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ALMOST TRANSITIVITY IN $\mathcal{C}_0$ SPACES OF VECTOR-VALUED FUNCTIONS

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Abstract By means of M-structure and dimension theory, we generalize some known results and obtain some new ones on almost transitivity in $\mathcal{C}_0(L, X)$. For instance, if X has the strong Banach–Stone property, then almost transitivity of $\mathcal{C}_0(L, X)$ is divided into two weaker properties, one of them depending only on topological properties of L and the other being closely related to the covering dimensions of L and X. This leads to some non-trivial examples of almost transitive $\mathcal{C}_0(L, X)$ spaces.

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1. Introduction

Every topological space considered here will be Hausdorff, even if it is not explicitly stated. If X is a Banach space, $\mathcal{G}(X)$ denotes the group of surjective linear isometries from X onto itself. Unless otherwise stated, in $\mathcal{G}(X)$ we consider the strong operator topology, which it inherits from $\mathcal{CL}(X, X)$. Given a topological space $\mathcal{T}$, by $\mathcal{BC}(\mathcal{T}, X)$ we denote the Banach space of bounded continuous functions from $\mathcal{T}$ into X, endowed with the sup norm. L will denote a locally compact topological space, $\hat{L}$ will be its one-point compactification and we will suppose that $L \subset \hat{L}$. Then $\mathcal{C}_0(L, X)$ indicates the subspace of $\mathcal{BC}(L, X)$ whose functions vanish at the infinity point ∞. If $f \in \mathcal{C}_0(L, X)$, we will denote by $\hat{f}$ the only continuous extension of f to $\hat{L}$.

Definition 1.1. Given $x \in X$, the *orbit* of x is the set $\mathcal{G}(x) = \{Tx : T \in \mathcal{G}\}$. It is said that

(i) X is *transitive* if, for every $x, y \in S_X$, there exists $T \in \mathcal{G}$ with $Tx = y$ (in other words, the orbit of every element with norm one is S_X);

(ii) X is *almost transitive* if, for every $x, y \in S_X$ and $\varepsilon > 0$, there exists $T \in \mathcal{G}$ with $\|Tx - y\| < \varepsilon$ (in other words, the orbit of every element with norm one is dense in S_X).

The study of almost transitivity in $\mathcal{C}_0(L,\mathbb{K})$, where $\mathbb{K}$ is the real or complex scalar field, was begun by Wood in [**20**], where he poses the natural question about the triviality of this study. In other words, if L has more than one point, can $\mathcal{C}_0(L,\mathbb{K})$ be almost transitive? In [**7**], Greim and Rajagopalan answer this question in the negative in the real case, and obtain significant restrictions in the complex case. Further advances on the subject are due to Cabello [**3**–**5**]. Finally, it is proved that the answer in the complex case is affirmative [**10**, **18**].

Thus, considering the general case, almost transitivity in $\mathcal{C}_0(L,X)$ becomes more meaningful. Until now, the only results on that matter are due to Greim, Jamison and Kaminska [**8**]. Our purpose is to shed more light on the subject.

Next we will introduce some topological results that will be necessary later.

Proposition 1.2 (see [12]). *Let $\mathcal{T}$ be a normal topological space and let X be a finite-dimensional Banach space. Given a closed set $F \subset \mathcal{T}$ and a continuous function $f : F \to B_X$, there exists $\bar{f} : \mathcal{T} \to B_X$, which is a continuous extension of f.*

Proposition 1.3 (Dowker, see [15]). *Let $\mathcal{T}$ be a collectionwise normal topological space and let X be a Banach space. Given a closed set $F \subset \mathcal{T}$ and a continuous function $f : F \to X$, there exists $\bar{f} : \mathcal{T} \to X$, which is a continuous extension of f.*

Actually, from the previous proposition we only need the following corollary. It is straightforward to prove taking into account that every compact space is collectionwise normal and that, if X is an infinite-dimensional Banach space, then S_X is a retract of B_X.

Corollary 1.4. *Let K be a compact space and let X be an infinite-dimensional Banach space. If $F \subset K$ is closed and $f : F \to S_X$ is continuous, then there exists $\bar{f} : K \to S_X$, which is a continuous extension of f.*

From these results we deduce the following lemma.

Lemma 1.5. *Let L be a locally compact space and let X be a Banach space. If $K \subset L$ is compact and $f : K \to S_X$ is continuous, then there exists $\bar{f} \in \mathcal{C}_0(L,X)$, which is a continuous extension of f with $\|\bar{f}\| = 1$.*

2. Some facts from dimension theory

The concept 'covering dimension' will not be defined here. The covering dimension of a completely regular topological space $\mathcal{T}$ will be denoted by $\dim \mathcal{T}$; besides, the algebraic dimension of a Banach space X as a vector space over $\mathbb{K}$ will be denoted by $\dim_{\mathbb{K}} X$. It is well known that if X is finite dimensional, then $\dim_{\mathbb{R}} X = \dim X$. If $\dim_{\mathbb{K}} X = \infty$, n is a natural number and L is any locally compact space, we will agree to write that $\dim L \leqslant \dim_{\mathbb{K}} X - n$.

Proposition 2.1 (see [14]). *If $\mathcal{T}$ is a topological space and $F \subset \mathcal{T}$ is closed, then $\dim F \leqslant \dim \mathcal{T}$.*

Proposition 2.2 (see [14]). *Let $\mathcal{T}$ be a normal topological space and let $n \geqslant 0$ be an integer. If $\{F_i : i \in \mathbb{N}\}$ is a closed covering of $\mathcal{T}$ with $\dim F_i \leqslant n$ for every $i \in \mathbb{N}$, then $\dim \mathcal{T} \leqslant n$.*

Definition 2.3. Let $\mathcal{T}$ be a topological space.

(i) It is said that $A \subset \mathcal{T}$ is *functionally closed* if there exists a continuous function $f : \mathcal{T} \to [0,1]$ such that $A = f^{-1}(0)$.

(ii) Let A, B be disjoint subsets of $\mathcal{T}$. A closed set $L \subset \mathcal{T}$ is called a *partition* between A and B if there exist disjoint open sets $U, V \subset \mathcal{T}$ such that $A \subset U$, $B \subset V$ and $\mathcal{T} \setminus L = U \cup V$.

Lemma 2.4 (see [16]). *Let $\mathcal{T}$ be a topological space. If A and B are functionally closed, disjoint subsets of $\mathcal{T}$ and $a, b \in \mathbb{R}$ with $a < b$, then there exists a continuous function $\varphi : \mathcal{T} \to [a,b]$ such that $A = \varphi^{-1}(\{a\})$ and $B = \varphi^{-1}(\{b\})$.*

Lemma 2.5 (see [6]). *Let $\mathcal{T}$ be a completely regular topological space and let $n \geqslant 0$ be an integer. The following assertions are equivalent.*

(1) $\dim \mathcal{T} \leqslant n$.

