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STRUCTURE OF NONUNITAL PURELY INFINITE SIMPLE RINGS

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In this note, we study the notion of purely infinite simple ring in the case of nonunital rings, and we obtain an analog to Zhang's Dichotomy for σ-unital purely infinite simple C*-algebras in the purely algebraic context.

Key Words: Purely infinite simple ring; Zhang's Dichotomy.

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INTRODUCTION

In 1981, Cuntz introduced the concept of a purely infinite simple C*-algebra (Cuntz, 1981). This notion has played a central role in the development of the theory of C*-algebras in the last two decades. A large series of contributions, due to Blackadar, Brown, Lin, Pedersen, Phillips, Rørdam and Zhang, among others, reflect the interest in the structure of such algebras. A particular interest deserves Zhang's result (Zhang, 1992), dividing σ-unital purely infinite simple C*-algebras in two types: unital and stable. This result—known as Zhang's Dichotomy for σ-unital purely infinite simple C*-algebras—played a central role in the study of the structure of corona and multiplier algebras for C*-algebras with real rank zero.

In 2002, Ara, Goodearl and Pardo (Ara et al., 2002) introduced a suitable definition of a purely infinite simple ring for unital rings, which agrees with that of Cuntz in the case of C*-algebras, and studied K_0 and K_1 groups of a purely infinite simple ring, specially in the case of von Neumann regular rings lying in this class. The natural generalization of this definition to the context of nonunital rings was already considered in Ara and Perera (2000), and also in Ara (2004), where Ara showed that every (non-necessarily unital) purely infinite simple ring is an exchange ring.

In this note, we study the notion of nonunital purely infinite simple ring considered in Ara (2004). We start by comparing this notion with a different one, inspired by Ara et al. (2002, Theorem 1.6), which turns out to be equivalent to the former one for C*-algebras (Cuntz, 1981; Lin and Zhang, 1991). We conclude

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that the original definition is stronger that the new one, but it is not clear whether both definitions are equivalent in the algebraic context. Finally, using the definition introduced in Ara (2004), we are able to prove an algebraic version of Zhang's result, dividing σ-unital purely infinite simple rings in unital and stable ones.

We need to fix some definitions. Given a ring R, we denote by $M_\infty(R) = \varinjlim M_n(R)$, under the maps $M_n(R) \to M_{n+1}(R)$ defined by $x \mapsto \mathrm{diag}(x, 0)$. Notice that $M_\infty(R)$ can also be described as the ring of countable infinite matrices over R with only finitely many nonzero entries. Given $p, q \in M_\infty(R)$ idempotents, we say that p and q are equivalent, denoted $p \sim q$, if there exist elements $x, y \in M_\infty(R)$ such that $xy = p$ and $yx = q$. We also write $p \leq q$, provided that $p = pq = qp$, $p \lesssim q$ if there exists an idempotent $r \in M_\infty(R)$ such that $p \sim r \leq q$, and $p \prec q$ if there exists an idempotent $r \in M_\infty(R)$ such that $p \sim r < q$. Given idempotents $p, q \in M_\infty(R)$, we define the direct sum of p and q as $p \oplus q = \left(\begin{smallmatrix} p & 0 \\ 0 & q \end{smallmatrix}\right)$. Also, for an idempotent $p \in M_\infty(R)$ and a positive integer n, we denote by $n \cdot p$ the direct sum of n copies of p. Two idempotents e, f are said to be orthogonal (denoted $e \perp f$), provided that $ef = fe = 0$. In that case, $e + f$ is an idempotent, and $(e + f)R = eR \oplus fR$. An idempotent e in a ring R is infinite if there exist orthogonal idempotents $f, g \in R$ such that $e = f + g$, while $e \sim f$ and $g \neq 0$.

1. BASIC CONCEPTS

In this section we study the notion of purely infinite simple ring in the case of nonunital rings. By analogy with the C*-algebra case, we consider two notions, that turn out to be equivalent for C*-algebras. The first one is that introduced in Ara et al. (2002) as a basic definition, and used in Ara (2004).

