{L}_1$ are:

$$\mathbf{v}_1 + \lambda\mathbf{v}_2, \mathbf{v}_2, \mathbf{v}_3, \quad \lambda \in \mathscr{R}.$$

(ii) If $a\neq 0$, $b=0$, the corresponding generators of the optimal system of subalgebras for $\mathscr{L}_2$ are:

$$\mathbf{v}_1 + \lambda\mathbf{v}_2, \mathbf{v}_2, \lambda\mathbf{v}_1 + \mathbf{v}_4^1,$$

$$\lambda\mathbf{v}_2 + 2\mathbf{v}_3 + \mathbf{v}_4^1, \mathbf{v}_3 + \lambda\mathbf{v}_4^1, \quad \lambda\neq\tfrac{1}{2}, \ \lambda \in \mathscr{R}.$$

(iii) If $a=0$, $b\neq 0$, the corresponding generators of the optimal system of subalgebras for $\mathscr{L}_3$ are:

$$\mathbf{v}_1 + \lambda\mathbf{v}_2, \mathbf{v}_2, \lambda\mathbf{v}_1 + \mathbf{v}_4^2,$$

$$\lambda\mathbf{v}_2 + 2\mathbf{v}_3 + \mathbf{v}_4^2, \mathbf{v}_3 + \lambda\mathbf{v}_4^2, \quad \lambda\neq\tfrac{1}{2}, \ \lambda \in \mathscr{R}.$$

(iv) If $a=0$, $b=0$, the corresponding generators of the optimal system of subalgebras for $\mathscr{L}_4$ are:

$$\mathbf{v}_1 + \lambda\mathbf{v}_2, \mathbf{v}_2, \lambda\mathbf{v}_1 + \mathbf{v}_4^3,$$

$$\lambda\mathbf{v}_2 + 2\mathbf{v}_3 + \mathbf{v}_4^3, \mathbf{v}_3 + \lambda\mathbf{v}_4^3, \quad \lambda\neq\tfrac{1}{2}, \ \lambda \in \mathscr{R}.$$

We remark that Eq. (3) does not admit Virasoro-type subalgebra. In the following, we list the similarity variables and similarity solutions as well as the systems of PDEs obtained when the $(2+1)$-dimensional equation (3) is reduced by means of $\{\mathbf{u}_i\}$. These generators $\{\mathbf{u}_i\}$, are obtained by adding to the generators of the optimal system the infinite-dimensional generator $\mathbf{v}_f$ and $\mathbf{v}_k$.

2.1. *Reductions for $a\neq 0$ and $b\neq 0$*

Reduction 1. By using the generator $\mathbf{v}_1 + \lambda\mathbf{v}_2 + \mathbf{v}_f$, we obtain the similarity variables and similarity solution

$$z_1 = x - \int f(t)\,\mathrm{d}t, \quad z_2 = z - \lambda t, \quad u = h(z_1, z_2) \tag{8}$$

and the PDE E_1

$$\begin{aligned} &8h^3h_{z_1z_1}h_{z_2}\lambda - 8h^3h_{z_1}h_{z_1z_2}\lambda - 2h^2h_{z_1}h_{z_1z_1z_1}h_{z_2} + 2h^2h_{z_1z_1}^2h_{z_2} \\ &\quad + 6hh_{z_1}^2h_{z_1z_1}h_{z_2} - 2a^2h^5h_{z_1z_1}h_{z_2} - 4abh^3h_{z_1z_1}h_{z_2} - 2b^2hh_{z_1z_1}h_{z_2} \\ &\quad - 6h_{z_1}^4h_{z_2} + 6a^2h^4h_{z_1}^2h_{z_2} - 6b^2h_{z_1}^2h_{z_2} + 2h^3h_{z_1}h_{z_1z_1z_1z_2} \\ &\quad - 2h^3h_{z_1z_1}h_{z_1z_1z_2} - 4h^2h_{z_1}^2h_{z_1z_1z_2} - 2h^2h_{z_1}h_{z_1z_1}h_{z_1z_2} + 6hh_{z_1}^3h_{z_1z_2} \\ &\quad + 2a^2h^5h_{z_1}h_{z_1z_2} + 4abh^3h_{z_1}h_{z_1z_2} + 2b^2h_{z_1}h_{z_1z_2} = 0. \end{aligned} \tag{9}$$

Reduction 2. By using the generator $\mathbf{v}_2+\mathbf{v}_f$, we obtain the similarity variables and similarity solution

$$z_1 = x - zf(t), \quad z_2 = t, \quad u = h(z_1, z_2) \tag{10}$$

and the PDE E_2

$$\begin{aligned}&-8h^3h_{z_1z_1}h_{z_2} - 2f(z_2)h^3h_{z_1}h_{z_1z_1z_1z_1} + 2f(z_2)h^3h_{z_1z_1}h_{z_1z_1z_1} + 6f(z_2)h^2h_{z_1}^2h_{z_1z_1z_1} - 12f(z_2)hh_{z_1}^3h_{z_1z_1}\\ &\quad + 8h^3h_{z_1}h_{z_1z_2} + 6f(z_2)h_{z_1}^5 - 6a^2f(z_2)h^4h_{z_1}^3 + 6b^2f(z_2)h_{z_1}^3 = 0.\end{aligned} \tag{11}$$

Reduction 3. By using the generator $\mathbf{v}_3+\mathbf{v}_f$, we obtain the similarity variables and similarity solution

$$z_1 = x - \int \frac{f(t)}{t}\,\mathrm{d}t, \quad z_2 = \frac{z}{t}, \quad u = h(z_1, z_2) \tag{12}$$

and the PDE E_3

$$\begin{aligned}&(8h^3h_{z_1z_1}h_{z_2} - 8h^3h_{z_1}h_{z_1z_2})z_2 - 2h^2h_{z_1}h_{z_1z_1z_1}h_{z_2} + 2h^2h_{z_1z_1}^2h_{z_2}\\ &\quad + 6hh_{z_1}^2h_{z_1z_1}h_{z_2} - 2a^2h^5h_{z_1z_1}h_{z_2} - 4abh^3h_{z_1z_1}h_{z_2} - 2b^2\,hh_{z_1z_1}h_{z_2}\\ &\quad - 6h_{z_1}^4h_{z_2} + 6a^2h^4h_{z_1}^2h_{z_2} - 6b^2h_{z_1}^2h_{z_2} + 2h^3h_{z_1}h_{z_1z_1z_1z_2}\\ &\quad - 2h^3h_{z_1z_1}h_{z_1z_1z_2} - 4h^2h_{z_1}^2h_{z_1z_1z_2} - 2h^2h_{z_1}h_{z_1z_1}h_{z_1z_2} + 6hh_{z_1}^3h_{z_1z_2}\\ &\quad + 2a^2\,h^5h_{z_1}h_{z_1z_2} + 4abh^3h_{z_1}h_{z_1z_2} + 2\,b^2h_{z_1}h_{z_1z_2} = 0.\end{aligned} \tag{13}$$