(2) *Given $n+1$ pairs $(A_1, B_1), \ldots, (A_{n+1}, B_{n+1})$ of functionally closed subsets of $\mathcal{T}$ such that $A_k \cap B_k = \varnothing$ for every $k \in \{1, \ldots, n+1\}$, there exist functionally closed sets $L_1, \ldots, L_{n+1} \subset \mathcal{T}$ such that $L_1 \cap \cdots \cap L_{n+1} = \varnothing$ and L_k is a partition between A_k and B_k for every $k \in \{1, \ldots, n+1\}$.*

The next theorem collects some results which can be found in [**16**]. The theorem that follows it is a finite-dimensional version, which also relies strongly on those results.

Theorem 2.6. *If X is an infinite-dimensional Banach space and $\mathcal{T}$ is a completely regular topological space, then*

(1) *the set $\{f \in \mathcal{BC}(\mathcal{T}, X) : f$ does not vanish$\}$ is dense in $\mathcal{BC}(\mathcal{T}, X)$;*

(2) *given a closed set $F \subset \mathcal{T}$, every continuous mapping $f : F \to S_X$ that has a continuous extension $g : \mathcal{T} \to B_X$ also has a continuous extension $\bar{f} : \mathcal{T} \to S_X$.*

Theorem 2.7. *Let X be a finite-dimensional Banach space and let $\mathcal{T}$ be a completely regular topological space. The following statements are equivalent.*

(1) $\dim \mathcal{T} \leqslant \dim X - 1$.

(2) *The set $\{f \in \mathcal{BC}(\mathcal{T}, X) : f$ does not vanish$\}$ is dense in $\mathcal{BC}(\mathcal{T}, X)$.*

(3) *Given a closed set $F \subset \mathcal{T}$, every continuous mapping $f : F \to S_X$ that has a continuous extension $g : \mathcal{T} \to B_X$ also has a continuous extension $\bar{f} : \mathcal{T} \to S_X$.*

(4) *Given a closed set $F \subset \mathcal{T}$, for every continuous mapping $f : F \to S_X$ that has a continuous extension $g : \mathcal{T} \to B_X$ and every $\varepsilon > 0$ there exists a mapping $\bar{f} : \mathcal{T} \to S_X$ such that $\|\bar{f}(t) - f(t)\| < \varepsilon$ for every $t \in F$.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) was proved in [**16**] and (3) $\Rightarrow$ (4) is obvious; we will have finished if we prove (4) $\Rightarrow$ (1).

Besides, it is straightforward to see that our hypothesis is preserved by isomorphisms, so we can suppose that X is $(\mathbb{R}^n, \|\cdot\|_\infty)$. Let $(A_1, B_1), \dots, (A_n, B_n)$ be n pairs of functionally closed subsets of $\mathcal{T}$ such that $A_k \cap B_k = \varnothing$ for every $k \in \{1, \dots, n\}$. By Lemma 2.4, for every k there exists a continuous function $\varphi_k : \mathcal{T} \to [-1, 1]$ such that $A_k = \varphi_k^{-1}(-1)$ and $B_k = \varphi_k^{-1}(1)$. Let $g : \mathcal{T} \to B_X$ be defined by $g(t) = (\varphi_1(t), \dots, \varphi_n(t))$; we have that g is continuous and $F = \{t \in \mathcal{T} : \|g(t)\|_\infty = 1\}$ is closed, moreover A_k and B_k are subsets of F for every k. Take $f = g_{|F}$. Using the hypothesis we deduce that there exists $\bar{f} : \mathcal{T} \to S_X$ with $\|f(t) - \bar{f}(t)\|_\infty < 1$ for every $t \in F$. Denoting by $\bar{f}_1, \dots, \bar{f}_n$ the coordinate functions of $\bar{f}$, for every $k \in \{1, \dots, n\}$ we define

$$L_k = \bar{f}_k^{-1}(0), \quad U_k = \bar{f}_k^{-1}((-\infty, 0)) \quad \text{and} \quad V_k = \bar{f}_k^{-1}((0, \infty))$$

and then $A_k \subset U_k$, $B_k \subset V_k$, $U_k \cap V_k = \varnothing$ and $\mathcal{T} \setminus L_k = U_k \cup V_k$, therefore every L_k is a partition between A_k and B_k. Moreover, if $x \in L_1 \cap \dots \cap L_n$, then $\bar{f}(x) = 0$, which is impossible. Consequently, $L_1 \cap \dots \cap L_n = \varnothing$, and applying Lemma 2.5 we deduce $\dim \mathcal{T} \leqslant n - 1$. □

3. Some facts from *M*-structure theory

The concepts 'centralizer' and 'M-finite space' will not be defined here. The centralizer of a Banach space X will be denoted by $Z(X)$. Every result and notion that appears in this section has been taken from [**2**]. Proposition 3.2 does not appear explicitly in that book, but it is a direct consequence of the results exposed there.

Definition 3.1. Let X be a Banach space. It is said that X has the strong Banach–Stone (SBS) property if for any two locally compact spaces L, L' and every surjective linear isometry $T : \mathcal{C}_0(L, X) \to \mathcal{C}_0(L', X)$ there exist a homeomorphism $\sigma : L' \to L$ and a continuous mapping $h : L' \to \mathcal{G}(X)$ such that $Tf(t) = h(t)(f(\sigma(t)))$ for every $t \in L'$ and $f \in \mathcal{C}_0(L, X)$.

Every Banach space X with $\dim_{\mathbb{K}} Z(X) = 1$ (in particular, every smooth or strictly convex space) has property SBS.

A Banach space X is M-finite if and only if it is linearly isometric to a finite product, with the sup norm, of spaces with one-dimensional centralizer. Such a product can be written in canonic form by grouping the factors that are linearly isometric (the process is exactly the same as in the fundamental theorem of arithmetic). Two M-finite spaces are linearly isometric if and only if they have the same canonic form. Every space without a copy of c_0 is M-finite. A $\mathcal{C}_0(L)$ space is M-finite if and only if L is finite.

The next proposition presents a characterization of the surjective linear isometries in $\mathcal{C}_0(L, X)$ when X is an M-finite space.

Proposition 3.2. *Let L be a locally compact space. If X is an M-finite Banach space with canonic decomposition $\prod_{i=1}^{r} X_i^{n_i}$, then for every surjective linear isometry*

$T : \mathcal{C}_0(L, X) \to \mathcal{C}_0(L, X)$ *there exist i varying in $\{1, \ldots, r\}$, homeomorphisms*

$$t^i : L \times \{1, \ldots, n_i\} \to L \times \{1, \ldots, n_i\}$$

and continuous mappings $u^i : L \times \{1, \ldots, n_i\} \to \mathcal{G}(X_i)$ such that, for every $f = (f_1^1, f_2^1, \ldots, f_{n_1}^1, \ldots, f_1^r, \ldots, f_{n_r}^r) \in \mathcal{C}_0(L, X)$ and every $l \in L$, we have

$$Tf(l) = (u^1(l,1) f^1_{t^1_2(l,1)}(t^1_1(l,1)), \ldots, u^1(l,n_1) f^1_{t^1_2(l,n_1)}(t^1_1(l,n_1)), \ldots, u^r(l,1) f^r_{t^r_2(l,1)}(t^1_1(l,1)), \ldots, u^r(l,n_r) f^r_{t^r_2(l,n_r)}(t^r_1(l,n_r))).$$

4. Almost transitivity in $\mathcal{C}_0(L, X)$: the general case

The intention of the next proposition is to contribute some information about some of the cases when X is not an M-finite space.