Definition 1.1 (Ara et al., 2002, Definition 1.2)**.** A ring R is said to be purely infinite simple if it is simple and every nonzero right ideal contains an infinite idempotent.

The second one is the alternative definition of purely infinite simple unital ring that rises from Ara et al. (2002, Theorem 1.6), adapted to the nonunital case. We borrow the name from Cohn (1995, pp. 241–242).

Definition 1.2. A nonzero ring R is 1-simple if for every nonzero elements $x, y \in R$ there exist $z, t \in R$ such that $zxt = y$.

Remark 1.3. (1) It is easy to see that the definition of purely infinite simple ring is right-left symmetric.

(2) It is clear that, by definition, any 1-simple ring is simple.

(3) If R is a unital 1-simple ring, then it is either a division ring or a purely infinite simple ring Ara et al. (2002, Theorem 1.6).

Now we study the relation between these definitions in the purely algebraic context.

Proposition 1.4. *If R is a* (*nonunital*) *purely infinite simple ring, then it is* 1*-simple.*

Proof. Let $x, y \in R$ be nonzero elements. By hypothesis, there exists an infinite idempotent $e \in xR$, so that $e = xr$ for some $r \in R$. Since R is simple, every nonzero finitely generated projective module is a generator of the category Mod-R. Since e is infinite and R is simple, it is easy to show that, for any natural number n, there exists a module epimorphism $\varphi_n : eR \to n(eR)$. Now, by simplicity, $y \in ReR$, so that $y = \sum_{i=1}^{m} z_i e t_i$ for some $z_1, \ldots, z_m, t_1, \ldots, t_m \in R$. Hence, multiplication by $(z_1, \ldots, z_m)$ defines a module homomorphism $\pi : m(eR) \to R$ such that $y \in \mathrm{Im}(\pi)$. Thus, $\rho = \pi \circ \varphi_m$ defines a module homomorphism from eR to R such that $y \in \mathrm{Im}(\rho)$. In particular, $y = \rho(et)$ for some $t \in R$. Since $e = e^2$, for any $a \in R$ we have $\rho(ea) = \rho(e)ea$. Hence

$$y = \rho(et) = \rho(e)et = \rho(e)x(rt),$$

as desired. □

The converse of Proposition 1.4 holds whenever R contains an infinite idempotent.

Proposition 1.5. *If R is a 1-simple ring containing an infinite idempotent, then it is purely infinite simple.*

Proof. Let $y \in R$ be a nonzero element, and let $e \in R$ be the infinite idempotent. By hypothesis, there exists $z, t \in R$ such that $e = zyt$. Without loss of generality, we can assume $z = ez$ and $t = te$. Set $f = ytz$. Then, $f^2 = (ytz)(ytz) = yt(zyt)z = ytez = ytz = f$, so that it is an idempotent. Clearly, $f \in yR$, and since $f = (yt)z$ and $e = z(yt)$, we have that $e \sim f$, whence f is an infinite idempotent, as desired. □

On one side, Lin and Zhang (1991, Theorem 2.2, Theorem 1.2) imply that a 1-simple C*-algebra contains a nontrivial idempotent. Hence, in the case of infinite dimensional C*-algebras, purely infinite simple is equivalent to 1-simple. On the other side, it is not clear whether a 1-simple ring has nonzero idempotents, whence the whole equivalence remains unsolved.

2. ALGEBRAIC ZHANG'S DICHOTOMY

In this section we will show that an analog of Zhang's Dichotomy for purely infinite simple C*-algebras (Zhang, 1992, Theorem 1.2) holds in the purely algebraic context.