2.2. Reductions for $a \neq 0$ and $b = 0$

Besides the previous reductions we obtain

Reduction 4. By using the generator $\lambda\mathbf{v}_1+\mathbf{v}_4^1+\mathbf{v}_f$, with $\lambda \neq 0$, we obtain the similarity variables and similarity solution

$$z_1 = x\mathrm{e}^{-t/\lambda} - \frac{1}{\lambda}\int \mathrm{e}^{-t/\lambda}f(t)\,\mathrm{d}t, \quad z_2 = z\mathrm{e}^{2t/\lambda}, \quad u = h(z_1, z_2)\mathrm{e}^{-t/\lambda} \tag{14}$$

and the PDE E_4

$$\begin{aligned}&-16h^3h_{z_1z_1}h_{z_2}z_2\lambda^{-1} + 16h^3h_{z_1}h_{z_1z_2}z_2\lambda^{-1} + 8h^4h_{z_1z_1}\lambda^{-1} - 16h^3h_{z_1}^2\lambda^{-1}\\ &\quad - 2h^2h_{z_1}h_{z_1z_1z_1}h_{z_2} + 2h^2h_{z_1z_1}^2h_{z_2} + 6hh_{z_1}^2h_{z_1z_1}h_{z_2} - 2a^2h^5h_{z_1z_1}h_{z_2}\\ &\quad - 6h_{z_1}^4h_{z_2} + 6a^2h^4h_{z_1}^2h_{z_2} + 2h^3h_{z_1}h_{z_1z_1z_1z_2} - 2h^3h_{z_1z_1}h_{z_1z_1z_2}\\ &\quad - 4h^2h_{z_1}^2h_{z_1z_1z_2} - 2h^2h_{z_1}h_{z_1z_1}h_{z_1z_2} + 6hh_{z_1}^3h_{z_1z_2} + 2a^2\,h^5h_{z_1}h_{z_1z_2} = 0.\end{aligned} \tag{15}$$

Reduction 5. By using the generator $\mathbf{v}_4^1+\mathbf{v}_f$, we obtain the similarity variables and similarity solution

$$z_1 = (x + f(t))\sqrt{z}, \quad z_2 = t, \quad u = h(z_1, z_2)\sqrt{z} \tag{16}$$

and the PDE E_5

$$\begin{aligned}&(h^3h_{z_1}h_{z_1z_1z_1z_1} - h^3h_{z_1z_1}h_{z_1z_1z_1} - 3h^2h_{z_1}^2h_{z_1z_1z_1} + 6hh_{z_1}^3h_{z_1z_1} - 3h_{z_1}^5 + 3a^2h^4h_{z_1}^3)z_1 - 8h^3h_{z_1z_1}h_{z_2}\\ &\quad + 3h^3h_{z_1}h_{z_1z_1z_1} - 2h^3h_{z_1z_1}^2 - 5h^2h_{z_1}^2h_{z_1z_1} - a^2h^6h_{z_1z_1} + 8h^3h_{z_1}h_{z_1z_2} + 3hh_{z_1}^4 + 5a^2h^5h_{z_1}^2 = 0.\end{aligned} \tag{17}$$

Reduction 6. By using the generator $\lambda\mathbf{v}_2+2\mathbf{v}_3+\mathbf{v}_4^1+\mathbf{v}_f$, we obtain the similarity variables and similarity solution

$$z_1 = xt^{-1/2} - \frac{1}{2}\int f(t)t^{-3/2}\,\mathrm{d}t, \quad z_2 = z - \frac{\lambda}{2}\ln t, \quad u = h(z_1, z_2)t^{-1/2} \tag{18}$$

and the PDE E_6

$$\begin{aligned}&2h^3h_{z_1z_1}h_{z_2}\lambda - 2h^3h_{z_1}h_{z_1z_2}\lambda - h^2h_{z_1}h_{z_1z_1z_1}h_{z_2} + h^2h^2_{z_1z_1}h_{z_2}\\&\quad + 3hh^2_{z_1}h_{z_1z_1}h_{z_2} - a^2h^5h_{z_1z_1}h_{z_2} - 3h^4_{z_1}h_{z_2} + 3a^2h^4h^2_{z_1}h_{z_2}\\&\quad + h^3h_{z_1}h_{z_1z_1z_1z_2} - h^3h_{z_1z_1}h_{z_1z_1z_2} - 2h^2h^2_{z_1}h_{z_1z_1z_2} - h^2h_{z_1}h_{z_1z_2}h_{z_1z_1}\\&\quad + 2h^4h_{z_1z_1} + 3hh^3_{z_1}h_{z_1z_2} + a^2h^5h_{z_1}h_{z_1z_2} - 4h^3h^2_{z_1} = 0.\end{aligned} \tag{19}$$

Reduction 7. By using the generator $\mathbf{v}_3 + \lambda\mathbf{v}_4^1 + \mathbf{v}_f$, $\lambda \neq \frac{1}{2}$, we obtain the similarity variables and similarity solution

$$z_1 = xt^{-\lambda} - \int f(t)t^{-\lambda-1}\,dt, \quad z_2 = zt^{2\lambda-1}, \quad u = h(z_1, z_2)t^{-\lambda} \tag{20}$$

and the PDE E_7

$$\begin{aligned}&-16h^3h_{z_1z_1}h_{z_2}z_2\lambda + 16h^3h_{z_1}h_{z_1z_2}z_2\lambda + 8h^4h_{z_1z_1}\lambda - 16h^3h^2_{z_1}\lambda\\&\quad + 8h^3h_{z_1z_1}h_{z_2}z_2 - 8h^3h_{z_1z_2}h_{z_1}z_2 - 2h^2h_{z_1}h_{z_1z_1z_1}h_{z_2} + 2h^2h^2_{z_1z_1}h_{z_2}\\&\quad + 6hh^2_{z_1}h_{z_1z_1}h_{z_2} - 2a^2h^5h_{z_1z_1}h_{z_2} - 6h^4_{z_1}h_{z_2} + 6a^2h^4h^2_{z_1}h_{z_2}\\&\quad + 2h^3h_{z_1}h_{z_1z_1z_1z_2} - 2h^3h_{z_1z_1}h_{z_1z_1z_2} - 4h^2h^2_{z_1}h_{z_1z_1z_2}\\&\quad - 2h^2h_{z_1}h_{z_1z_1}h_{z_1z_2} + 6hh^3_{z_1}h_{z_1z_2} + 2a^2h^5h_{z_1}h_{z_1z_2} = 0.\end{aligned} \tag{21}$$

