Two-graded absolute valued algebras ☆

Antonio J. Calderón Martín [a,*], Cándido Martín González [b]

[a] *Departamento de Matemáticas, Universidad de Cádiz, 11510 Puerto Real, Cádiz, Spain*
[b] *Departamento de Algebra, Geometría y Topología, Universidad de Málaga,*
Apartado 59, 29080 Málaga, Spain

Received 24 May 2004

Available online 10 August 2005

Communicated by Georgia Benkart

Abstract

We study two-graded absolute valued algebras. These are two-graded algebras satisfying the absolute value multiplicative property only on homogeneous elements. Thus, hexagonions (also called sedenions) and other sixteen-dimensional algebras arise as examples of these algebras. The even parts of two-graded absolute valued algebras are the absolute valued algebras, while the odd parts are exactly the absolute valued triple systems. So, in a way these two-graded algebras give a unifying viewpoint of both structures. We also study the simplicity and give several ways to construct two-graded absolute valued algebras. We also provide a description of isomorphism classes for two-graded absolute valued algebras of dimensions 1, 2 and 4.

1. Introduction and preliminaries

1.1. Let $\mathbb{K}$ denote the field of real or complex numbers. An absolute valued algebra over $\mathbb{K}$ is a non-zero algebra A (not necessarily associative or unital), over $\mathbb{K}$ provided

* Corresponding author.
E-mail addresses: ajesus.calderon@uca.es (A.J. Calderón Martín), candido@apncs.cie.uma.es (C. Martín González).

with a norm $|\cdot|$ which endows the underlying vector space A with a normed structure, and such that satisfies the absolute value multiplicative property in the sense that $|xy| = |x||y|$ for all $x, y \in A$. Some examples of absolute valued algebras are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (the algebra of Hamilton quaternions), and $\mathbb{O}$ (the algebra of Cayley numbers), with norms equal to their usual absolute values. Since the early paper of A. Albert [4] where it is proved that the only finite-dimensional absolute valued algebra is $\mathbb{C}$ in the complex case and $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ in the real one, absolute valued algebras have been intensively studied by many authors. The work [30], by Angel Rodríguez Palacios, is an excellent survey of the actual state of the art. The following references are also fundamental for the reader: [4,5,15,19,21–23,28,29]. In some cases, the results arising in the literature give conditions on an absolute valued algebra assuring that such an algebra is finite-dimensional. All such results rely more or less deeply on the famous Urbanik–Wright Theorem [33] asserting that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ are the unique absolute valued real algebras with a unit.

1.2. Clearly, any finite-dimensional absolute valued algebra is a division algebra, conversely, absolute valued division algebras are finite-dimensional [34]. It is easy to see that if two norms on a finite-dimensional algebra convert it into an absolute valued algebra, then they must coincide (see, for instance, [15]). From here, it is also clear that any isomorphism between two finite-dimensional absolute valued algebras is isometric. A precise determination of isomorphism classes for absolute valued real algebras of dimensions 1 and 2 is given in [29], where the number of classes reduces to 1 and 4, respectively, while a detailed determination for the four-dimensional ones appears in [28].

1.3. A *two-graded algebra* A is a $\mathbb{K}$-algebra which splits into the direct sum $A = A_0 \oplus A_1$ of $\mathbb{K}$-submodules (called the even and the odd part respectively) satisfying $A_\alpha A_\beta \subset A_{\alpha+\beta}$ for all α, β in $\mathbb{Z}_2$. The notions of homomorphism, subalgebra and ideal in the graded sense will be used with its usual meaning. However:

Definition 1.1. A two-graded absolute valued algebra (two-graded a.v. algebra), is a non-zero two-graded algebra $A = A_0 \oplus A_1$ over $\mathbb{K}$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, endowed with two norms $|\cdot| : A_i \to \mathbb{K}$, $i = 0, 1$, such that $|x_i x_j| = |x_i||x_j|$, for any $x_i, x_j \in A_0 \cup A_1$.

Let us note that the absolute value condition on the product only holds for the homogeneous elements in A. Clearly, if we fix $x_i \in A_i$, $i = 0, 1$, with $|x_i| = 1$, then the restrictions to the homogeneous parts A_j of the *left and right product operators* $L(x_i)$, $R(x_i) : A \to A$, defined by $L(x_i)(y) := x_i y$ and $R(x_i)(y) = y x_i$, are isometric in the finite-dimensional case.

Two-graded a.v. algebras are a particular type of superalgebras. Of course Lie superalgebras are not absolute valued. A classification of associative two-graded a.v. algebras is not difficult to improvise (it could be extracted from standard results on the classification of prime associative superalgebras with non-zero socle). On the other hand, since alternative prime superalgebras in characteristic different from two are associative or their odd parts are zero, the study of the alternative superalgebras which are two-graded a.v. algebras reduces to the associative case. Jordan superalgebras are different. A description of those Jordan non-commutative superalgebras which admit a two-graded absolute valued algebra

structure would be interesting. Such algebras are necessarily finite-dimensional since their even parts are absolute valued Jordan algebras, hence finite-dimensional. Relationships with other classes of superalgebras can not be discarded and probably they will deserve consideration in future works.

From now on, *all the two-graded a.v. algebras considered in this work, will be real and finite-dimensional*, unless otherwise stated.

1.4. A notion related to that of two-graded a.v. algebras, is the concept of an absolute valued triple system (a.v. triple system in the sequel). Let T be a vector space over $\mathbb{K}$. We shall say that T is a *triple system* if it is endowed with a trilinear map

$$\langle\,\rangle : T \times T \times T \to T,$$

called the *triple product* of T. Let T, T' be triple systems, a bijective linear map $f : T \to T'$ is called an *isomorphism* of triple systems if it satisfies

$$f\big(\langle xyz\rangle\big) = \big\langle f(x)f(y)f(z)\big\rangle$$

for any $x, y, z \in T$. Triple systems appear in the literature as the natural ternary extension of algebras and have been studied in the associative [12,31,32], non-associative [8,9,14,17, 18] and general context [13]. An absolute valued triple system is defined as follows.

Definition 1.2. An absolute valued triple system (a.v. triple system), is a non-zero triple system T over $\mathbb{K}$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, endowed with a norm $|\cdot|$ satisfying $|\langle xyz\rangle| = |x||y||z|$ for any $x, y, z \in T$.

Trivially, the odd part of any two-graded a.v. algebra can be endowed with an a.v. triple system structure in many different ways. We shall investigate this relation in Section 4.