Proposition 4.1. *If X is a Banach space and L, L' are locally compact spaces, then $\mathcal{C}_0(L, \mathcal{C}_0(L', X))$ is linearly isometric to $\mathcal{C}_0(L \times L', X)$.*

Proof. It is straightforward to prove that the natural mapping $T : \mathcal{C}_0(L, \mathcal{C}_0(L', X)) \to \mathcal{C}_0(L \times L', X)$ defined by $Tf(t, t') = f(t)(t')$ is a well-defined surjective linear isometry. □

Taking into account that Wood's conjecture is true in the real case, it is deduced from the previous proposition that $\mathcal{C}_0(L, \mathcal{C}_0(L', \mathbb{R}))$ cannot be almost transitive if $L \times L'$ has more than one point.

In the general case we have already mentioned that Wood's conjecture is false, but the previous proposition can still be useful. For instance, if X is an SBS Banach space we have that $\mathcal{C}_0(L, c_0(X))$ is linearly isometric to $\mathcal{C}_0(L \times \mathbb{N}, X)$, reducing our problem to the study on SBS spaces that follows.

Definition 4.2. Let L be a locally compact space and let X be a Banach space. We say that $Y = \mathcal{C}_0(L, X)$

(i) is *almost positive transitive* for $x \in S_X$ if, given $\varepsilon > 0$ and $f_1, f_2 \in S_Y$ of the form $f_i(t) = \alpha_i(t)x$, with $\alpha_i \in \mathcal{C}_0(L, \mathbb{R}^+)$ and $\|\alpha_i\| = 1$ for every $i \in \{1, 2\}$, there exists $T \in \mathcal{G}(Y)$ with $\|Tf_1 - f_2\| < \varepsilon$;

(ii) admits *almost polar decompositions* for $x \in S_X$ if, given $\varepsilon > 0$ and $f \in S_Y$, there exists $T \in \mathcal{G}(Y)$ with $\|T|f|_x - f\| < \varepsilon$, where $|f|_x$ is defined by $|f|_x(t) = \|f(t)\|x$.

The previous definitions are inspired by [**7**]. Similar notions can be found in [**4**], [**5**] and [**9**]. The idea, which is to split the almost transitivity of the space into two weaker properties that are complementary in some sense, has turned out to be very fruitful.

It is straightforward to verify that, fixed $x \in S_X$, the space $\mathcal{C}_0(L, X)$ is almost transitive if and only if it is almost positive transitive for x and admits almost polar decompositions for x.

Let us recall that if X has the SBS property, then the elements of $\mathcal{G}(\mathcal{C}_0(L, X))$ are of the form $Tf(t) = h(t)(f(\sigma(t)))$, where $\sigma : L \to L$ is a homeomorphism and $h : L \to \mathcal{G}(X)$ is continuous. The next proposition (which is enunciated in [**7**] in a similar situation) shows that, in this case, almost positive transitivity 'depends' on σ and, admitting almost polar decompositions, on h.

Proposition 4.3. *Let L be a locally compact space, let X be an SBS Banach space and $x \in S_X$. Let us write $Y = \mathcal{C}_0(L, X)$. We have the following.*

(1) *If Y is almost positive transitive for x, then, given $\varepsilon > 0$ and $f_1, f_2 \in S_Y$ of the form $f_i(t) = \alpha_i(t)x$, with $\alpha_i \in \mathcal{C}_0(L, \mathbb{R}^+)$ for every $i \in \{1, 2\}$, there exists $T \in \mathcal{G}(Y)$ of the form $Tf(t) = f(\sigma(t))$ with $\|Tf_1 - f_2\| < \varepsilon$. In other words, in the isometry T the mapping h can be chosen to be always equal to the identity.*

(2) *If Y admits almost polar decompositions for x, then, given $\varepsilon > 0$ and $f \in S_Y$, there exists $T \in \mathcal{G}(Y)$ of the form $Tf(t) = h(t)(f(t))$ with $\|T|f|_x - f\| < \varepsilon$. In other words, in the isometry T the mapping σ can be chosen to be the identity.*

Proof. (1) Just take into account that

$$\begin{aligned}\|\alpha_1(\sigma(t))x - \alpha_2(t)x\| &= |\alpha_1(\sigma(t)) - \alpha_2(t)| \\ &= |\|h(t)(\alpha_1(\sigma(t))x)\| - \|\alpha_2(t)x\|| \\ &\leqslant \|h(t)(\alpha_1(\sigma(t))x) - \alpha_2(t)x\|.\end{aligned}$$

(2) Just take into account that

$$\begin{aligned}&\|h(t)(\|f(t)\|x) - f(t)\| \\ &\quad\leqslant \|h(t)(\|f(t)\|x) - h(t)(\|f(\sigma(t))\|x)\| + \|h(t)(\|f(\sigma(t))\|x) - f(t)\| \\ &\quad= |\|f(t)\| - \|f(\sigma(t))\|| + \|h(t)(\|f(\sigma(t))\|x) - f(t)\| \\ &\quad= |\|h(t)(\|f(\sigma(t))\|x)\| - \|\|f(t)x\|\|| + \|h(t)(\|f(\sigma(t))\|x) - f(t)\| \\ &\quad\leqslant 2\|h(t)(\|f(\sigma(t))\|x) - f(t)\|.\end{aligned}$$

□

Proposition 4.4. *Let L be a locally compact space and let X be a Banach space. If X is almost transitive and $\mathcal{C}_0(L, X)$ admits almost polar decompositions for an $x \in S_X$, then $\mathcal{C}_0(L, X)$ admits almost polar decompositions for every $y \in S_X$.*

Proof. Take $y \in S_X$ and $f \in S_Y$. Given $\varepsilon > 0$, there exists $T \in \mathcal{G}(\mathcal{C}_0(L, X))$ such that $\|Tf - |f|_x\| < \frac{1}{2}\varepsilon$. Let $S \in \mathcal{G}(X)$ be such that $\|Sx - y\| < \frac{1}{2}\varepsilon$, we define $\tilde{T} : \mathcal{C}_0(L, X) \to \mathcal{C}_0(L, X)$ by $\tilde{T}f = S \circ Tf$, and it is clear that $\tilde{T} \in \mathcal{G}(\mathcal{C}_0(L, X))$. For every $t \in L$ we have

$$\begin{aligned}\|\tilde{T}f(t) - |f|_y(t)\| &\leqslant \|S(Tf(t)) - S(\|f(t)\|x)\| + \|S(\|f(t)\|x) - \|f(t)\|y\| \\ &\leqslant \|Tf(t) - \|f(t)\|x\| + \|Sx - y\| < \varepsilon.\end{aligned}$$

□

The next proposition shows that it has not been too restrictive to ask for X to be almost transitive. Moreover, the first part should not be surprising, since in Proposition 4.3 we have already proved that if X is SBS, then almost positive transitivity of $\mathcal{C}_0(L,X)$ is an eminently topological property, because it depends strongly on the group of homeomorphisms of L.