2.3. Reductions for $a = 0$ and $b \neq 0$

These reductions can be obtained from the previous case due to the invariance of Eq. (3) under the transformation

$$u \to \frac{1}{u} \quad a \to b.$$

2.4. Reductions for $a = 0$ and $b = 0$

Besides the previous reductions we obtain

Reduction 8. By using the generator $\mathbf{v}_1 + \lambda\mathbf{v}_2 + \mathbf{v}_f + \mathbf{v}_k$, we obtain the similarity variables and similarity solution

$$z_1 = x - \int f(t)\,dt, \quad z_2 = z - \lambda t, \quad u = h(z_1, z_2)e^{1/\lambda\int k(z)\,dz} \tag{22}$$

and the PDE E_1 (9) with $a = 0$ and $b = 0$.

Reduction 9. By using the generator $\mathbf{v}_2 + \mathbf{v}_f + \mathbf{v}_k$, we obtain the similarity variables and similarity solution

$$z_1 = x - zf(t), \quad z_2 = t, \quad u = h(z_1, z_2)e^{\int k(z)\,dz} \tag{23}$$

and PDE (11) with $a = 0$ and $b = 0$.

Reduction 10. By using the generator $\lambda\mathbf{v}_1 + \mathbf{v}_4^3 + \mathbf{v}_f + \mathbf{v}_k$, $\lambda \neq 0$ we obtain the similarity variables and similarity solution

$$z_1 = xe^{-t/\lambda} - \frac{1}{\lambda}\int e^{-t/\lambda}f(t)\,dt, \quad z_2 = ze^{2t/\lambda}, \quad u = h(z_1, z_2)e^{-1/2\int\frac{k(z)}{z}\,dz} \tag{24}$$

and PDE E_4 (15) with $h = 1/h$ and $a = 0$.

Reduction 11. By using the generator $\mathbf{v}_4^3 + \mathbf{v}_f + \mathbf{v}_k$, we obtain the similarity variables and similarity solution

$$z_1 = (x + f(t))\sqrt{z}, \quad z_2 = t, \quad u = h(z_1, z_2)e^{-1/2\int \frac{k(z)}{z}\,dz} \tag{25}$$

and the PDE E_5 (17) with $a = 0$.

Reduction 12. By using the generator $\lambda\mathbf{v}_2 + 2\mathbf{v}_3 + \mathbf{v}_4^3 + \mathbf{v}_f + \mathbf{v}_k$, $\lambda \neq 0$ we obtain the similarity variables and similarity solution

$$z_1 = xt^{-1/2} - \frac{1}{2}\int f(t)t^{-3/2}\,dt, \quad z_2 = z - \frac{\lambda}{2}\ln t, \quad u = h(z_1, z_2)e^{1/\lambda\int k(z)\,dz} \tag{26}$$

and the PDE E_6 (19) with $h \to 1/h$ and $a = 0$.

Reduction 13. By using the generator $\mathbf{v}_3 + \lambda\mathbf{v}_4^3 + \mathbf{v}_f + \mathbf{v}_k$, we obtain the similarity variables and similarity solution

$$z_1 = xt^{-\lambda} - \int f(t)t^{-\lambda-1}\,dt, \quad z_2 = zt^{2\lambda-1}, \quad u = h(z_1, z_2)e^{1/(1-2\lambda)\int \frac{k(z)}{z}\,dz} \tag{27}$$

and the PDE E_7 (21) with $h \to 1/h$ and $a = 0$.

We remark that equation E_2 with $f(z_2) \equiv 1$ can be integrated once with respect to z_1 and becomes the $(1+1)$-dimensional (CDF) equation.

3. Symmetry reductions to ODEs

In several cases, the reduced PDEs in $(1+1)$ variables admit symmetries which lead to further reductions to ODEs, we shall use again the techniques of Lie group theory.

(1) Equation $\mathbf{E_1}$, admits the following symmetries

$$\mathbf{v}_{11} = \frac{\partial}{\partial z_1}, \quad \mathbf{v}_\alpha = \alpha(z_2)\frac{\partial}{\partial z_2}, \tag{28}$$

where $\alpha(z_2)$ is an arbitrary function of z_2. By using $\mathbf{v}_{11} + \mathbf{v}_\alpha$ we obtain the similarity variable and similarity solutions

$$w = z_1 - \int \frac{1}{\alpha(z_2)}\,dz_2, \quad h = g(w), \tag{29}$$

and the autonomous ODE

$$-g^3g'g'''' + g^3g''g''' + 3g^2(g')^2g''' - 6g(g')^3g'' + 3(g')^5 - 3a^2g^4(g')^3 + 3b^2(g')^3 = 0. \tag{30}$$

By dividing by $g^2(g')^2$, integrating once with respect to w and then multiplying by $g^3(g')^2$ Eq. (30) can be reduced to the following second-order autonomous ODE:

$$g'' = \frac{3}{2}\frac{(g')^2}{g} - \frac{a^2}{2}g^3 + \frac{3b^2}{2}\frac{1}{g} + k_1g + k_2. \tag{31}$$

By multiplying by $g^{-3}g'$, integrating once with respect to w we get

$$(g')^2 = -a^2g^4 + 2k_1g^2 + k_2g + b^2 + 2k_3g^3. \tag{32}$$

The integration may be completed in terms of elliptic functions.