The study of two-graded a.v. algebras arises naturally from that of a.v. triple systems. It supposes not only a way of extending the theory of absolute valued algebras to other algebraic structures but also the research of other objects related to them. Thus, the Lie group aut(A) (and its Lie algebra der(A)), when A is a two-graded a.v. algebra or an a.v. triple system, is an object of possible independent interest.

2. Construction of two-graded a.v. algebras

2.1. If A is a two-graded a.v. algebra, its even part is an a.v. algebra. But does the converse hold? In other words: Given any a.v. algebra, is there a two-graded a.v. algebra with non-zero odd part whose even part is the original given a.v. algebra? The answer is: For any a.v. algebra A we can consider the two-graded a.v. algebra $A \times A$ with product

$$(x, y)(z, t) = (xz + yt, xt + yz), \quad x, y, z, t \in A,$$

even part $A \times 0 \cong A$ and odd part $0 \times A$. This construction allows us to consider two-graded a.v. algebras of double dimension than the dimension of any a.v. algebra. Therefore,

Albert's result in 1.1 gives us that there exist two-graded a.v. algebras of dimensions 1 and 2 in the complex case, and 1, 2, 4, 8 and 16 in the real one. Reciprocally, let A be any finite-dimensional two-graded a.v. algebra $A = A_0 \oplus A_1$. If $A_1 = 0$ then Albert's results in [4] implies $\dim(A) = \dim(A_0) \in \{1, 2, 4, 8\}$. If on the contrary we have $A_1 \neq 0$, let us fix $v \in A_1$, $v \neq 0$. Then the left product operators $L(v) : A_0 \to A_1$ and $L(v) : A_1 \to A_0$ are linear monomorphisms and so $\dim(A_0) = \dim(A_1)$, now Albert's result completes the proof that A has dimension 1, 2, 4, 8 or 16. Moreover, in the complex case $\dim(A) \in \{1, 2\}$.

2.2. The construction in 2.1 suggests a slightly more general method for building two-graded a.v. algebras.

Let A be a normed space and $\alpha : A \times A \to A$ a bilinear map satisfying $|\alpha(x, y)| = |x||y|$ for all $x, y \in A$. Of course A is an a.v. algebra relative to the product α (this will be denoted by (A, α)). Suppose now that we have absolute valued products $\alpha, \beta, \gamma, \delta$ on the underlying normed space of A and define on $A \times A$ the product

$$(x, y)(z, t) = \big(\alpha(x, z) + \beta(y, t), \gamma(x, t) + \delta(y, z)\big), \quad x, y, z, t \in A.$$

Then $A \times A$ is a two-graded a.v. algebra with even part $A \times 0$ (isomorphic to A) and odd part $0 \times A$. Thus for every a.v. algebra A there is (possibly) an infinity of two-graded a.v. algebras B such that $B_0 = A$ (take $B = A \times A$ and any collection of absolute valued products $\{\alpha, \beta, \gamma, \delta\}$, for instance, $\alpha = \beta = \gamma = \delta$ agreeing with the product of A).

Let us denote by $A_{\alpha, \beta, \gamma, \delta}$ the two-graded a.v. algebra described above. Next we prove that any two-graded a.v. algebra with non-zero odd part, is of the form $A_{\alpha, \beta, \gamma, \delta}$. Let C be a division composition real algebra (so that $C = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$). Let $\alpha \in SO(E)$ be a rotation of the underlying Euclidean space E of C. Then there exist $\beta, \gamma \in SO(E)$ such that

$$\alpha(xy) = \beta(x)\gamma(y) \quad \text{for all } x, y \in C. \tag{1}$$

This triality property is easy to establish for $C = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. For octonions, it is a direct application of triality (see, for instance, [27, Proposition 4, p. 227, and Proposition 2, p. 275]). By [4, Section 3, p. 497], we know that any a.v. algebra A is isomorphic with $C = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$ with product $(x, y) \mapsto \alpha(x)\beta(y)$ for suitable linear isometries α, β of the underlying Euclidean space E of C. We can suppose that the underlying Euclidean space of A is also E. If the isomorphism is $f : A \to C$ then for any two elements $x, y \in A$ we have $xy = f^{-1}(\alpha(f(x))\beta(f(y)))$. Suppose now that we are working in the highest-dimensional case: $C = \mathbb{O}$. If f turns out to be an element in $SO(8)$, then applying triality we have $xy = \beta(x)\gamma(y)$ for certain $\beta, \gamma \in O(8)$. If f is not a rotation then $f = g \circ -$ where $x \mapsto \bar{x}$ is the Cayley involution. In this case $xy = g^{-1}(\alpha(f(x))\beta(f(y)))$, and applying triality to g we get the product of A to be of the form $(x, y) \mapsto \beta(y)\gamma(x)$ for certain isometries $\beta, \gamma \in O(8)$. As a corollary we have:

Theorem 2.1. *Any two-graded a.v. algebra, with non-zero odd part, B is of the form $C \times C$ where C is an a.v. algebra mounted over $\mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$, and the product in B is*

$$(x, y)(z, t) = \big(\alpha_1(x, z) + \alpha_2(y, t), \alpha_3(x, t) + \alpha_4(y, z)\big).$$

The products $\alpha_i : A \times A \to A$ $(i = 1, 2, 3, 4)$ take one of the possible forms:

(1) $(x, y) \mapsto \beta_i(x)\gamma_i(y)$, *or*
(2) $(x, y) \mapsto \beta_i(y)\gamma_i(x)$,

for some linear isometries β_i and γ_i of C.

Unfortunately the general form of a two-graded a.v. algebra is too complex so as to allow a classification in the general case.

The proof of the following result is straightforward:

Theorem 2.2. *We have $A_{\alpha,\beta,\gamma,\delta} \cong A_{\alpha',\beta',\gamma',\delta'}$ if and only if there exist linear isometries $f, g : A \to A$ such that $f\alpha = \alpha'(f \times f)$, $f\beta = \beta'(g \times g)$, $g\gamma = \gamma'(f \times g)$, and $g\delta = \delta'(g \times f)$.*

Of course the meaning of $f \times f$ is the map $A \times A \to A \times A$ such that $(a, b) \mapsto (f(a), f(b))$. The isometric character of the linear map $f, g : A \to A$ is that $|f(a)| = |g(a)| = |a|$ for all $a \in A$. Observe that the condition $f\alpha = \alpha'(f \times f)$ is just the assertion that f is an isomorphism of a.v. algebras $(A, \alpha) \to (A, \alpha')$.