Proposition 4.5. *Let L be a locally compact space, let X be an SBS Banach space and $x \in S_X$. We have that*

(1) *$\mathcal{C}_0(L,X)$ is almost positive transitive for x if and only if $\mathcal{C}_0(L,\mathbb{R})$ is almost positive transitive;*

(2) *if $\mathcal{C}_0(L,X)$ admits almost polar decompositions for x, then X is almost transitive.*

Proof. (1) It is immediate from Proposition 4.3.

(2) Take $\varepsilon > 0$. Given $y \in S_X$, let $f \in S_Y$ be such that for certain $t_0 \in L$ we have $f(t_0) = y$. There exists a continuous mapping $h : L \to \mathcal{G}(X)$ such that $\|h(t)(\|f(t)\|x) - f(t)\| < \varepsilon$ for every $t \in L$. In particular, $\|h(t_0)(\|f(t_0)\|x) - f(t_0)\| = \|h(t_0)(x) - y\| < \varepsilon$. Therefore, we have proved that the orbit of x is dense in S_X. This implies immediately that X is almost transitive. □

As a consequence, if X has property SBS, in $\mathcal{C}_0(L,X)$ we will deal with properties 'to admit almost polar decompositions' and 'to be almost positive transitive' without specifying which $x \in S_X$.

It is well known that a finite-dimensional Banach space is almost transitive if and only if it is a Hilbert space [**19**]. This leads to the following corollary.

Corollary 4.6. *Let L be a locally compact space and let X be an SBS Banach space. If X is finite dimensional and $\mathcal{C}_0(L,X)$ admits almost polar decompositions, then X is a Hilbert space.*

In the following theorem, points (1) and (2) are slight generalizations of known results [**7**, **8**, **20**], although the proofs are essentially the same. Points (3) and (4) are new.

Theorem 4.7. *Let L be a locally compact space with more than one point. If $\mathcal{C}_0(L,\mathbb{R})$ is almost positive transitive, then*

(1) *L has no compact open subset (in other words, L is not compact and $\hat{L}$ is connected);*

(2) *if $K \subset L$ is compact and connected, then $\operatorname{Int}^L K = \varnothing$.*

If, moreover, L is not connected, then

(3) *no connected component of L is open (in particular, L has infinitely many connected components);*

(4) *given a compact set $K \subset L$ and a clopen set $O \subset L$, there always exists a connected component of O which does not intersect K.*

Proof. (1) If $K \subset L$ is a compact open proper subset, then there exist $f, g \in \mathcal{C}_0(L, \mathbb{R}^+)$ which satisfy $\|f\| = \|g\| = 1$, $f(L) = \{0, 1\}$ and $\frac{1}{2} \in g(L)$. It is clear that there does not exist a homeomorphism $\sigma : L \to L$ with $\|f(t) - g(\sigma(t))\| < \frac{1}{2}$ for every $t \in L$. If L is compact, there exists $f \in \mathcal{C}_0(L, \mathbb{R}^+)$ with $\{0, 1\} \subset f(L)$ and $\|f\| = 1$. Obviously, we do not have $\|f(t) - 1\| < 1$ for every $t \in L$.

(2) If $K \subset L$ is compact and connected with $U = \operatorname{Int}^L K \neq \varnothing$, as L is not compact we have $L \setminus K \neq \varnothing$ and there exist $f, g \in \mathcal{C}_0(L, \mathbb{R}^+)$ which satisfy $\|f\| = \|g\| = 1$, $f(K) \subset [\frac{1}{3}, 1]$, $f(L \setminus K) \subset [0, \frac{2}{3}]$, $g(K \setminus U) = 0$ and $1 \in f(U) \cap g(U) \cap g(L \setminus K)$. Let $\sigma : L \to L$ be a homeomorphism with $|f(t) - g(\sigma(t))| < \frac{1}{3}$ for every $t \in L$. There exists $t \in L$ such that $\sigma(t) \in U$ and $g(\sigma(t)) = 1$, then $f(t) > \frac{2}{3}$ and therefore $t \in K$ and $\sigma(t) \in \sigma(K)$, thus $\sigma(K) \cap U \neq \varnothing$. Analogously, $\sigma(K) \cap L \setminus K \neq \varnothing$. Besides, if $t \in K$, then $f(t) \geqslant \frac{1}{3}$ and $g(\sigma(t)) > 0$, which implies $\sigma(t) \notin K \setminus U$. We deduce that $\sigma(K) \subset U \cup (L \setminus K)$, and this contradicts the connectedness of K.

Now we assume that L is not connected.

(3) Let us suppose that C is an open connected component of L, as L is not connected we have that C and $L \setminus C$ are non-empty clopen subsets of L, and there exist $f, g \in \mathcal{C}_0(L, \mathbb{R}^+)$ which satisfy $\|f\| = \|g\| = 1$, $f(L \setminus C) = \{0\}$ and $1 \in f(C) \cap g(C) \cap g(L \setminus C)$. If $\sigma : L \to L$ is a homeomorphism with $|f(t) - g(\sigma(t))| < 1$ for every $t \in L$, then $\sigma(C) \cap C \neq \varnothing$ and $\sigma(C) \cap L \setminus C \neq \varnothing$, which contradicts the connectedness of C.

(4) Let us suppose that there exist a clopen set $O \subset L$ and a compact set $K \subset L$ such that K intersects every connected component of O. We can assume that $K \subset O$. If O were connected, it would be an open connected component of L; thus, there exist U, V non-empty clopen sets such that $U \cap V = \varnothing$ and $U \cup V = O$. Taking into account that $K \cap U$ and $K \cap V$ are compact sets which intersect every connected component of U and V, respectively, there exist $f, g \in \mathcal{C}_0(L, \mathbb{R}^+)$ which satisfy $\|f\| = \|g\| = 1$, $f(K \cap U) = \frac{1}{2}$, $f(U) \subset [0, \frac{1}{2}]$, $g(K) = \{1\}$ and $g(L \setminus O) = \{0\}$. Let $\sigma : L \to L$ be a homeomorphism and let M be a connected component of U. Thus, $f(M) \subset [0, \frac{1}{2}]$ and there exists $x \in M$ such that $f(x) = \frac{1}{2}$. Since U is clopen, M will also be a connected component of L. This implies that $\sigma(M)$ is a connected component either of $L \setminus O$, in which case it would be $g(\sigma(M)) = 0$ and in particular $g(\sigma(x)) = 0$, or of O, in which case there would exist $y \in K \cap \sigma(M)$, and we would obtain $f(\sigma^{-1}(y)) \leqslant \frac{1}{2}$ but $g(y) = 1$. In both cases, it is impossible that $|f(t) - g(\sigma(t))| < \frac{1}{2}$ for every $t \in L$. □

It is very tempting to affirm that there does not exist a locally compact space which satisfies (1), (2), (3) and (4) in the previous theorem and, as a consequence, that if $\mathcal{C}_0(L, \mathbb{R})$ is almost positive transitive, then L must be connected. However, we have not been able to prove or disprove it.