(2) Equation $\mathbf{E_1}$, for $b = 0$ and $\lambda \neq 0$, admits symmetries (28). For $\lambda = 0$ besides the previous ones admits the following generator:

$$\mathbf{v}_{12} = z_1\frac{\partial}{\partial z_1} - h\frac{\partial}{\partial h}. \tag{33}$$

By using $\mathbf{v}_{11}+\mathbf{v}_\alpha$ we obtain the similarity variable and similarity solutions (29) and the ODE (30) with $b=0$.

By using $\mathbf{v}_{12}+\mathbf{v}_\alpha$ we obtain the similarity variable and similarity solutions

$$w=z_1 e^{-\int \frac{1}{\alpha(z_2)}\,dz_2},\quad h=g(w)e^{-\int \frac{1}{\alpha(z_2)}\,dz_2}, \tag{34}$$

and the ODE

$$\begin{aligned}&-g^3g'g''''w+g^3g''g'''w+3g^2(g')^2g'''w-6g(g')^3g''w+3(g')^5w-3a^2g^4(g')^3w\\&-3g^3g'g'''+2g^3(g'')^2+5g^2(g')^2g''+a^2g^6g''-3g(g')^4-5a^2g^5(g')^2=0.\end{aligned} \tag{35}$$

By dividing (35) by $g^2(g')^2$, integrating once with respect to w, and then multiplying by $g^{-3}(g')$ and integrating again with respect to w we arrive at Painlevé III (PIII)

$$g''=\frac{(g')^2}{g}-\frac{g'}{w}-a^2g^3+\frac{k_1}{2w}-\frac{k_2g^2}{w}. \tag{36}$$

(3) Equation $\mathbf{E_2}$, admits the following symmetries:

$$\mathbf{v}_\zeta=\zeta(z_2)\frac{\partial}{\partial z_1},\quad \mathbf{v}_\beta=\frac{1}{f(z_2)}\frac{\partial}{\partial z_2},\quad f(z_2)\neq 0, \tag{37}$$

where $\zeta(z_2)$ and $f(z_2)$ are arbitrary functions of z_2. By using $\mathbf{v}_\zeta+\mathbf{v}_\beta$ we obtain the similarity variable and similarity solutions

$$w=z_1-\int \zeta(z_2)f(z_2)\,dz_2,\quad h=g(w), \tag{38}$$

and ODE (30) that can be integrated in terms of elliptic functions.

(4) Equation $\mathbf{E_2}$, with $b=0$ admits symmetries (38) and

$$\mathbf{v}_\gamma=z_1\frac{\partial}{\partial z_1}+\frac{3}{f(z_2)}\int f(z_2)\,dz_2\frac{\partial}{\partial z_2}-h\frac{\partial}{\partial h},\quad f(z_2)\neq 0. \tag{39}$$

By using $\mathbf{v}_\zeta+\mathbf{v}_\beta+\mathbf{v}_\gamma$ we obtain the similarity variable and similarity solutions

$$\begin{aligned}&w=z_1\left(1+3\int f(z_2)\,dz_2\right)^{1/3}-\int\frac{f(z_2)\alpha(z_2)\,dz_2}{(1+3\int f(z_2)\,dz_2)^{4/3}},\\&h=g(w)\left(1+3\int f(z_2)\,dz_2\right)^{-1/3}\end{aligned} \tag{40}$$

and the ODE

$$-g^3g'g''''+g^3g''g'''+3g^2(g')^2g'''-6g(g')^3g''+4g^4g''+3(g')^5-3a^2g^4(g')^3-8g^3(g')^2=0. \tag{41}$$

By dividing (41) by $g^3(g')^2$, integrating once with respect to w, multiplying by g' and integrating again with respect to w we arrive to the following second-order ODE:

$$g''=\frac{3}{2}\frac{(g')^2}{g}-\frac{a^2}{2}g^3+k_2g-4wg+k_1. \tag{42}$$

The change of variables $g=y^{-1}$ leads to

$$y''=\frac{1}{2}\frac{(y')^2}{g}-\frac{a^2}{2y}+k_2y^2+4wy+k_1y. \tag{43}$$

By making the change of variables $y=\alpha V(Z)$ with $Z=\beta w$ leads to (44),

$$V''=\frac{1}{2}\frac{(V')^2}{V}+4cV^2-ZV-\frac{1}{2}\frac{1}{V}, \tag{44}$$

the solutions can be written in terms of the second Painlevé equation (PII) (see Ref. [28]).

(5) Equation $\mathbf{E_3}$, only admits the following translation symmetry $\mathbf{v}_{11}$, that leads to a trivial reduction.

(6) Equation $\mathbf{E_4}$, admits the following symmetries:

$$\mathbf{v}_{11}, \quad \mathbf{v}_{41} = z_1 \frac{\partial}{\partial z_1} - 2z_2 \frac{\partial}{\partial z_2} - h \frac{\partial}{\partial h}. \tag{45}$$

By using $\mathbf{v}_{41}$ we obtain the similarity variable and similarity solutions

$$w = z_1 \sqrt{z_2}, \quad h = g(w)\sqrt{z_2} \tag{46}$$

and the ODE (35) that leads to (PIII).

(7) Equation $\mathbf{E_5}$, admits the following symmetries:

$$\mathbf{v}_{51} = \frac{\partial}{\partial z_2}, \quad \mathbf{v}_{52} = z_1 \frac{\partial}{\partial z_1} + 2z_2 \frac{\partial}{\partial z_2} - h \frac{\partial}{\partial h}.$$

By using $\mathbf{v}_{51}$ we obtain the similarity variable and similarity solutions

$$w = z_1, \quad h = g(w) \tag{47}$$

and the ODE (35) that leads to (PIII).

By using $\mathbf{v}_{52}$ we obtain the similarity variable and similarity solutions

$$w = z_1/\sqrt{z_2}, \quad h = g(w)/\sqrt{z_2} \tag{48}$$

and the ODE

$$\begin{aligned} &g^3 g' g'''' w - g^3 g'' g''' w - 3g^2 (g')^2 g''' w + 6g(g')^3 g'' w - 3(g')^5 w + 3a^2 g^4 (g')^3 w + 3g^3 g' g''' \\ &\quad - 2g^3 (g'')^2 - 5g^2 (g')^2 g'' - a^2 g^6 g'' + 4g^4 g'' + 3g(g')^4 + 5a^2 g^5 (g')^2 - 8g^3 (g')^2 = 0. \end{aligned} \tag{49}$$

By dividing (49) by $g^2(g')^2$, integrating once with respect to w, setting the integrating constant $k_1 = 0$, multiplying by $wg^{-2}g'$ and integrating again with respect to w we get the following second-order ODE:

$$g'' = \frac{3}{4} \frac{(g')^2}{g} + \left(\frac{2k_2^2}{w^2} - 4 \right) g + a^2. \tag{50}$$

The change of variables $g = \eta^4(w) V(\xi(w))$ leads to (44) where $\xi_w^2 \eta^4 + a^2 = 0$ and η must satisfy the linear equation

$$\eta'' + \left(1 - \frac{k_2}{2w^2} \right) \eta = 0$$

whose solutions are expressed in term of Bessel functions.