The previous theorem can be summarized in the formula

$$A_{\alpha,\beta,\gamma,\delta} \cong A_{f\alpha(f \times f)^{-1}, f\beta(g \times g)^{-1}, g\gamma(f \times g)^{-1}, g\delta(g \times f)^{-1}},$$

which has as a particular case

$$A_{\alpha,\beta,\gamma,\delta} \cong A_{\alpha, \beta(g \times g)^{-1}, g\gamma(1 \times g)^{-1}, g\delta(g \times 1)^{-1}}, \tag{2}$$

taking $f = 1_A$ (then $\alpha = \alpha'$). To study degrees of freedom we have for choosing β (once α has been fixed), we must study the action of the group of isometries $O(E)$ of the underlying normed space E of A, on a certain set. Denoting by $O(E)$ this group, that is,

$$O(E) = \{ h \in \mathrm{End}_{\mathbb{K}}(E) \colon |h(x)| = |x|, \ \forall x \in E \},$$

we can define M as the set of bilinear maps $\rho : E \times E \to E$ such that $|\rho(x, y)| = |x||y|$ for all $x, y \in E$. That, is M is the set of all a.v. products. There is a natural action $O(E) \times M \to M$ given by $h \cdot \rho := \rho(h \times h)^{-1}$, $h \in O(E)$, $\rho \in M$. So Eq. (2) says that $A_{\alpha,\beta,\gamma,\delta} \cong A_{\alpha,\beta',\gamma',\delta'}$ when β' is in the orbit of β under $O(E)$, and for suitable γ' and δ'.

2.3.   As a particular case, which will be useful in last section, take the real a.v. two-graded algebra $A = \mathbb{C}$, so that $E = \mathbb{R}^2$ with the Euclidean inner product, define $\alpha(x, y) := \sigma_n(x)\sigma_m(x)$, and $\beta(x, y) := v_1\sigma_i(x)\sigma_j(y)$ where $n, m, i, j \in \{-1, 1\}$, σ_1 denotes the identity map, σ_{-1} the complex conjugation map and $|v_1| = 1$. Then it is easy to see that β is in the same orbit as $\beta'(x, y) := v_1'\sigma_{-i}(x)\sigma_{-j}(y)$ for convenient v_1'. Indeed, define $h : \mathbb{C} \to \mathbb{C}$ to be the complex-conjugation, then for any $a, b \in A$ we have:

$$h \cdot \beta(a, b) = \beta(h \times h)^{-1}(a, b) = \beta(\bar{a}, \bar{b}) = v_1 \sigma_i(\bar{a}) \sigma_j(\bar{b}) = v_1 \sigma_{-i}(a) \sigma_{-j}(b) = \beta'(a, b),$$

so that $h \cdot \beta = \beta'$ taking $v_1' = v_1$.

2.4. Once we have defined two-graded a.v. algebras we must exhibit examples in order to have an idea of the amplitude the definition has. We have mentioned before the fact that the even part of a two-graded a.v. algebra is an a.v. algebra, while its odd part is an a.v. triple system. In a way, two-graded a.v. algebras contain a.v. algebras and also a.v. triple systems. But if we want to give concrete examples of two-graded a.v. algebras, we must do it at three or four different levels of complexity.

Trivial two-graded a.v. algebras

In the first level we have the *trivial two-graded a.v. algebras*. These are the two-graded a.v. algebras with zero odd part (that is, they are simply a.v. algebras). These are not interesting for us since we are interested in real gradings not just the trivial ones. Of course the finite-dimensional two-graded a.v. algebras in this item, have dimensions 1, 2, 4 and 8; the simplest example being the base field $\mathbb{R}$. Some more examples of algebras in this case are $\mathbb{C}, \overset{*}{\mathbb{C}}, \mathbb{H}, \overset{*}{\mathbb{H}}, \mathbb{O}, \overset{*}{\mathbb{O}}$ and $\mathbb{P}$. To recall the definition of these algebras, we must mention that for any real composition algebra C with Cayley involution $x \mapsto \bar{x}$, the algebra $\overset{*}{C}$ is the one whose underlying normed space agrees with that of C, its product being the one given by $x \cdot y = \bar{x}\bar{y}$ (juxtaposition denoting products in C). The algebra $\mathbb{P}$ of pseudo-octonions was introduced by S. Okubo in [26]. This is nothing more than the subspace of $\mathcal{M}_3(\mathbb{C})$ formed by the zero trace matrices fixed by the involution $m \mapsto \bar{m}^t$ (conjugating and transposing). The product in this space of matrices is given by $x \cdot y := \mu xy + (1 - \mu)yx - \frac{1}{3}\mathrm{tr}(xy)1$, where μ is any of the root of $3\mu(1 - \mu) = 1$. This algebra $\mathbb{P}$ is absolute valued for the absolute value coming from the inner product $(x|y) := \frac{1}{6}\mathrm{tr}(xy)$.

Non-simple two-graded a.v. algebras

In the second level of complexity we find the two-graded a.v. algebras constructed as in 2.1. As we prove later, these are the only two-graded a.v. algebras which are not simple (in the ungraded sense). Furthermore, the algebras in this level of complexity are not prime, and zero divisors live comfortably in them. These two-graded a.v. algebras suggest another way in which the class of a.v. algebras can be embedded in that of two-graded a.v. algebras. The classification of these non-simple two-graded a.v. algebras is equivalent to the classification of a.v. algebras. The finite-dimensional two-graded a.v. algebras in this item, have dimensions 2, 4, 8 or 16.

Cayley–Dickson process

In the third level of complexity we find those two-graded a.v. algebras constructed from an a.v. algebra with involution by the Cayley–Dickson process. So, if $(A, -)$ is an a.v. algebra with involution $a \mapsto \bar{a}$, then we shall denote by $\mathrm{CD}(A, -, \mu)$, $\mu \in \pm 1$, the algebra whose underlying vector space is $A \times A$ with the product

$$(x, y)(x', y') := (xx' + \mu y'\bar{y}, \bar{x}y' + x'y),$$

being $A \times 0$ the even part and $0 \times A$ the odd part. The algebras in this level have dimensions 2, 4, 8 or 16 again. Among them, we find the real division algebras $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ and the hexagonions (or sedenions) $\mathbb{X}$ constructed by applying the Cayley–Dickson process to $\mathbb{O}$ with the scalar -1. We also find the algebras of real split quaternions, the algebra of real split octonions and that of real split hexagonions $\mathbb{X}_s$. To investigate the simplicity of these algebras we shall need the fact that the only a.v. commutative algebras, are up to isomorphism: $\mathbb{R}$, $\mathbb{C}$ or $\overset{*}{\mathbb{C}}$. This fact, which is known as the commutative Urbanik–Wright Theorem, is not difficult to prove directly in the finite-dimensional case, but it can be seen also as a corollary of the well-known classification of flexible a.v. algebras [21]: The flexible a.v. algebras are $\mathbb{R}$, $\mathbb{C}$, $\overset{*}{\mathbb{C}}$, $\mathbb{H}$, $\overset{*}{\mathbb{H}}$, $\mathbb{O}$, $\overset{*}{\mathbb{O}}$, and $\mathbb{P}$.