Question 4.8. Does there exist a locally compact space satisfying conditions (1), (2), (3) and (4) of Theorem 4.7?

Question 4.9. Let L be a locally compact space and let X be an SBS Banach space. If $\mathcal{C}_0(L,X)$ is almost positive transitive, is L necessarily connected?

Of course, a negative answer to Question 4.8 implies automatically an affirmative answer to Question 4.9.

Proposition 4.10. *Let L be a locally compact space and let X be a Banach space. If $\mathcal{C}_0(L,X)$ is almost transitive and X is M-finite, with canonic decomposition $X=\prod_{i=1}^{r}X_i^{n_i}$, then $r=1$.*

Proof. Suppose that $r\geqslant 2$. For every $i\in\{1,2\}$, fix $\eta^i\in\mathcal{C}_0(L,X_i)$ with $\|\eta^i\|=1$, and let $\theta^i : L\to X_i$ be the constant function zero.

Take $f\in S_{\mathcal{C}_0(L,X)}$, which satisfies $f_1^1=\eta^1$ and $f_j^2=\theta^2$ for every $j\in\{1,\dots,n_2\}$, and $g\in S_{\mathcal{C}_0(L,X)}$, which satisfies $g_1^2=\eta^2$. Taking into account Proposition 3.2, it is clear that there does not exist $T\in\mathcal{G}(\mathcal{C}_0(L,X))$ with $\|Tf-g\|<1$. □

Let us note that in the previous proposition we deduce $X=X_1^{n_1}$ and therefore $\mathcal{C}_0(L,X)$ is linearly isometric to $\mathcal{C}_0(L\times\{1,\dots,n_1\},X_1)$. This proves that, when studying almost positive transitivity of $\mathcal{C}_0(L,X)$, where X is an M-finite space, we can restrict ourselves to the case when X has property SBS. Moreover, we might formulate the following question.

Question 4.11. Let L be a locally compact space and let X be a Banach space. If $\mathcal{C}_0(L,X)$ is almost positive transitive and X is M-finite, can we affirm that X has property SBS?

Of course, an affirmative answer to Question 4.9 would automatically imply an affirmative answer to Question 4.11.

We will now study the role that is played by admitting almost polar decompositions. A curious relationship between the covering dimensions of L and X appears. The importance of covering dimension in this problem had already been noted in [**5**].

Proposition 4.12. *Let L be a locally compact space, let X be a Banach space and $x\in S_X$. Let us consider the following assertions.*

(a) *$\mathcal{C}_0(L,X)$ admits almost polar decompositions for x.*

(b) *Given a compact set $K\subset L$, $\varepsilon>0$ and a continuous mapping $f:K\to S_X$, there exists $h:L\to\mathcal{G}(X)$ continuous with $\|h(t)(x)-f(t)\|<\varepsilon$ for every $t\in K$.*

(c) *Given a compact set $K\subset L$, $\varepsilon>0$ and a continuous mapping $f:L\to S_X$, there exists $h:L\to\mathcal{G}(X)$ continuous with $\|h(t)(x)-f(t)\|<\varepsilon$ for every $t\in K$.*

(d) $\dim L\leqslant\dim X-1$.

(e) $\dim K\leqslant\dim X-1$ *for every compact set $K\subset L$.*

We have

(1) *If X has property SBS, then $(a)\Rightarrow(b)$.*

(2) (b) $\Rightarrow$ ((c) $\wedge$ (e)).

(3) ((c) $\wedge$ (d)) $\Rightarrow$ (a).

(4) (d) $\Rightarrow$ (e) *and, if* L *is* σ*-compact, then* (d) $\Leftrightarrow$ (e).

Proof. (1) By Lemma 1.5, there exists $g \in S_{\mathcal{C}_0(L,X)}$ such that $g_{|K} = f$. Using the hypothesis, we deduce that there exists a continuous mapping $h : L \to \mathcal{G}(X)$ with $\|g(t) - h(t)(\|g(t)\|x)\| < \varepsilon$ for every $t \in L$. If $t \in K$, then

$$\|f(t) - h(t)(x)\| = \|g(t) - h(t)(\|g(t)\|x)\| < \varepsilon.$$

(2) (b) $\Rightarrow$ (c) is trivial. To prove (e) from (b), we can assume that X is finite dimensional. Let $F \subset K$ be closed, let $f : F \to S_X$ be continuous and $\varepsilon > 0$. Since F is compact, there exists $h : L \to \mathcal{G}(X)$ continuous with $\|h(t)(x) - f(t)\| < \varepsilon$ for every $t \in F$. Let $\rho : K \to S_X$ be given by $\rho(t) = h(t)(x)$. We have $\|\rho(t) - f(t)\| < \varepsilon$ for every $t \in F$. Thus $\dim K \leqslant \dim X - 1$.

(3) Take $f \in S_{\mathcal{C}_0(L,X)}$ and $\varepsilon \in (0, 2]$. The set $K = \{t \in \hat{L} : \|\hat{f}(t)\| \geqslant \frac{1}{2}\varepsilon\}$ is a compact subset of L. By Lemma 1.5, there exists $g : L \to B_X$ continuous and such that $g(t) = f(t)/\|f(t)\|$ for every $t \in K$. By Theorems 2.6 and 2.7, there also exists $\bar{f} : L \to S_X$ continuous and such that $\bar{f}(t) = f(t)/\|f(t)\|$ if $t \in K$. Let $h : L \to \mathcal{G}(X)$ be a continuous mapping with $\|h(t)(x) - \bar{f}(t)\| < \varepsilon$ for every $t \in K$. We define $T : \mathcal{C}_0(L, X) \to \mathcal{C}_0(L, X)$ by $Tu(t) = h(t)(u(t))$, and we have that, if $t \in K$, then

$$\begin{aligned}\|T|f|_x(t) - f(t)\| &= \|\|f(t)\|h(t)(x) - f(t)\|\\ &= \|\|f(t)\|h(t)(x) - \|f(t)\|\bar{f}(t)\| < \varepsilon,\end{aligned}$$

and, if $t \notin K$, then

$$\|T|f|_x(t) - f(t)\| \leqslant 2\|f(t)\| < \varepsilon.$$

Therefore, $\|T|f|_x - f\| < \varepsilon$.