(8) Equation $\mathbf{E_6}$, admits, the following symmetries:

$$\mathbf{v}_{11}, \quad \mathbf{v}_{51}, \quad \lambda \neq 0. \tag{51}$$

Besides the previous symmetries, $\mathbf{E_6}$ admits, for $\lambda = 0$, the generator $\mathbf{v}_{41}$.

By using $\mathbf{v}_{41}$ we obtain the similarity variable and similarity solutions (46) and the ODE (49).

By using $\mu \mathbf{v}_{11} + \mathbf{v}_{51}$ we obtain the similarity variable and similarity solutions

$$w = z_1 - \mu z_2, \quad h = g(w) \tag{52}$$

and the ODE

$$-g^3 g' g'''' \mu + g^3 g'' g''' \mu + 3g^2 (g')^2 g''' \mu - 6g(g')^3 g'' \mu + 3(g')^5 \mu - 3a^2 g^4 (g')^3 \mu + 2g^4 g'' - 4g^3 (g')^2 = 0. \tag{53}$$

By dividing by $g^2(g')^2$, integrating once with respect to w, multiplying then by $g^{-2}g'$ and integrating again with respect to w, Eq. (53) can be reduced to the following second-order ODE:

$$g'' = \frac{3}{2} \frac{(g')^2}{g} - \frac{a^2}{2} g^3 + (k_2 - 2w) \frac{g}{\mu} - \frac{k_1}{\mu}. \tag{54}$$

The change of variables $g = V^{-1}$ leads to (44) whose solutions are expressed in term of (PII).

(9) Equation $\mathbf{E}_7$, admits for $\lambda \neq 0$ symmetries (45). Equation $\mathbf{E}_7$, for $\lambda = \frac{1}{2}$ also admits $\mathbf{v}_{51}$.
By using $\mathbf{v}_{41}$ we obtain the similarity variable and similarity solutions (46) and ODE (49).
By using $\mu\mathbf{v}_{11} + \mathbf{v}_{71}$ with $\lambda = \frac{1}{2}$, we obtain the similarity variable and similarity solutions (52) and ODE (53).

4. Some travelling wave solutions and other explicit solutions

In the following we present some explicit solutions of the second-order ODEs as well as the corresponding travelling solution of the $(2+1)$ CDF equation. Eq. (32) can be integrated in terms of elliptic functions. Setting in (32) $b = 0$ and $k_2 = -4$ an exact solution is given in terms of the Weirstrass $\mathscr{P}$ function. Clearly any of the rational, hyperbolic or trigonometric degenerations of the $\mathscr{P}$ functions also give solutions. In particular, solitary waves result:

Setting $k_1 = -2c_2^2$, $k_2 = 0$, $k_3 = -\frac{1}{c_1}(2c_1c_2 - a)(2c_1c_2 + a)$, we get

$$g = \frac{1}{c_1 - (c_1 + \frac{a^2}{4c_1c_2^2})\cosh^2(c_2 w)}.$$

By considering the corresponding symmetry reductions (8) and (29) we obtain that a "curve" soliton solution for the (CDF) equation in $(2+1)$ dimensions can be written as

$$u = \frac{1}{c_1 - (c_1 + \frac{a^2}{4c_1c_2^2})(\cosh^2(c_2 x - \varphi(t) - \delta(z - \lambda t)))}, \tag{55}$$

with

$$\varphi(t) = c_2 \int f(t)\,\mathrm{d}t, \quad \delta(z_2) = c_2 \int \frac{\mathrm{d}z_2}{\alpha(z_2)}, \quad z_2 = z - \lambda t. \tag{56}$$

In Fig. 1 we can see solution (55) with $c_1 = -1$, $c_2 = 1$, $a = 4$, $\delta(z - \lambda t) = \sin(z - t)$ and $\varphi(t) = -t$ for $t = 1$.

By considering the corresponding symmetry reductions (10) and (38) we obtain that a soliton solution for the (CDF) equation in $(2+1)$ dimensions can be written as

$$u = \frac{1}{c_1 - (c_1 + \frac{a^2}{4c_1c_2^2})(\cosh^2(c_2 x - zf(t) - \psi(t)))}, \tag{57}$$

with

$$\psi(t) = c_2 \int \delta(t) f(t)\,\mathrm{d}t. \tag{58}$$

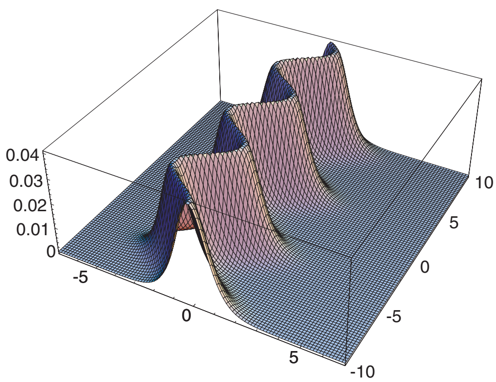

Fig. 1. Curve soliton $t = 1$.

In Fig. 2 we can see (57) with $c_1 = 1$, $c_2 = 1$, $a = 4$, $f(t) = t^4$ and $\psi(t) = t$ for $t = 0$ and 1, respectively, we observe that this soliton solution is "rotating".