In order to state the results, we need to recall some definitions. Recall that a ring R is said to be *exchange* if for every element $a \in R$ there exists and idempotent $e \in R$ and elements $r, s \in R$ such that $e = ra = a + s - sa$ (Ara, 1997). This definition reduces to the Goodearl-Nicholson characterization of exchange rings in case R is a unital ring: a unital ring R is said to be *exchange* if for every element $a \in R$ there exists and idempotent $e \in aR$ such that $(1 - e) \in (1 - a)R$. The next definitions are borrowed from Ara and Perera (2000). Given a semiprime ring R, we say that a double centralizer for R is a pair (f, g) such that $f : R \to R$ is a right module morphism, $g : R \to R$ is a left module morphism, satisfying $g(x)y = xf(y)$ for all $x, y \in R$. Notice that for any element $a \in R$, the pair (f_a, g_a), where the maps are left/right multiplication by a respectively, is a double centralizer. The set of double centralizers over R, endowed with the componentwise addition and the product defined by the rule $(f_1, g_1) \cdot (f_2, g_2) = (f_1 \cdot f_2, g_2 \cdot g_1)$, has the structure of ring with unit (Id, Id), and it is called the ring of multipliers of R, denoted $\mathcal{M}(R)$. Notice that R is an ideal of $\mathcal{M}(R)$ through the identification of $a \in R$ with $(f_a, g_a) \in \mathcal{M}(R)$; moreover, $\mathcal{M}(R)$ coincides with R whenever R is a unital ring. A net $(x_\lambda)_{\lambda\in\Lambda} \subset \mathcal{M}(R)$ converges in the strict topology to $x \in \mathcal{M}(R)$ if for every $a \in R$ there exists λ_0 such that $(x_\lambda - x)a = a(x_\lambda - x) = 0$ for $\lambda \geq \lambda_0$. We say that a net $\{a_i\} \subset R$ is *an approximate unit for* R provided that it converges to 1 in the strict topology. An approximate unit consisting of idempotents is called a *local unit*. We can assume that an approximate (local) unit is increasing (Ara and Perera, 2000, Lemma 1.5). A ring with an approximate unit is called *s-unital*. A s-unital ring with a countable approximate unit is called *σ-unital*. A ring has a *countable unit* if it is σ-unital and has a local unit. This is equivalent to the fact that there exists an increasing sequence of idempotents $\{e_n\}_{n\in\mathbb{N}}$ such that $R = \bigcup_{n\in\mathbb{N}} e_n R e_n$ (Ara and Perera, 2000, p. 3366).

Theorem 2.1 (Ara, 2004, Theorem 1.1). *Every purely infinite simple ring is an exchange ring.*

We thank P. Ara for the proof of the following result.

Lemma 2.2. *Every s-unital exchange ring is a ring with local units.*

Proof. Given a finite number of elements $x_1, \ldots, x_n \in R$, we must find an idempotent $h \in R$ such that $x_i \in hRh$ for all i. Since R is s-unital, there is $y \in R$ such that $x_i y = x_i$ for all i.

Let us work in $R^1 = R \oplus \mathbb{Z}$ the unitization of R. By the exchange property of R, there is $e = e^2 \in R$ such that $e \in yR$ and $1 - e \in (1 - y)R^1$. Choose $t \in R$ such that $1 - e = (1 - y)(1 - t)$. We then have

$$x_i(1 - e) = x_i(1 - y)(1 - t) = 0.$$

Now there is $z \in R$ such that $zx_i = x_i$ for all i and $ze = e$. Since the exchange property is left-right symmetric, there is an idempotent $g \in R$ such that $(1 - g)x_i = 0$ for all i and $(1 - g)e = 0$. Now take $h = e + g - eg$. Then h is an idempotent in R and $x_i \in hRh$ for all i, as desired. □