Suppose now that B is a two-graded a.v. algebra, $B = \mathrm{CD}(A, -, \mu)$ where $(A, -)$ is an a.v. algebra with involution and $\mu = \pm 1$. Let $I \lhd B$ be a proper non-zero ideal of B. If $I \cap B_0 \neq 0$, as B is a division algebra in the graded sense, we immediately conclude $B \subset I$ contradicting the fact that I is proper. If $I \cap B_1 \neq 0$ we obtain $I = B$ as before. Thus we have $I \cap B_0 = I \cap B_1 = 0$. As a consequence, for fixed $i, j \in \{0, 1\}$, $i \neq j$, we have that for any $x \in B_i$ there is a unique $y \in B_j$ such that $x + y \in I$. If we denote by $\pi : A \to A$ the map such that $(x, \pi(x)) \in I$ for each $x \in A$, then $I = \{(x, \pi(x)) : x \in A\}$. Since I is an ideal we have $(x, \pi(x))(a, b) \in I$, for all $a, b, x \in A$. But

$$(x, \pi(x))(a, b) = \left(xa + \mu b\overline{\pi(x)}, \bar{x}b + a\pi(x)\right) \in I.$$

Taking $b = 0$ we get $(xa, a\pi(x)) \in I$ implying $\pi(xa) = a\pi(x)$. Analogously $(a, b)(x, \pi(x)) \in I$ implying $(ax + \mu\pi(x)\bar{b}, \bar{a}\pi(x) + xb) \in I$. For $b = 0$ we get $\pi(ax) = \bar{a}\pi(x)$. Thus we have proved $\pi(xa) = a\pi(x)$ and $\pi(ax) = \bar{a}\pi(x)$ for all $a, x \in A$. Taking $x = 1$ in the previous equalities we get $\pi(a) = a\pi(1) = \bar{a}\pi(1)$ for any $a \in A$. This implies $\pi(1) = 0$ or $\bar{a} = a$ for each $a \in A$. The first possibility would imply $\pi = 0$ and then $I = A \times 0 = B_0$ contradicting the fact $I \cap B_0 = 0$. Hence necessarily $\bar{a} = a$ for any $a \in A$. In this case A is commutative and applying the above observation, we have $A \cong \mathbb{R}$, $\mathbb{C}$ or $\overset{*}{\mathbb{C}}$. Thus we only have to worry about the non-simple algebras obtained as $\mathrm{CD}(C, -, \pm 1)$ for $C = \mathbb{R}, \mathbb{C}$ or $\overset{*}{\mathbb{C}}$. We shall see (paragraph above Theorem 3.1), that these non-simple algebras are isomorphic to $C \times C$ with componentwise operations, even part the diagonal elements $\Delta = \{(x, x) : x \in C\}$ and odd part the antidiagonal elements $\Delta' = \{(x, -x) : x \in C\}$. However, we are interested in the exploration of the different algebras $\mathrm{CD}(C, -, \pm 1)$. So we analyze the different cases.

(1) For $C = \mathbb{R}$ the only involution is the identity, so the only algebras arising from $\mathrm{CD}(\mathbb{R}, 1, \pm 1)$ are $\mathbb{C} = \mathrm{CD}(\mathbb{R}, 1, -1)$ and $\mathbb{C}_s = \mathrm{CD}(\mathbb{R}, 1, 1)$ (the algebra with base $\{1, i\}$ such that $i^2 = 1$, isomorphic to $\mathbb{R} \times \mathbb{R}$ with componentwise operations and exchange involution). The only non-simple one in this case is of course $\mathbb{C}_s$.

(2) For $C = \mathbb{C}$ with the identity involution we have $\mathrm{CD}(\mathbb{C}, 1, 1) \cong \mathrm{CD}(\mathbb{C}, 1, -1) = \mathbb{C}_{\mathbb{C}} \cong \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ ($\mathbb{C}_{\mathbb{C}}$ is the complex algebra $\mathbb{C} \times \mathbb{C}$ with componentwise operations). In our case, we restrict scalars so as to consider $\mathbb{C}_{\mathbb{C}}$ as a real algebra. This is non-simple indeed.

(3) Consider now $C = \mathbb{C}$ with the complex conjugation as involution. The algebras arising from $\mathrm{CD}(\mathbb{C}, -, \pm 1)$ are $\mathbb{H}_s = \mathrm{CD}(\mathbb{C}, -, 1) \cong \mathcal{M}_2(\mathbb{R})$ (split quaternions), and $\mathbb{H} =$

$CD(\mathbb{C}, -, -1)$ division quaternions. Both are simple so this case does not yield non-simple examples.

(4) Consider now the case $C = \overset{*}{\mathbb{C}}$. How many involutions can we find in $\overset{*}{\mathbb{C}}$? Any involution in $\mathbb{C}$ is also an involution in $\overset{*}{\mathbb{C}}$, but there are some more involutions in this algebra. It is not difficult to check that there are six automorphisms (hence antiautomorphisms given the commutativity of the algebra). Precisely

$$\mathrm{aut}(\overset{*}{\mathbb{C}}) = \big\{1, \omega 1, \omega^2 1, -, \omega-, \omega^2-\big\},$$

where $\omega = \exp 2\pi i/3$ is a primitive cubic root of 1, and for $j = 0, 1, 2$, by $\omega^j 1$ we denote the map $x \mapsto \omega^j x$, while $\omega^j -$ is the map $x \mapsto \omega^j \bar{x}$. It is easy to prove that there are only four involutions in $\overset{*}{\mathbb{C}}$: $1, -, \omega-$ and ω^2-. We have to consider then the eight algebras $CD(\overset{*}{\mathbb{C}}, \sigma, \pm 1)$ with $\sigma \in \{1, -, \omega-, \omega^2-\}$. In order to rule out possible isomorphisms among them, we have computed the set of idempotents in the algebras, as well as the sets of tripotents and antitripotents of the triple systems $\langle xyz \rangle := (xy)z$, in each case. We recall that an element x of a triple system T is called a *tripotent* (respectively *antitripotent*), if $\langle xxx \rangle = x$ (respectively $\langle xxx \rangle = -x$). A complete description of this study is given in Fig. 1. The algebra $CD(\overset{*}{\mathbb{C}}, \omega-, \epsilon)$ is isomorphic (in graded sense) to $CD(\overset{*}{\mathbb{C}}, \omega^2-, \epsilon)$ for $\epsilon = \pm 1$, the isomorphism is $(x, y) \mapsto (\omega^2 x, \omega^2 y)$.