Setting in (32) $a = 0$, $b = 0$, $k_1 = -2c_1$, $k_2 = 0$, $k_3 = 4c_2$ we get the solution

$$g = -\frac{c_1}{c_2 \cosh^2 \sqrt{c_1} w}.$$

By considering the corresponding symmetry reductions (8) and (29), we obtain that solutions of the CDF in $(2+1)$ dimensions can be written as

$$u = -\frac{c_1 \rho(z)}{c_2 \cosh^2(\sqrt{c_1}(x - \varphi(t) - \delta(z - \lambda t)))}. \tag{59}$$

By considering the corresponding symmetry reductions (10) and (38) we obtain that a solution for the (CDF) equation in $(2+1)$ dimensions can be written as

$$u = \frac{c_1 \rho(z)}{c_2 \cosh^2(\sqrt{c_1}(x - zf(t) - \psi(t)))}, \tag{60}$$

with

$$\psi(t) = c_2 \int \delta(t) f(t)\, dt. \tag{61}$$

In Fig. 3 we can see solution (59) with $\varphi(t) = 0$, $\delta(z - \lambda t) = z - t$, $-c_1\rho(z) = \cosh^{-2}(z)$, and $-c_1\rho(z) = \cosh^{-2}(z) + \cosh^{-2}(z+4)$, respectively, for $t = 1$. We observe that these dromions and coherent structures are localized in all directions. In Fig. 4 we can see solution (60) with $f(t) = t$, $\psi(t) = t$, $-c_1\rho(z) = \cosh^{-2}(z)$, respectively, for $t = 1$ and 2. We observe that the solution is localized in all directions and evolves "rotating" and changing its shape.

Setting in (32) $a = 0$, we get

$$g = \frac{\sqrt{k_2^2 - 8k_1 b^2}}{4k_1}(\sin(\sqrt{2k_1}(w + c) + k_2)), \quad g = \frac{\sqrt{k_2^2 - 8k_1 b^2}}{4k_1}(\cos(\sqrt{2k_1}(w + c) - k_2)),$$

$$g = \frac{\sqrt{-k_2^2 + 8k_1 b^2}}{4k_1}(\sinh(\sqrt{-2k_1}(w + c) + k_2)), \quad g = \frac{\sqrt{k_2^2 - 8k_1 b^2}}{4k_1}(\cosh(\sqrt{-2k_1}(w + c) + k_2)).$$

By considering the corresponding symmetry reductions (8) and (29) we obtain that some exact solutions for the (CDF) equation in $(2+1)$ dimensions can be written as

$$u = -\frac{\sqrt{k_2^2 - 8k_1 b^2}}{4k_1}(\sin(\sqrt{2k_1}((x - \varphi(t) - \delta(z - \lambda t) + k_2)))), \tag{62}$$

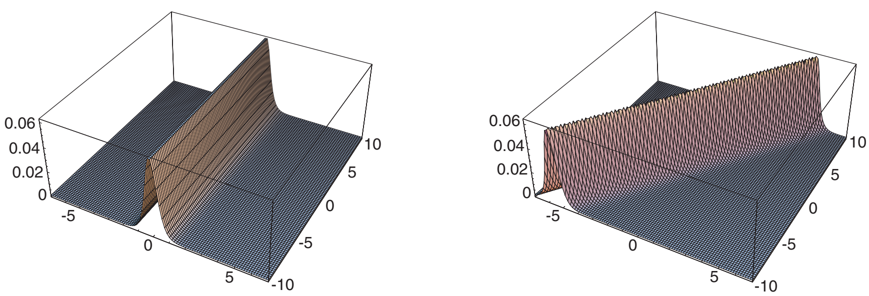

Fig. 2. Evolution of the rotating soliton with $t = 0$ and 1.

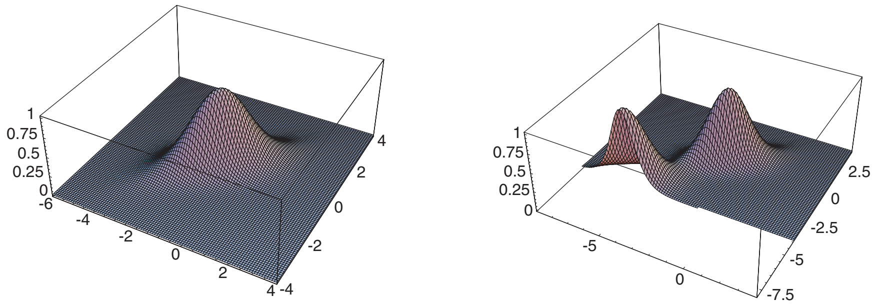

Fig. 3. Dromion and coherent structure $t = 1$.

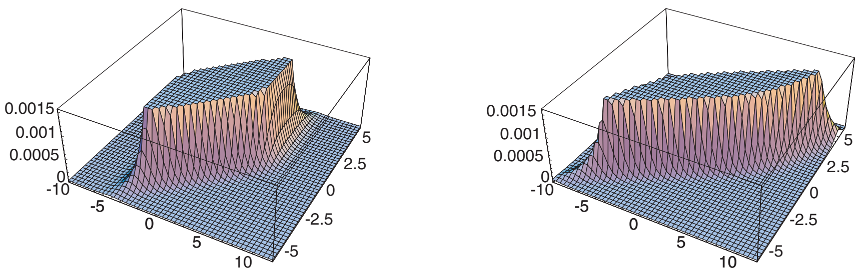

Fig. 4. Evolution of solution (60) with $t = 1$ and 2.

$$u = -\frac{\sqrt{k_2^2 - 8k_1 b^2}}{4k_1}(\cos(\sqrt{2k_1}((x - \varphi(t) - \delta(z - \lambda t) - k_2)))), \tag{63}$$

$$u = -\frac{\sqrt{-k_2^2 + 8k_1 b^2}}{4k_1}(\sinh(\sqrt{-2k_1}((x - \varphi(t) - \delta(z - \lambda t) - k_2)))), \tag{64}$$

$$u = \frac{\sqrt{k_2^2 - 8k_1 b^2}}{4k_1}(\cosh(\sqrt{-2k_1}((x - \varphi(t) - \delta(z - \lambda t) - k_2)))). \tag{65}$$

In Fig. 5 we can see the periodic solution (62) with $k_1 = 1$, $k_2 = 4b$, $b = 2$, $\varphi(t) = t$, $\delta(z_2) = z - \lambda t$ and $\delta(z_2) = \sin(z - \lambda t)$ for $t = 1$, respectively.