Corollary 2.3. *Every σ-unital exchange ring is a ring with countable unit.*

The next result fills the gap to get the desired dichotomy. In order to prove it, we need to recall a few things of K-Theory. For a ring R, we denote by $V(R)$ the Abelian monoid of equivalence classes of idempotents in $M_\infty(R)$ under the relation $\sim$ defined above, with the operation $[p] + [q] = [p \oplus q]$. We consider this monoid endowed with the algebraic pre-ordering, denoted by $\leq$, that corresponds to the ordering induced by the relation $\lesssim$; in particular, $<$ corresponds to the relation $\prec$. Given a ring R, it is easy to see that $V(R)$ is conical, and if R is simple, then so is $V(R)$. If R is purely infinite simple (non-necessarily unital), then the argument in the proof of Ara et al. (2002, Proposition 2.1) implies that $V(R)^*$ is a group. Hence, for every $e, f \in R$ nonzero idempotents in a purely infinite simple ring, we have $[e] < [f]$, and thus $e \prec f$.

Lemma 2.4. *Let R be a σ-unital, nonunital, purely infinite simple ring. For any sequence of nonzero orthogonal idempotents $\{p_n\}_{n\geq 1}$ such that $\sum_{i=1}^{n} p_i \to P \in \mathscr{M}(R)$ in the strict topology, $P \sim 1 \in \mathscr{M}(R)$.*

Proof. By Theorem 2.1 and Corollary 2.3, R has a countable unit. Let $\{e_n\}_{n\geq 1}$ be an increasing countable unit in R. Since R is purely infinite simple,

$$e_1 \prec p_1 + p_2 \prec e_3 \prec p_1 + p_2 + p_3 + p_4 \prec \dots.$$

It means that there exists an idempotent $h_1 \in R$ such that $h_1 \sim e_1$ and $h_1 < p_1 + p_2$. Hence, $p_1 + p_2 - h_1 \prec e_3 - e_1$, and thus there exists an idempotent $g' \in R$ with $g' \sim p_1 + p_2 - h_1$ and $g' < e_3 - e_1$. Defining $g_2 = e_1 \oplus g' \in R$, we have $e_1 < e_1 \oplus g' = g_2 < e_1 + e_3 - e_1 = e_3$, and $g_2 \sim h_1 + p_1 + p_2 - h_1 = p_1 + p_2$. By recurrence on this argument, we get two sequences of idempotents $\{g_{2j}\}_{j\in\mathbb{N}}$ and $\{h_{2j+1}\}_{j\in\mathbb{N}}$ such that, for each $j \in \mathbb{N}$, $e_{2j-1} < g_{2j} < e_{2j+1}$, $g_{2j} \sim p_1 + \cdots + p_{2j}$, with $p_1 + \cdots + p_{2j} < h_{2j+1} < p_1 + \cdots + p_{2(j+1)}$, and $h_{2j+1} \sim e_{2j+1}$. So we have:

$$\begin{array}{ccccccccc} h_1 & < & p_1 + p_2 & < & h_3 & < & p_1 + p_2 + p_3 + p_4 & < & \cdots \\ \wr & & \wr & & \wr & & \wr & & \\ e_1 & < & g_2 & < & e_3 & < & g_4 & < & \cdots. \end{array}$$

For each $n \in \mathbb{N}$, define

$$g_n = \begin{cases} 0, & n = 0; \\ e_n, & n \text{ odd}; \\ g_n, & n \text{ even}; \end{cases}$$

$$h_n = \begin{cases} 0, & n = 0; \\ h_n, & n \text{ odd}; \\ p_1 + \cdots + p_n, & n \text{ even}. \end{cases}$$