(4) This is immediately deduced from Propositions 2.1 and 2.2. □

5. Almost transitivity in $\mathcal{C}_0(L, X)$, where X is a Hilbert space

Proposition 5.1. *Given* $n \in \{1, 2, 4, 8\}$ *and* $2n$ *symbols*

$$\{a_1, a_2, \ldots, a_n, -a_1, -a_2, \ldots, -a_n\},$$

there exists a square matrix of order n *which satisfies the following conditions.*

(i) *Its elements belong to the set of symbols.*

(ii) *Its first column is* $(a_1, a_2, \ldots, a_n)$.

(iii) *It is formally orthogonal.*

Proof.

$$(a),\quad \begin{pmatrix} a & -b \\ b & a \end{pmatrix},\quad \begin{pmatrix} a & -b & c & d \\ b & a & -d & c \\ c & -d & -a & -b \\ d & c & b & -a \end{pmatrix}$$

$$\text{and}\quad \begin{pmatrix} a & b & c & d & e & f & g & h \\ b & -a & d & -c & -f & e & -h & g \\ c & -d & -a & b & g & -h & -e & f \\ d & c & -b & -a & h & g & -f & -e \\ e & f & -g & -h & -a & -b & c & d \\ f & -e & h & -g & b & -a & d & -c \\ g & h & e & f & -c & -d & -a & -b \\ h & -g & -f & e & -d & c & b & -a \end{pmatrix}.$$

□

We cannot extend Proposition 5.1 to any $n \in \{16, 32, \dots\}$. This is an easy consequence of the next assertion.

Proposition 5.2 (see [1]). *Given $n \in \mathbb{N}$, the following statements are equivalent.*

(i) *There exist $n-1$ continuous mappings $v_1, v_2, \dots, v_{n-1} : S_{\mathbb{R}^n} \to S_{\mathbb{R}^n}$ such that $v_i(x)$ is orthogonal to x for every $i \in \{1, \dots, n-1\}$ and $x \in S_{\mathbb{R}^n}$, and the vectors $v_1(x), v_2(x), \dots, v_{n-1}(x)$ are linearly independent for every $x \in S_{\mathbb{R}^n}$.*

(ii) $n \in \{1, 2, 4, 8\}$.

Now we will give six preliminary results, dealing with S_X-valued and $\mathcal{G}(X)$-valued functions, towards Theorem 5.10.

Lemma 5.3. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements of l_2 convergent to $x_0 \in l_2$. Then, for every $k \in \mathbb{N}$, the sequence $(z_n)_{n\in\mathbb{N}}$ defined by $z_n(j) = x_0(k)x_0(j)(1 + x_n(1)) - x_n(k)x_n(j)(1 + x_0(1))$ is included in l_2 and converges to 0.*

Proof. For the sake of simplicity we will write x_{ab} instead of $x_a(b)$. For every $j \in \mathbb{N}$ we have

$$\begin{aligned}
&(x_{0k}x_{0j}(1 + x_{n1}) - x_{nk}x_{nj}(1 + x_{01}))^2 \\
&\quad = x_{0k}^2(1 + x_{n1})^2 x_{0j}^2 + x_{nk}^2(1 + x_{01})^2 x_{nj}^2 - 2x_{0k}x_{nk}(1 + x_{01})(1 + x_{n1})x_{0j}x_{nj} \\
&\quad = x_{0k}^2(1 + x_{n1})^2 x_{0j}^2 + x_{nk}^2(1 + x_{01})^2 x_{nj}^2 - x_{0k}x_{nk}(1 + x_{01})(1 + x_{n1})(x_{0j}^2 + x_{nj}^2) \\
&\qquad\quad + x_{0k}x_{nk}(1 + x_{01})(1 + x_{n1})(x_{0j} - x_{nj})^2 \\
&\quad = (x_{0k}^2(1 + x_{n1})^2 - x_{0k}x_{nk}(1 + x_{01})(1 + x_{n1}))x_{0j}^2 + (x_{nk}^2(1 + x_{01})^2 \\
&\qquad\quad - x_{0k}x_{nk}(1 + x_{01})(1 + x_{n1}))x_{nj}^2 + x_{0k}x_{nk}(1 + x_{01})(1 + x_{n1})(x_{0j} - x_{nj})^2 \\
&\quad = A_n x_{0j}^2 + B_n x_{nj}^2 + C_n(x_{0j} - x_{nj})^2,
\end{aligned}$$

where $(A_n)_{n\in\mathbb{N}}, (B_n)_{n\in\mathbb{N}}$ are sequences convergent to zero and $(C_n)_{n\in\mathbb{N}}$ is a bounded sequence. Therefore,

$$\begin{aligned}\|z_n\|^2 &= \sum_{j=1}^{\infty}(A_n x_{0j}^2 + B_n x_{nj}^2 + C_n(x_{0j} - x_{nj})^2)\\ &= A_n\|x_0\|^2 + B_n\|x_n\|^2 + C_n\|x_n - x_0\|^2\end{aligned}$$

and from this we deduce the desired result. □

Proposition 5.4. *Let X be a Hilbert space and $e_1 \in S_X$. There exists $H : S_X \setminus \{-e_1\} \to \mathcal{G}(X)$ continuous and such that $H(x)(e_1) = x$ for every $x \in S_X \setminus \{-e_1\}$.*

Proof. The case $\dim X = 1$ is trivial. Otherwise, let $(e_1, (e_\alpha)_{\alpha\in\Lambda})$ be an orthonormal basis of X. Take $\Gamma = \Lambda \cup \{1\}$. We define $H(x)(e_1) = x$ and, for $\beta \in \Lambda$,

$$H(x)(e_\beta) = \left(-x_\beta, \left(\delta_{\alpha\beta} - \frac{x_\alpha x_\beta}{x_1 + 1}\right)_{\alpha\in\Lambda}\right)$$

(where $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$) and we extend $H(x)$ by linearity. Let us see that H satisfies the required properties.

Take $x \in S_X \setminus \{-e_1\}$. If $\alpha, \beta \in \Gamma$, it is straightforward to prove that

$$(H(x)(e_\alpha) \mid H(x)(e_\beta)) = \delta_{\alpha\beta},$$

from which $(H(x)(e_\alpha))_{\alpha\in\Gamma}$ is an orthonormal basis of X. If $y \in X$, then, by Pythagoras's theorem,

$$\begin{aligned}\|H(x)(y)\|^2 &= \left\|\sum_{\alpha\in\Gamma} y_\alpha H(x)e_\alpha\right\|^2\\ &= \sum_{\alpha\in\Gamma} |y_\alpha|^2\|H(x)e_\alpha\|^2\\ &= \sum_{\alpha\in\Gamma} |y_\alpha|^2 = \|y\|^2.\end{aligned}$$