Setting in (32) $a = 1$ and $b = 0$,

$$g = -\frac{2\tan(\frac{\sqrt{3}}{2}w)}{\tan(\frac{\sqrt{3}}{2}w) + 3}.$$

Setting in (32) $a = \sqrt{2}i/\sqrt{3}$, $k_1 = 0$, $k_2 = \frac{8}{3}$ and $b = 0$ we get

$$g = \frac{2\,\mathrm{sech}^2 w - 2}{2\,\mathrm{sech}^2 w + 1}.$$

Setting in (32) $b = 0$, $k_1 = k_2 = 0$ and $\lambda = k_3/a^2$, we get

$$g = \frac{2\lambda}{1 + a^2\lambda^2 w^2}.$$

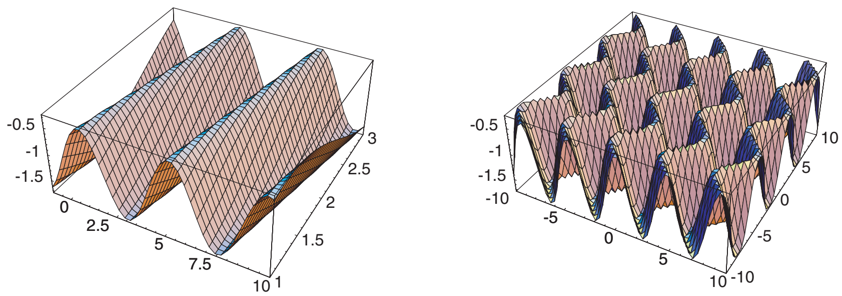

Fig. 5. Periodic solution (62) for $t = 1$.

Setting in (32) $a = 0$, $b = 0$ and $k_2 = k_3 = 0$ we get

$$g = \rho(z)e^{\sqrt{-2k_1}w}.$$

By considering the corresponding symmetry reductions (8) and (29) we obtain that some exact solutions for the (CDF) equation in $(2+1)$ dimensions can be written as

$$u = -\frac{2\tan(\frac{\sqrt{3}}{2}(x-\varphi(t)-\delta(z-\lambda t)))}{\tan\frac{\sqrt{3}}{2}(x-\varphi(t)-\delta(z-\lambda t))+3}, \tag{66}$$

$$u = \frac{2\,\mathrm{sech}^2(x-\varphi(t)-\delta(z-\lambda t))-2}{2\,\mathrm{sech}^2(x-\varphi(t)-\delta(z-\lambda t))+1}, \tag{67}$$

$$u = \frac{2\lambda}{1+a^2\lambda^2(x-\varphi(t)-\delta(z-\lambda t))^2}, \tag{68}$$

$$u = \rho(z)e^{\sqrt{-2k_1}(x-\varphi(t)-\delta(z-\lambda t))}, \tag{69}$$

with

$$\varphi(t) = c_2\int f(t)\,dt, \quad \delta(z_2) = c_2\int \frac{dz_2}{\alpha(z_2)}, \quad z_2 = z-\lambda t. \tag{70}$$

In Fig. 6 we can see the "curve" soliton solution (68) with $a = 1$, $c = 1$, $\varphi(t) = t$, $\delta(z_2) = (z-\lambda t)^2$ and $\delta(z_2) = Ai(z-\lambda t)$ for $t = 1$, respectively.

In Fig. 7 we can see solution (69) with $\delta(z_2) = (z-\lambda t)$ $\varphi(t) = t$, $\rho(z) = e^{-z^2}$ and $\rho(z) = \tanh(z)$, respectively, for $t = 1$. We observe that setting $\rho(z)$ in a convenient form, we can modulate the solution.

Setting in (32) $b = 0$ and $k_2 = 0$ we get

$$g = \frac{4k_1}{\sqrt{k_4^2-8k_1a^2}(\cosh(\sqrt{-2k_1}(w+c)-k_4))}, \quad g = \frac{4k_1}{\sqrt{k_4^2-8k_1a^2}(\cos(\sqrt{2k_1}(w+c)-k_4))}.$$

By considering the corresponding symmetry reductions (8) and (29) we obtain that some exact solutions for the (CDF) equation in $(2+1)$ dimensions can be written as

$$u = \frac{4k_1}{\sqrt{k_4^2-8k_1a^2}(\cosh(\sqrt{-2k_1}((x-\varphi(t)-\delta(z-\lambda t)+c))-k_4))}, \tag{71}$$

$$u = \frac{4k_1}{\sqrt{k_4^2-8k_1a^2}(\cos(\sqrt{2k_1}((x-\varphi(t)-\delta(z-\lambda t)+c))-k_4))}. \tag{72}$$

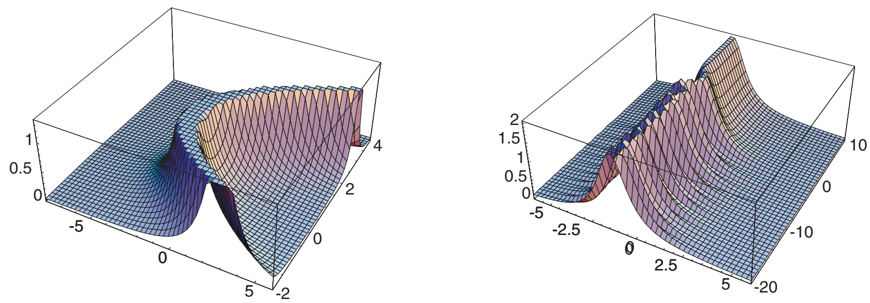

Fig. 6. Curve solitons (68) with $t = 1$.

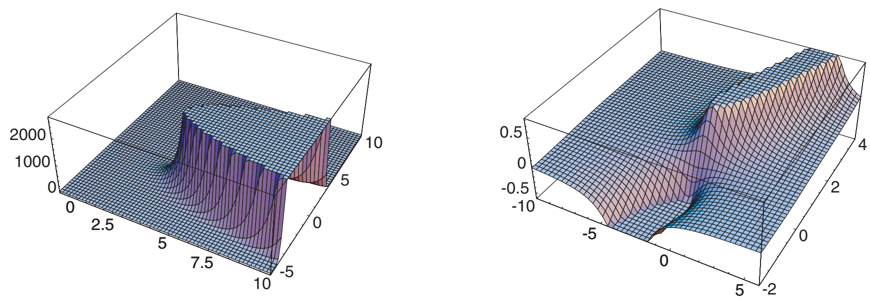

Fig. 7. Solution (69) with $t = 1$.