Then, we have two ascending sequences of idempotents, $\{g_n\}_{n\in\mathbb{N}}$ and $\{h_n\}_{n\in\mathbb{N}}$, such that $g_n \sim h_n$ for each $n \in \mathbb{N}$. Notice that $h_{2n} = \sum_{i=1}^{2n} p_i$ in $\mathscr{M}(R)$. Also notice that, given any $a \in R$, there exists $n \in \mathbb{N}$ such that, for any $m \geq n$, $h_{2m}a = Pa$. Since

$$h_{2m+2} = (h_{2m+2} - h_{2m+1}) + (h_{2m+1} - h_{2m}) + h_{2m},$$

defining $\tilde{p} = (h_{2m+2} - h_{2m+1})$ and $\hat{p} = (h_{2m+1} - h_{2m})$, we have

$$\tilde{p}a + \hat{p}a + h_{2m}a = h_{2m+2}a = Pa = h_{2m}a.$$

Thus, $\tilde{p}a + \hat{p}a = 0$, and since $\tilde{p} \perp \hat{p}$, we have $\tilde{p}a = \hat{p}a = 0$. Thus, for any $m \geq n$, $h_{2m+1}a = \hat{p}a + h_{2m}a = Pa$. Hence, $h_n \to P$ in $\mathscr{M}(R)$. Similarly, we get $g_n \to 1$ in $\mathscr{M}(R)$.

Since R is purely infinite simple, $V(R)^*$ is a group (Ara et al., 2002, Proposition 2.1). So, for $i \in \mathbb{N}$, since $h_i + (h_{i+1} - h_i) = h_{i+1} \sim g_{i+1} = g_i + (g_{i+1} - g_i)$, we have $h_{i+1} - h_i \sim g_{i+1} - g_i$. Thus, there exist $x_i \in (g_{i+1} - g_i)R(h_{i+1} - h_i)$, $y_i \in (h_{i+1} - h_i)R(g_{i+1} - g_i)$ with $x_i y_i = g_{i+1} - g_i$, $y_i x_i = h_{i+1} - h_i$. As $\{\sum_{i=0}^n (g_{i+1} - g_i)\}_{n\in\mathbb{N}} \to 1$ and $\{\sum_{i=0}^n (h_{i+1} - h_i)\}_{n\in\mathbb{N}} \to P$, by Ara and Perera (2000, Lemma 1.7), $\{\sum_{i=0}^n x_i\}_{n\in\mathbb{N}} \to x$ and $\{\sum_{i=0}^n y_i\}_{n\in\mathbb{N}} \to y$, for some $x \in \mathcal{M}(R)P$ and $y \in P\mathcal{M}(R)$. By Ara and Perera (2000, Lemma 1.3), $xy = 1$ and $yx = P$. Hence, $P \sim 1$ in $\mathcal{M}(R)$. □

Finally, we get the main result in this article.

Theorem 2.5. *Let R be a σ-unital, nonunital, purely infinite simple ring. Then*:

(1) $R \cong M_\infty(R)$;
(2) *For every nonzero idempotent $q \in R$, we have $R \cong M_\infty(qRq)$.*

Proof. By Theorem 2.1 and Corollary 2.3, R has a countable unit. Let $\{e_n\}_{n\geq 1}$ be an increasing countable unit in R. Fix a nonzero idempotent $q \in R$. We define a sequence of idempotents by recurrence, as follows:

$$q_0 = 0;$$
$$q_n = e_n - e_{n-1}, \qquad n \in \mathbb{N} \quad (e_0 = 0).$$

Since R is purely infinite simple, q_n is an infinite idempotent for any $n \in \mathbb{N}$. Moreover, $q \lesssim q_n$. Hence, for each $n \in \mathbb{N}$, there exists an idempotent $p_n \in R$ such that $p_n \leq q_n$ and $p_n \sim q$.

By construction, $e_n = \sum_{i=0}^n q_i$, and $\{e_n\}_{n\in\mathbb{N}} = \{\sum_{i=0}^n q_i\}_{n\in\mathbb{N}}$ converges to $1 \in \mathcal{M}(R)$ in the strict topology of R; in particular, it is a Cauchy sequence. Since R is simple, it is semiprime, and

$$\left(\sum_{i=0}^n p_i - \sum_{i=0}^m p_i\right) = \sum_m^n p_i \leq \sum_m^n q_i$$

implies that $\left(\sum_m^n p_i\right)_{n\in\mathbb{N}}$ is also a Cauchy sequence. By Ara and Perera (2000, Proposition 1.6), $\mathcal{M}(R)$ is complete, so that $\left(\sum_m^n p_i\right)_{n\in\mathbb{N}}$ converges to some $P \in \mathcal{M}(R)$.