(a) The algebra $CD(\overset{*}{\mathbb{C}}, 1, 1)$ is isomorphic to $\overset{*}{\mathbb{C}} \times \overset{*}{\mathbb{C}}$ with componentwise operations, even part the diagonal Δ and odd part the antidiagonal Δ'. This is of course non-simple but the direct sum of the ideals $\overset{*}{\mathbb{C}} \times 0$ and $0 \times \overset{*}{\mathbb{C}}$. Since $\overset{*}{\mathbb{C}}$ has four idempotents, this algebra has sixteen. Its odd part is the a.v. triple system $\mathbb{C}$ with the triple product $\langle abc \rangle = (ab)c = ab\bar{c}$. The tripotents of this a.v. triple systems are 0 and the whole unit sphere S^1 of $\mathbb{C}$. The only antitripotent is zero.

(b) The algebra $CD(\overset{*}{\mathbb{C}}, 1, -1)$, which is simple, has only four idempotents and so this algebra is not isomorphic to the previous one.

(c) Let us consider now the algebra $CD(\overset{*}{\mathbb{C}}, -, \epsilon)$ with $\epsilon = \pm 1$. The odd part is the a.v. triple system $\mathbb{C}$ with triple product $\langle abc \rangle = \epsilon a\bar{b}\bar{c}$. This has three tripotents hence it is not isomorphic to any of the previous cases. It can be seen that this is a simple two-graded a.v. algebra. The algebras $CD(\overset{*}{\mathbb{C}}, -, 1)$ and $CD(\overset{*}{\mathbb{C}}, -, -1)$ have isomorphic even parts and also the a.v. triple systems extracted from their odd parts are isomorphic. However they are not isomorphic since the sets of idempotents are very different from a topological viewpoint. If fact in both cases the set of idempotents is infinite. But in $CD(\overset{*}{\mathbb{C}}, -, 1)$ the set is not bounded, in the sense that it is not contained in a closed ball of the underlying Euclidean space, while in $CD(\overset{*}{\mathbb{C}}, -, -1)$ it is. This implies the non-isomorphic character of these algebras since an isomorphism would induce a homeomorphism between both sets of idempotents.

(d) The algebras $CD(\overset{*}{\mathbb{C}}, \omega-, \pm 1)$ are non-isomorphic to the previous ones. One can prove that there is no graded isomorphism between them.

Summarizing, the different algebras obtained from $\overset{*}{\mathbb{C}}$ by applying the Cayley–Dickson process are the given in Fig. 1.

Algebra	Idempotents	Tripotents	Antitripotents
$\mathrm{CD}(\overset{*}{\mathbb{C}}, 1, 1)$	16	$S^1 \cup \{0\}$	1
$\mathrm{CD}(\overset{*}{\mathbb{C}}, 1, -1)$	4	1	$S^1 \cup \{0\}$
$\mathrm{CD}(\overset{*}{\mathbb{C}}, -, \pm 1)$	∞	3	3
$\mathrm{CD}(\overset{*}{\mathbb{C}}, \omega-, \pm 1)$	4	1	1

Fig. 1. Algebras obtained from $\overset{*}{\mathbb{C}}$.

A sixteen-dimensional two-graded a.v. algebra

One of the relevant facts about two-graded a.v. algebras is that one can exhibit sixteen-dimensional examples. Any algebra obtained from an eight-dimensional absolute valued algebra by the Cayley–Dickson process is a sixteen-dimensional two-graded a.v. algebra. Consider, for instance, the a.v. algebra $(\mathbb{O}, -)$ where $x \mapsto \bar{x}$ is the Cayley involution. Then $\mathbb{X} = \mathrm{CD}(\mathbb{O}, -, -1)$ is the so called hexagonions which has been considered previously by many authors (see, for instance, [1,2,6,16,20,24]).

If we take a non-zero element $x = (a, b) \in \mathbb{X}$ then the standard involution in $\mathbb{X}$ maps x to $\bar{x} := (\bar{a}, -b)$. Therefore

$$x\bar{x} = (a, b)(\bar{a}, -b) = (a\bar{a} + b\bar{b}, -\bar{a}b + \bar{a}b) = (a\bar{a} + b\bar{b}, 0).$$

Since the octonionic norm $z \mapsto z\bar{z}$ is positive definite, the quadratic map $\mathbb{X} \to \mathbb{R}1$ given by $x \mapsto |x| := x\bar{x}$ is also positive definite hence for a non-zero $x \in X$ we have $x\bar{x} = \bar{x}x = |x|$ and any non-zero element has an inverse. Though this algebra has multiplicative inverses, it is not a division algebra, since an explicit computation proves that the hexagonions have zero divisors. In fact, the zero divisors of norm one in the hexagonions form a subspace that is homeomorphic to the exceptional Lie group G_2 (see [16,24]).

There is also a way to introduce hexagonions by giving a suitable basis and its multiplication table (see also [1]). Consider the real algebra with a basis $\{e_0, \ldots, e_{15}\}$ in which e_0 is the unit of the algebra and for $i, j > 0$ we have the multiplication rules

$$e_i e_j = -\delta_{ij} e_0 + \sum_{k=1}^{15} \epsilon_{ijk} e_k, \tag{3}$$

where δ_{ij} stands for the Kronecker delta, and ϵ_{ijk} is the totally antisymmetric tensor such that $\epsilon_{ijk} = 1$ for (i, j, k) being one of the following triplets:

$$(1, 2, 3), \quad (1, 4, 5), \quad (2, 4, 6), \quad (3, 4, 7), \quad (2, 5, 7), \quad (1, 7, 6), \quad (3, 6, 5),$$
$$(1, 8, 9), \quad (2, 8, 10), \quad (3, 8, 11), \quad (4, 8, 12), \quad (5, 8, 13), \quad (6, 8, 14), \quad (7, 8, 15),$$
$$(1, 11, 10), \quad (1, 13, 12), \quad (1, 14, 15), \quad (2, 9, 11), \quad (2, 14, 12), \quad (2, 15, 13), \quad (3, 10, 9),$$
$$(4, 9, 13), \quad (4, 10, 14), \quad (4, 11, 15), \quad (3, 15, 12), \quad (3, 13, 14), \quad (5, 12, 9) \quad (5, 10, 15),$$
$$(5, 14, 11), \quad (6, 15, 9), \quad (6, 12, 10), \quad (6, 11, 13), \quad (7, 9, 14), \quad (7, 13, 10), \quad (7, 12, 11),$$