In Fig. 8 we can see the solitary wave (71) with $\varphi(t) = t$, $\delta(z_2) = (z - \lambda t)$ and $\delta(z_2) = (z^3 - \lambda t)$ for $t = 1$, respectively. We observe that setting $k_4 = 1$ we have in terms of k_1 a one parameter family of solitary waves. Eq. (3) support two kinds of solitary waves: a solitary family that decays exponentially and a solitary wave which decays algebraically.

We remark that by considering the corresponding symmetry reductions (10) and (38) we obtain that some exact solutions for the (CDF) equation in $(2+1)$ dimensions can be written, as

$$u = \frac{\sqrt{-2k_1}}{|a|}\operatorname{sech}(\sqrt{-2k_1}(x - zf(t) - \psi(t))), \quad u = -\frac{\sqrt{2k_1}}{|a|}\operatorname{cosech}(\sqrt{-2k_1}(x - zf(t) - \psi(t))),$$

$$u = -\frac{\sqrt{k_2^2 - 8k_1b^2}}{4k_1}\sin(\sqrt{2k_1}(x - zf(t) - \psi(t)) + k_2), u = -\frac{\sqrt{k_2^2 - 8k_1b^2}}{4k_1}\cos(\sqrt{2k_1}(x - zf(t) - \psi(t)) - k_2),$$

$$u = -\frac{\sqrt{-k_2^2 + 8k_1b^2}}{4k_1}\sinh(\sqrt{-2k_1}(x - zf(t) - \psi(t)) - k_2), u = \frac{4k_1}{\sqrt{k_4^2 - 8k_1a^2}\cosh(\sqrt{-2k_1}(x - zf(t) - \psi(t)) - k_4)},$$

$$u = \frac{\sqrt{k_2^2 - 8k_1b^2}}{4k_1}\cosh(\sqrt{-2k_1}(x - zf(t) - \psi(t)) - k_2), u = \frac{4k_1}{\sqrt{k_4^2 - 8k_1a^2}\cos(\sqrt{2k_1}(x - f(t) - \psi(t)) - k_4)},$$

$$u = -\frac{2\tan(\frac{\sqrt{3}}{2}(x - zf(t) - \psi(t)))}{\tan\frac{\sqrt{3}}{2}(x - zf(t) - \psi(t)) + 3}, \quad u = \frac{2\operatorname{sech}^2(x - zf(t) - \psi(t)) - 2}{2\operatorname{sech}^2(x - zf(t) - (z\psi t)) + 1},$$

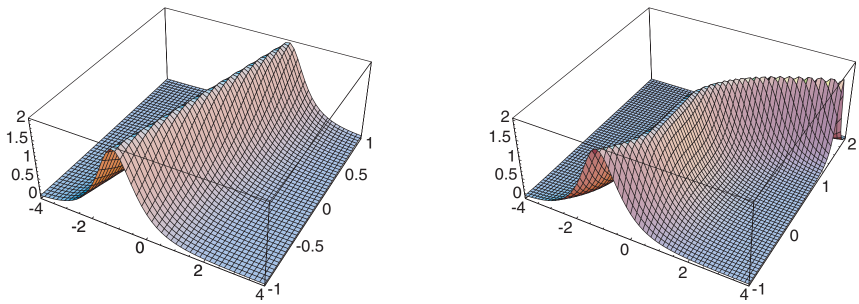

Fig. 8. Solitary waves (71) $t = 1$.

$$u = \frac{2\lambda}{1 + a^2\lambda^2(x - zf(t) - \psi(t))^2}, \quad u = \rho(z)\mathrm{e}^{\sqrt{-2k_1}(x - zf(t) - \psi(t))},$$

with

$$\psi(t) = c_2 \int \zeta(t) f(t)\,\mathrm{d}t. \tag{73}$$

Setting $k_1 = 0$, $k_2 = 0$ in Eq. (36) some particular solutions arise:

$$g = \frac{2c_1c_2w^{c_1-1}}{w^{2c_1} + c_2^2a^2}, \quad g = \frac{c_2}{w\cosh(c_2a\log(w) + c_1)}. \tag{74}$$

By considering the corresponding symmetry reductions (14) and (46), we obtain that a family of solutions for the CDF in $(2+1)$ dimensions can be written as

$$u = \frac{2c_1c_2(x - \varphi(t))^{c_1-1}z^{c_1/2}}{(x - \varphi(t))^{2c_1}z^{c_1} + c_2^2a^2}, \tag{75}$$

$$u = \frac{c_2}{(x - \varphi(t))\cosh(c_2a(\frac{\log(z)}{2} + \log(x - \varphi(t)) + c_1))}. \tag{76}$$

Solution (75) depending on the choice of c_1 and c_2, becomes a singular solution. In particular if c_1 is a rational number the solution becomes singular on an algebraic curve. It is interesting that this solution possesses quite rich structure because of the entrance of an arbitrary function. For example, by choosing $c_1 = 4$, $c_2 = 1$ and $\varphi(t) = \sin(t)$ the solution becomes

$$u = \frac{8(x - \sin(t))^3z^2}{(x - \sin(t))^8z^4 + a^2}, \tag{77}$$

which is, for $a \neq 0$, a bounded regular solution for any values of x, t and z.

Solution (76) becomes singular on an infinite number of curves in the plane XZ moving with t.

The most interesting solutions are the soliton solutions, the entrance of some arbitrary functions $\rho(z)$, $\varphi(t)$, $\delta(z - \lambda t)$ allows a wide variety of qualitative and physical behaviour for these solutions.

5. Conclusions

In this work, we have carried out a detailed Lie symmetry analysis of the $(2+1)$-dimensional integrable generalization of the CDF equation. Through this invariance analysis we obtain $(1+1)$-dimensional PDEs. The invariance study of these PDEs and further reductions lead to second-order integrable ODEs whose solutions are all expressible in terms of known functions, some of them expressible in terms of the second and third Painlevé trascendents. For the CDF equation in $(2+1)$ dimensions we obtained families of solutions which have a rich variety of qualitative behaviors. This is due to the freedom in the choice of the arbitrary functions $\varphi(t)$ $\rho(z)$ and $\delta(z-\lambda t)$. Among them we have obtained bounded solutions such as soliton solutions. Because of these arbitrary functions which are included in the single soliton solution, the solution is localized on a curve and the curve may have quite a free form.

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