Clearly, $\{q_n\}_{n\in\mathbb{N}}$ is a family of orthogonal idempotents. Then, by Ara and Perera (2000, Lemma 1.3)

$$P^2 = \left(\lim_n \sum_{i=1}^n p_i\right)\left(\lim_n \sum_{j=1}^n p_j\right) = \lim_n \left(\sum_{i=1}^n p_i\right)\left(\sum_{j=1}^n p_j\right) = \lim_n \sum_{i=1}^n p_i = P,$$

whence P is an idempotent of $\mathcal{M}(R)$. By Lemma 2.4, $P \sim 1 \in \mathcal{M}(R)$. In particular, there exist $u \in P\mathcal{M}(R)$ and $v \in \mathcal{M}(R)P$ such that $uv = P, vu = 1$. Notice that, since R is nonunital, $P \notin R$.

Thus, we can define two ring morphisms, $\rho : R \to PRP$ by the rule $\rho(r) = urv$, and $\psi : PRP \to R$ by the rule $\psi(r) = vru$. Clearly, they are mutually inverses, so that,

$$R \cong PRP. \tag{1}$$

Define $t_n = \sum_{i=1}^{n} p_i$. Since $\{t_n\}_{n\in\mathbb{N}}$ converges to $P \in \mathcal{M}(R)$, we have $PRP = \bigcup_{n\in\mathbb{N}} t_nRt_n$. Then, $t_{n+1} - t_n = \sum_{i=1}^{n+1} p_i - \sum_{i=1}^{n} p_i = p_{n+1} \sim q$, and since $t_n = (t_n - t_{n-1}) \oplus (t_{n-1} - t_{n-2}) \oplus \cdots \oplus (t_1 - t_0) \sim nq$, we get

$$t_nRt_n \cong End_R(t_nR) \cong End_R(n(qR)) \cong M_n(qRq).$$

Under this identification, $t_nRt_n \hookrightarrow t_{n+1}Rt_{n+1}$ is the map

$$M_n(qRq) \longrightarrow M_{n+1}(qRq)$$
$$a \longmapsto \mathrm{diag}(a, 0)$$

so that

$$PRP = \bigcup_{n\in\mathbb{N}} t_nRt_n \cong \bigcup_{n\in\mathbb{N}} M_n(qRq) = M_\infty(qRq). \tag{2}$$

Finally, if $q \in R$ is a nonzero idempotent, qRq is a unital, purely infinite simple ring. Then, (1) and (2) imply $R \cong PRP \cong M_\infty(qRq)$. Hence, $M_\infty(R) \cong M_\infty(M_\infty(qRq)) \cong M_\infty(qRq) \cong R$, as desired. □

Then, we get the corresponding dichotomy result, analog to Zhang (1992, Theorem 1.2(i)). We say that a (nonunital) ring R is stable if there exists a ring S such that $R \cong M_\infty(S)$.

Corollary 2.6. *Let R be a σ-unital purely infinite simple ring. Then it is either unital or stable.*

Remark 2.7. Notice that we cannot guarantee that a nonunital, purely infinite simple ring has s-unit. For example, given a field K, consider, for $n \geq 2$, the Leavitt algebra

$$R = K\left\langle x_1, \ldots, x_n, y_1, \ldots, y_n \mid x_iy_j = \delta_{ij}, \sum_{i=1}^{n} y_ix_i = 1 \right\rangle.$$

This is a purely infinite simple ring (see Ara et al., 2002), so that any right ideal of R is a nonunital purely infinite simple ring. Then, it is easy to see that the right ideal $L = y_1R$ is a nonunital, purely infinite simple ring with no s-unit.

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