and $\epsilon_{ijk} = 0$ in the remaining cases. It is not difficult to prove that this algebra is isomorphic to $\mathbb{X}$. In order to see this, we can consider the basis of $\mathbb{H} \subset \mathbb{O} \subset \mathbb{X}$ given by $\{e_0, \ldots, e_3\}$

such that e_0 is the unit of $\mathbb{H}$ and for $i, j = 1, 2, 3$ we have $e_i e_j = -\delta_{ij} e_0 + \sum_{k=1}^{3} \epsilon_{ijk} e_k$ where δ_{ij} is as before and ϵ_{ijk} is the totally antisymmetric tensor such that $\epsilon_{123} = 1$. Consider now any e_4 in $\mathbb{O} - \mathbb{H}$, orthogonal to e_i for $i = 0, \ldots, 3$, and satisfying $|e_4| = 1$. Define then $e_{4+i} := e_i e_4$ ($i = 1, \ldots, 3$). Thus we obtain a basis $\{e_0, \ldots, e_7\}$ of $\mathbb{O}$ whose multiplication relations are $e_0 e_i = e_i e_0 = e_0$ for all i, and for $i, j > 0$, $e_i e_j = -\delta_{ij} e_0 + \sum_{k=1}^{7} \epsilon_{ijk} e_k$ where δ_{ij} is as before the Kronecker delta, and ϵ_{ijk} is the totally antisymmetric tensor such that $\epsilon_{ijk} = 1$ for (i, j, k) equal to any of the following triplets

$$(1, 2, 3), \ (1, 4, 5), \ (2, 4, 6), \ (3, 4, 7), \ (2, 5, 7), \ (1, 7, 6), \ (3, 6, 5),$$

and $\epsilon_{ijk} = 0$ in the remaining cases. Finally, let us take an element $e_8 \in \mathbb{X} - \mathbb{O}$ orthogonal to e_i for $i = 0, \ldots, 7$ and satisfying $|e_8| = 1$. Define $e_{8+i} := e_i e_8$ for $i = 1, \ldots, 7$. We obtain in this case a basis $\{e_0, \ldots, e_{15}\}$ of $\mathbb{X}$ whose multiplicative relations are given by Eq. (3).

Some other algebras are also of interest from different viewpoints. For instance, the sixteen-dimensional two-graded a.v. algebras $CD(\mathbb{O}, -, 1)$ of split hexagonions, or the algebras $CD(\overset{*}{\mathbb{O}}, \sigma, \epsilon)$, with σ an involution of $\overset{*}{\mathbb{O}}$ and $\epsilon = \pm 1$, or $CD(\mathbb{P}, \sigma, \epsilon)$. Of course the twisted versions of these algebras are also worth to consider.

3. Simplicity of two-graded a.v. algebras

Finite-dimensional a.v. algebras are always division algebras. The bad news about two-graded a.v. algebras is that they are not necessarily simple. In fact, if A is an a.v. algebra, then $B := A \times A$ with the product

$$(x, y)(z, t) = (xz + yt, xt + yz)$$

is two-graded with even part $A \times 0$ and odd part $0 \times A$. However the subspace $\Delta = \{(x, x) : x \in A\}$ is a proper non-zero ideal of B. Moreover, $\Delta' := \{(x, -x) : x \in A\}$ is also a proper non-zero ideal, $B = \Delta \oplus \Delta'$ and $\Delta \Delta' = \Delta' \Delta = 0$, so zero divisors exist in abundance.

But two-graded a.v. finite-dimensional algebras are division algebras in the graded sense: For any homogeneous element a, the left and right multiplication operators $L(a)$, $R(a)$, are invertible on the homogeneous parts. From this, it is a corollary the fact that two-graded a.v. algebras are simple in the graded sense. We pose the following question: What conditions imply that a two-graded a.v. algebra is simple in the ungraded sense?

To answer this question take a two-graded a.v. algebra A and a non-zero proper ideal $I \triangleleft A$. If $I \cap A_0 \neq 0$, then as A is a division algebra in the graded sense, we immediately conclude $A \subset I$ contradicting the fact that I is proper. If $I \cap A_1 \neq 0$ we obtain $I = A$ as before. Thus we have $I \cap A_0 = I \cap A_1 = 0$. Next we can take

$$\pi_0(I) := \{a_0 \in A_0 : \exists a_1 \in A_1; \ a_0 + a_1 \in I\}$$

which is a non-zero ideal of A_0. Then $\pi_0(I) = A_0$. Following this idea one can prove that $\pi_1(I)$ (defined similarly) is also the whole A_1. Define now the map $\theta : A_0 \to A_1$ as fol-

lows: For any $a_0 \in A_0$, one define $\theta(a_0) := a_1$ the unique element such that $a_0 + a_1 \in I$. It is easy to see now that $I = \{a_0 + \theta(a_0): a_0 \in A_0\}$. Symmetrically we can define $\theta' : A_1 \to A_0$ such that $a_1 \mapsto a_0$ (the unique even element such that $a_0 + a_1 \in I$). Finally we can define a linear map $\Omega : A \to A$ extending θ and θ' and satisfying $\Omega^2 = 1_A$. Furthermore, this map satisfies the identities

$$\Omega(x_i x_j) = \Omega(x_i) x_j = x_i \Omega(x_j) \quad \text{and} \quad x_i x_j = \Omega(x_i) \Omega(x_j)$$

for homogeneous elements x_i and x_j.[1] The ideal I agrees with the set of all $a_0 + \Omega(a_0)$ and then, defining J as the set of all $a_0 - \Omega(a_0)$ one checks that J is a proper non-zero ideal of A such that $IJ = JI = I \cap J = 0$ and $A = I \oplus J$ (this last being a consequence of:

$$a_0 = \left(\frac{a_0}{2} + \frac{\Omega(a_0)}{2} \right) + \left(\frac{a_0}{2} - \frac{\Omega(a_0)}{2} \right),$$

$$a_1 = \left(\frac{\Omega(a_1)}{2} + \frac{a_1}{2} \right) - \left(\frac{\Omega(a_1)}{2} - \frac{a_1}{2} \right),$$

for $a_i \in A_i$, $i = 0, 1$). Next we consider the two-graded a.v. algebra $A_0 \times A_0$ with the product $(x, y)(z, t) = (xz + yt, xt + yz)$, even part $A_0 \times 0$, odd part $0 \times A_0$ and absolute values $|(a_0, 0)| := |a_0|$, $|(0, a_0)| := |a_0|$. Taking into account that $A = I \oplus J$, we can define the map $\phi : A \to A_0 \times A_0$ given by $\phi(a_0 + \Omega(a_0)) := (a_0, a_0)$ and $\phi(a_0 - \Omega(a_0)) := (a_0, -a_0)$ for any $a_0 \in A_0$. It is straightforward to prove that ϕ is an isometric isomorphism of two-graded a.v. algebras. Thus we have proved:

Theorem 3.1. *Let A be a two-graded a.v. finite-dimensional algebra. Then we have only one of the following possibilities*:

(1) *A is simple as ungraded algebra.*
(2) *$A \cong B \times B$ for some a.v. algebra B, the product in A being $(x, y)(x', y') = (xx' + yy', xy' + yx')$ for $x, x', y, y' \in A$, even part $A_0 = B \times 0$, odd part $A_1 = 0 \times B$ and absolute values $|(x, 0)| := |x|$, $|(0, y)| := |y|$, $x, y \in B$.*

As a consequence of the last theorem we can conclude that the worst two-graded a.v. algebras (those which are not simple) are the best known since their study reduces to that of a.v. algebras (with no gradings).

4. Two-graded a.v. algebras and a.v. triple systems

In this section we shall show how the theory of a.v. triple systems can be related to the one of two-graded a.v. algebras. We recall that the definition of a.v. triple system is given

---

[1] We deduce from these equations that Ω is in the centroid of A. The study of the centroid is a subject deserving some attention since it could provide alternative tools in the theory of two-graded a.v. algebras.

in 1.4. The reader also can find some aspects of the theory of a.v. triple system in [10,11]. In this section, T will denote a finite-dimensional a.v. triple system over $\mathbb{K}$.

The key reference for a.v. triple systems will be [25]. In the finite-dimensional case the absolute value comes from an inner product, so the results in [25] can be applied and are fundamental for our study.

Following the philosophy of [25], if $A = A_0 \oplus A_1$ is a two-graded a.v. algebra, then its odd part A_1 is an a.v. triple system by defining $\langle xyz \rangle := (xy)z$ (or $\langle xyz \rangle := x(yz)$) for $x, y, z \in A_1$. These are called the left and right standard triples products in A_1. Any permutation of these triple products also provides us with an a.v. triple system structure to A_1. These triple products defined as permutations of the standard ones will be called in the sequel, *standard triple products*. Two a.v. triples systems T and T' are called *isotopic* (denoted $T \sim T'$) if there are linear isometries $F_i : T \to T'$ ($i = 0, 1, 2, 3$) such that $F_0(\langle xyz \rangle) = \langle F_1(x) F_2(y) F_3(z) \rangle$ for any $x, y, z \in T$. The notion of isotopy, as introduced in [25], is more general. But working over the reals or complexes, it is equivalent to the previous one as the following result (inspired in [25]) proves.

Lemma 4.1. *Let T and T' be a.v. triple systems, and $F_i : T \to T'$ norm similarities, that is, $|F_i(x)| = a_i |x|$ for all $x \in T$, and some (necessarily positive) $a_i \in \mathbb{R}$, $i = 0, 1, 2, 3$. Suppose that $a_0 = a_1 a_2 a_3$, and $F_0(\langle xyz \rangle) = \langle F_1(x) F_2(y) F_3(z) \rangle$ for any $x, y, z \in T$. Then the maps $G_i := a_i^{-1} F_i$ ($i = 0, 1, 2, 3$), are linear isometries providing an isotopy $T \sim T'$.*

We have proved in 2.2 that for any a.v. algebra A, there are (possibly) many two-graded a.v. algebras whose even part is A. Thus the even part of a two-graded a.v. algebra does not characterize the whole algebra at all. However, the even parts of two-graded a.v. algebras exhaust the class of all a.v. algebras.

Let us specify now, three ways of constructing a.v. triple systems from two-graded a.v. algebras. If $A = A_0 \oplus A_1$ is a two-graded a.v. algebra, we can construct an a.v. triple system over A_1 by defining the triple product $\langle xyz \rangle := (xy)z$. The second example is the one given by the triple product $\langle xyz \rangle := x(yz)$, while the third one is provided with the triple product $\langle xyz \rangle := (xz)y$. We say that *an a.v. triple system comes from a two-graded a.v. algebra A* if it agrees with $(A_1, \langle \rangle)$ for some of the previous triple systems.

Now we pose the following question: Is there for any a.v. triple system T, any two-graded algebra whose odd part is T (perhaps up to isotopy)? In other terms, does the class of odd parts of two-graded a.v. algebras exhaust that of a.v. triple systems?

To answer this question we shall need the following:

Proposition 4.2. *Let T be an a.v. triple system for which there is a two-graded a.v. algebra A such that $T = A_1$ with the triple product $\langle xyz \rangle := (xy)z$ (respectively $\langle xyz \rangle := x(yz)$ or $\langle xyz \rangle := (xz)y$). Then, if $T \sim T'$ for an a.v. triple system $(T', \langle \rangle')$, there is a two-graded a.v. algebra A' whose odd part is T' and $\langle xyz \rangle' := (xy)z$ (respectively $\langle xyz \rangle' := x(yz)$ or $\langle xyz \rangle' := (xz)y$) for any $x, y, z \in T'$.*

Proof. Let consider the two-graded a.v. algebra A. As $\dim(A_0) = \dim(A_1)$ (see 2.1), we can suppose that the two-graded algebra whose odd part is T is $A = T \times T$ for a suit-

able product such that $A_0 = T \times 0$, $A_1 = 0 \times T$ and $(0, \langle abc \rangle) = ((0, a)(0, b))(0, c)$ for $a, b, c \in T$.

As $T \sim T'$, there are linear isometries $F_i : T \to T'$ ($i = 0, 1, 2, 3$) such that $F_0(\langle xyz \rangle) = \langle F_1(x) F_2(y) F_3(z) \rangle'$ for all $x, y, z \in T$. Define now the two-graded algebra $A' = T \times T'$ with the product

$$(x, x')(y, y') = \big(xy + F_1^{-1}(x') F_2^{-1}(y'), F_0\big(x F_3^{-1}(y')\big) + F_0\big(F_3^{-1}(x')y\big)\big).$$

This is a two-graded a.v. algebra with even part $T \times 0$ and odd part $0 \times T'$. In this algebra we have $(0, x')(0, y') = (F_1^{-1}(x') F_2^{-1}(y'), 0)$ and

$$\big((0, x')(0, y')\big)(0, z') = \big(F_1^{-1}(x') F_2^{-1}(y'), 0\big)(0, z') = \big(0, F_0\big(\big(F_1^{-1}(x') F_2^{-1}(y')\big) F_3^{-1}(z')\big)\big)$$
$$= \big(0, F_0\big(\langle F_1^{-1}(x') F_2^{-1}(y') F_3^{-1}(z') \rangle\big)\big) = \big(0, \langle x'y'z' \rangle'\big).$$

The rest of the cases are similar to the previous one. $\square$

We now return to the fact that any a.v. triple system appears as the odd part of some two-graded a.v. algebra. To see this, as a consequence of Proposition 4.2, we only need to prove the result for one representative in the isotopy class of every a.v. triple system. Thanks to K. McCrimmon [25], we can exhibit such a representative.