Hence, $H(x)$ is a surjective linear isometry.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence convergent to certain x in $S_X\setminus\{-e_1\}$. Let us fix $y \in X$, given $\varepsilon > 0$ there exists $M \subset \Gamma$ which is finite and such that $\sum_{i\in\Gamma\setminus M} |y_i|^2 < \frac{1}{16}\varepsilon^2$. Besides, by Lemma 5.3, for every $i \in M$ we have $\lim_n y_i(H(x_n)(e_i) - H(x)(e_i)) = 0$, therefore there exists $n_0 \in \mathbb{N}$ such that, if $n \geqslant n_0$, then $\sum_{i\in M} \|y_i(H(x_n)(e_i) - H(x)(e_i))\| < \frac{1}{2}\varepsilon$. Joining both inequalities, for every $n \geqslant n_0$ we have

$$\begin{aligned}\|H(x_n)(y) - H(x)(y)\| \leqslant{}& \left\|\sum_{i\in M} y_i(H(x_n)(e_i) - H(x)(e_i))\right\|\\ &+ \left\|\sum_{i\in\Gamma\setminus M} y_i(H(x_n)(e_i) - H(x)(e_i))\right\| < \varepsilon,\end{aligned}$$

from which we deduce that $H(x_n)$ converges to $H(x)$ in the strong operator topology. □

Corollary 5.5. *Let X be a Hilbert space and $e_1 \in S_X$. If $f : L \to S_X$ is continuous and nowhere equal to $-e_1$, then there exists $h : L \to \mathcal{G}(X)$ continuous with $h(t)(e_1) = f(t)$ for every $t \in L$.*

Corollary 5.6. *Let X be a Hilbert space. Given $e_1 \in S_X$, the following statements are equivalent.*

(1) *If $f : L \to S_X$ is continuous, there exists $h : L \to \mathcal{G}(X)$ continuous with $\|h(t)(e_1) - f(t)\| < 2$ for every $t \in L$.*

(2) *If $f : L \to S_X$ is continuous, there exists $h : L \to \mathcal{G}(X)$ continuous with $h(t)(e_1) = f(t)$ for every $t \in L$.*

Proof. Let $f : L \to S_X$ and $h : L \to \mathcal{G}(X)$ be continuous mappings with $\|h(t)(e_1) - f(t)\| < 2$ for every $t \in L$. The mapping $\rho : L \to S_X$ given by $\rho(t) = h(t)^{-1}(f(t))$ is continuous and satisfies $\|\rho(t) - e_1\| < 2$ for every $t \in L$, therefore ρ is nowhere equal to $-e_1$. Let $h_2 : L \to \mathcal{G}(X)$ be continuous with $h_2(t)(e_1) = \rho(t)$ for every $t \in L$. The mapping $h_3 = hh_2 : L \to \mathcal{G}(X)$ is continuous and satisfies $h_3(t)(e_1) = h(t)h_2(t)(e_1) = h(t)\rho(t) = f(t)$ for every $t \in L$. □

Proposition 5.7. *Let X be a Hilbert space and $e_1 \in S_X$. If $\dim L \leqslant \dim X - 2$, given $\varepsilon > 0$ and $f : L \to S_X$ continuous, there exists $g : L \to S_X$ continuous which is nowhere equal to $-e_1$ and such that $\|g(t) - f(t)\| < \varepsilon$ for every $t \in L$.*

Proof. Let $(e_1, (e_\alpha)_{\alpha\in\Lambda})$ be an orthonormal basis of X seen as a real space: in this way we can write $X = \mathbb{R} \oplus_2 Y$, where Y is a Hilbert space and $(e_\alpha)_{\alpha\in\Lambda}$ is a basis of Y. Let us consider $u = (f_\alpha)_{\alpha\in\Lambda} : L \to B_Y$. Since $\dim L \leqslant \dim X - 2 = \dim Y - 1$ there exists $v : L \to B_Y$ which does not vanish and with $\|u - v\| < \frac{1}{2}\varepsilon$. Now we define $g : L \to S_X$ by $g(t) = (f_1(t)^2 + \|v(t)\|^2)^{-1/2}(f_1(t), v(t))$, then g is continuous and for every $t \in L$ we have

$$\begin{aligned}
\|g(t) - f(t)\| &= \|(f_1(t)^2 + \|v(t)\|^2)^{-1/2}(f_1(t), v(t)) - (f_1(t), u(t))\| \\
&\leqslant |(f_1(t)^2 + \|v(t)\|^2)^{-1/2} - 1|\|(f_1(t), v(t))\| + \|(f_1(t), u(t)) - (f_1(t), v(t))\| \\
&\leqslant |(f_1(t)^2 + \|v(t)\|^2)^{1/2} - 1| + \|u - v\| \\
&= |\|(f_1(t), v(t))\| - \|(f_1(t), u(t))\|| + \|u - v\| \\
&\leqslant \|(0, u(t) - v(t))\| + \|u - v\| \leqslant 2\|u - v\| < \varepsilon.
\end{aligned}$$

Moreover, g is nowhere equal to $-e_1$ because v does not vanish. □

Corollary 5.8. *Let X be a Hilbert space and $e_1 \in S_X$. If $\dim L \leqslant \dim X - 2$ and $f : L \to S_X$ is continuous, then there exists $h : L \to \mathcal{G}(X)$ continuous with $h(t)(e_1) = f(t)$ for every $t \in L$.*

Corollary 5.9. *If X is an infinite-dimensional Hilbert space, then $\mathcal{C}_0(L, X)$ admits almost polar decompositions.*

Proof. It is a direct consequence of Proposition 4.12 together with Corollary 5.8. □

Theorem 5.10. *Let L be a locally compact space and $n \in \mathbb{N}$. Let us consider the following statements.*

(1) $\dim L \leqslant n \dim_{\mathbb{R}} \mathbb{K} + \chi_{\{1,2,4,8\}}(n) - 2$.

(2) $\mathcal{C}_0(L, (\mathbb{K}^n, \|\cdot\|_2))$ *admits almost polar decompositions.*

(3) $\dim K \leqslant n \dim_{\mathbb{R}} \mathbb{K} - 1$ *for every compact set* $K \subset L$.

We have (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Proof. (1) $\Rightarrow$ (2). Let us suppose first that $n \in \{1, 2, 4, 8\}$, and so $\dim L \leqslant n - 1$.

Take $f : L \to S_{\mathbb{K}^n}$. Let A be a formally orthogonal matrix with the properties mentioned in Proposition 5.1 and whose first column is $(f_1, f_2, \dots, f_n)$. If we define $h : L \to \mathcal{G}(\mathbb{K}^n)$ by

$$h(t) = A(t) = \begin{pmatrix} f_1(t) & * & \cdots & * \\ f_2(t) & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ f_n(t) & * & \cdots & * \end{pmatrix},$$

then it is clear that $h(t)(e_1) = f(t)$ for every $t \in L$. Now apply Proposition 4.12.

In the case when $n \in \mathbb{N} \setminus \{1, 2, 4, 8\}$ we have $\dim L \leqslant n - 2$ and the result follows from Corollary 5.8 and Proposition 4.12.

(2) $\Rightarrow$ (3). We have already proved this (Proposition 4.12). □

Note that, taking into account what we have proved up until now, if L is σ-compact and $n \in \{1, 2, 4, 8\}$, then the three statements in the previous theorem turn out to be equivalent.

Example 5.11. Let us see that if $n \in \mathbb{N} \setminus \{1, 2, 4, 8\}$, then $\mathcal{C}(S_{\mathbb{R}^n}, (\mathbb{R}^n, \|\cdot\|_2))$ does not admit almost polar decompositions, and therefore in the previous theorem (3) $\Rightarrow$ (2) is not true in general.

Indeed, if $\mathcal{C}(S_{\mathbb{R}^n}, (\mathbb{R}^n, \|\cdot\|_2))$ admits almost polar decompositions, by virtue of Corollary 5.6 together with Proposition 4.12, there would exist $h : S_{\mathbb{R}^n} \to \mathcal{G}(\mathbb{R}^n)$ continuous with $h(t)(e_1) = t$ for every $t \in S_{\mathbb{R}^n}$. Writing h in matricial form, and calling its columns $v_0, v_1, \dots, v_{n-1}$, we would have that $v_i(t)$ is orthogonal to $v_0(t) = t$ for every $i \in \{1, \dots, n-1\}$ and the vectors $v_1(x), v_2(x), \dots, v_{n-1}(x)$ are linearly independent for every $x \in S_{\mathbb{R}^n}$. This contradicts Proposition 5.2.

It is straightforward to see that if $\mathcal{C}_0(L, (\mathbb{C}^n, \|\cdot\|_2))$ admits almost polar decompositions, then $\mathcal{C}_0(L, (\mathbb{R}^{2n}, \|\cdot\|_2))$ also admits them, thus the previous example can be also used in the complex case (in dimensions one, two and four).

In order to see what happens with (2) $\Rightarrow$ (1) in Theorem 5.10 we need a previous result.

Proposition 5.12. *Let L be a locally compact space. The space $\mathcal{C}_0(L, \mathbb{R})$ admits almost polar decompositions if and only if* $\dim \hat{L} = 0$.

Proof. Necessity. Suppose that $\mathcal{C}_0(L,\mathbb{R})$ admits almost polar decompositions. If M is a connected subset of L and $x, y \in M$ are distinct points, then we can construct $f \in \mathcal{C}_0(L,\mathbb{R})$ with $f(x) = 1$ and $f(y) = -1$. There must exist a continuous mapping $h : L \to \{-1, 1\}$ such that $|h(t)|f(t)| - f(t)| < 2$ for every $t \in L$. In particular $h(x) = 1$, $h(y) = -1$ and therefore $h(M) = \{-1, 1\}$, which is a contradiction since M is connected.

From this we deduce that L is totally disconnected and it was proved in [**7**] that this implies that L is zero dimensional. This in turn implies that $\hat{L}$ is zero dimensional, and for compact Hausdorff spaces (see [**6**]) this is the same as having covering dimension zero.

Sufficiency. If $\dim \hat{L} = 0$ then, by Theorem 5.10, $\mathcal{C}(\hat{L},\mathbb{R})$ admits almost polar decompositions. If $f \in \mathcal{C}_0(L,\mathbb{R})$ and $\varepsilon > 0$, then there exists a continuous function $h : \hat{L} \to \{-1, 1\}$ such that $\|h(t)|\hat{f}(t)| - \hat{f}(t)\| < \varepsilon$ for every $t \in \hat{L}$, in particular $\|h(t)|f(t)| - f(t)\| < \varepsilon$ for every $t \in L$, so $\mathcal{C}_0(L,\mathbb{R})$ admits almost polar decompositions. □

Example 5.13. In [**17**], Rajagopalan constructs under the continuum hypothesis (CH) a locally compact space L which is scattered, countably compact, first countable and such that $\dim \hat{L} = 0$ but $\dim L \neq 0$. Therefore, under CH the implication (2) $\Rightarrow$ (1) of Theorem 5.10 is not true in general.

We do not know if there is a counterexample to (2) $\Rightarrow$ (1) without requiring CH; it should be noticed that Rajagopalan's example has many properties we do not need here.

6. Examples and some more questions

We introduce some terminology to gain convenience when managing the examples.

Definition 6.1. Let L be a locally compact space with more than one point.

(i) We will say that L is an L_0-space if $\mathcal{C}_0(L,\mathbb{C})$ is almost transitive.

(ii) We will say that L is an L_1-space if $\mathcal{C}_0(L,\mathbb{C})$ is transitive.

In [**7**] it is proved that if there exists an L_0-space, then there exists an L_1-space and that an L_1-space never satisfies the first axiom of countability. In [**18**] and [**10**] it is proved that if L is a locally compact, non-compact space and $\hat{L}$ is the topological space known as a pseudoarc, then L is an L_0-space. However, such an L cannot be an L_1-space because the pseudoarc is metrizable. Moreover, the aforementioned L is unique up to homeomorphisms, because the pseudoarc is a homogeneous topological space. Thus, the actual state of knowledge on L_i-spaces is summarized as: there exist at least two L_0-spaces, one of them is an L_1-space and the other is not.

Coming to the general case, and taking into account everything which has been proved up until now, we deduce the following proposition.

Proposition 6.2. *If X is a Hilbert space with $\dim X \geqslant 2$ and L is a σ-compact L_0-space, then $\mathcal{C}_0(L,X)$ is almost transitive.*

Proof. Proposition 4.5 plus Theorem 5.10 plus the observation after Theorem 5.10. □

It is not difficult to prove that if $\hat{L}$ is the pseudoarc, then L is connected (every composant of the pseudoarc is dense in it (see [**11**] or [**13**])), therefore we have the following proposition.

Proposition 6.3. *If L is a locally compact space, $\hat{L}$ is the pseudoarc and $n \in \mathbb{N}$, then none of the spaces $\mathcal{C}_0(L, (\mathbb{R}^n, \|\cdot\|_\infty))$ and $\mathcal{C}_0(L, (\mathbb{C}^{n+1}, \|\cdot\|_\infty))$ is almost transitive.*

Proof. The case of $\mathcal{C}_0(L, \mathbb{R})$ is a consequence of Wood's conjecture in the real case. Alternatively, we can deduce it from Theorem 5.10, because $\hat{L}$ is a metrizable continuum and therefore $\dim L = 1$. Since L is σ-compact, this contradicts the mentioned theorem together with the observation that follows it.

For the remaining cases, apply connectedness of L plus Proposition 4.1 plus Theorem 4.7. □

Nevertheless, we have the following question.

Question 6.4. If L is an L_1-space, is $\mathcal{C}_0(L, (\mathbb{C}^2, \|\cdot\|_\infty))$ almost transitive?

The authors' opinions on this question differ: A.A. believes the answer is 'no' while F.R. believes it is 'yes'. Note that a locally compact space L with $\mathcal{C}_0(L, (\mathbb{C}^2, \|\cdot\|_\infty))$ almost transitive is enough to answer Questions 4.9 and 4.11 in the negative.

Finally, a question that has not been completely solved and which was also treated in [**8**].

Question 6.5. If K is a compact space with more than one point, does there exist a Banach space X such that $\mathcal{C}(K, X)$ is almost transitive?

As observed before, in such a case, X cannot be M-finite.

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