In the complex case any finite-dimensional a.v. triple system is one-dimensional and isometrically isomorphic to $\mathbb{C}$ with the triple product $\langle xyz \rangle = xyz$, then it is immediate to check that this a.v. triple system is the odd part of the two-graded a.v. complex algebra $\mathbb{C} \times \mathbb{C}$ with product

$$(x, y)(x', y') = (xx' + yy', xy' + yx').$$

In the real case, any (finite-dimensional) a.v. triple system is isotopic to one of the following:

  (I) Dimension one: $T = \mathbb{R}$ with $\langle xyz \rangle = xyz$.
 (II) Dimension two: $T = \mathbb{C}$ with $\langle xyz \rangle = xyz$.
(III) Dimension four: $T = \mathbb{H}$ (real division quaternions) and $\langle xyz \rangle$ is one of
        (i) xyz,
        (ii) xzy,
        (iii) yxz.
 (IV) Dimension eight: $T = \mathbb{O}$ (real division octonions) and $\langle xyz \rangle$ is one of
        (i) $(xy)z$,
        (ii) $(xz)y$,
        (iii) $(yx)z$,
        (iv) $x(yz)$,
        (v) $x(zy)$,
        (vi) $y(xz)$.

So, it suffices for our purposes to prove that any of the a.v. triple systems above is the odd part of a two-graded a.v. algebra. Consider first the case in which $T = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$ with the triple product $\langle xyz \rangle = (xy)z$. This is the odd part of the two-graded a.v. algebra $T \times T$ with product

$$(x, y)(x', y') = (xx' + yy', xy' + yx'). \qquad (4)$$

As we check immediately, the identity $(0, \langle xyz \rangle) = ((0, x)(0, y))(0, z)$ holds for $x, y, z \in T$. This comprises cases (I), (II), (III)(i) and (IV)(i). If T is as in (III)(ii) or (IV)(ii) then T is the odd part of the two-graded a.v. algebra whose product is (4). In this case the triple product is related to the binary product of $T \times T$ by $(0, \langle xyz \rangle) = ((0, x)(0, z))(0, y)$. For cases (III)(iii) and (IV)(iii) we can take $T \times T$ with the product

$$(x, y)(x', y') = (xx' + y'y, xy' + yx'),$$

and $(0, \langle xyz \rangle) = ((0, x)(0, y))(0, z)$. The rest of the possibilities are similar. Summarizing all of this we can state:

Theorem 4.3. *Any a.v. triple system is the odd part of a two-graded a.v. algebra $A = A_0 \oplus A_1$ with some of the triple products*: $(xy)z, x(yz)$ *or* $(xz)y$.

5. On the classification of two-graded a.v. algebras

As we said in 1.2, there is in the literature a precise determination of isomorphism classes for (ungraded) a.v. algebras of dimensions 1, 2 and 4. These, joint with the results in Section 2, lead us to study in this section the isomorphism classes for two-graded a.v. algebras of dimensions 1, 2 and 4. From now on, given $x, y \in \mathbb{K}$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, the juxtaposition xy will mean the usual product in $\mathbb{K}$. The following lemma will be useful in our study.

Lemma 5.1. *Let $A = A_0 \oplus A_1$ be a two-graded a.v. algebra and let B_0 be an a.v. algebra such that $A_0 \cong B_0$. Then the direct sum $B := B_0 \oplus A_1$ can be endowed with a two-graded a.v. algebra structure, satisfying $A \cong B$.*

Proof. Let us denote by ϕ the isomorphism from A_0 onto B_0. Then it is easy to verify that the product $(b, x)(b', x') := (bb' + \phi(xx'), \phi^{-1}(b)x' + x\phi^{-1}(b'))$ endows $B = B_0 \oplus A_1$ with a structure of two-graded a.v. algebra, and that $\mu : A \to B$ defined by $\mu(x_0, x_1) := (\phi(x_0), x_1)$ is an isomorphism. $\square$

5.1. The results in 2.1 give the following:

Proposition 5.2. *Let $A = A_0 \oplus A_1$ be a one-dimensional two-graded a.v. algebra over $\mathbb{K}$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then $A \cong \mathbb{K} \oplus 0 \cong \mathbb{K}$ with the usual product and norm in $\mathbb{K}$.*

5.2.   In order to begin the study of two-dimensional two-graded a.v. algebras $A = A_0 \oplus A_1$, we shall fix some notation. The two-graded a.v. real algebra $\mathbb{R} \times \mathbb{R}$ with the product

$$(x, y)(u, v) = (xu + \epsilon_2 yv, \epsilon_3 xv + \epsilon_3 yu),$$

where $\epsilon_i \in \pm 1$, $i = 1, 2, 3$, and the usual Euclidean norm will be denoted by $A(\epsilon_1, \epsilon_2, \epsilon_3)$. The two-graded a.v. complex algebra $\mathbb{C} \times \mathbb{C}$ with the product

$$(x, y)(u, v) = (xu + p_1 yv, p_2 xv + p_3 yu),$$

where $p_i \in \mathbb{C}$ with $|p_i| = 1$, $i = 1, 2, 3$, and with the usual Euclidean norm, will be denote by $A(p_1, p_2, p_3)$.

If $A_1 = 0$ then $A = A_0$ is an a.v. algebra with dimension 2 and so it is well described in [29].

Then, let us suppose that $A_1 \neq 0$. As A_0 is an a.v. algebra and $\dim A_0 = \dim A_1$ (see 2.1), we have $\dim A_0 = \dim A_1 = 1$ and so it is easy to see that $A_0 \cong \mathbb{R}$ in the real case and $A_0 \cong \mathbb{C}$ in the complex one. Moreover, $A_1 \cong \mathbb{K}$ as vector spaces, so we can write $A = \mathbb{K} \times \mathbb{K}$ and, taking into account Lemma 5.1, the product in A can be expressed by

$$(x, y)(u, v) = (xu + y \circ v, x \,\square\, v + y \,\triangle\, u) \tag{5}$$

where the products $\mathbb{K} \times \mathbb{K} \to \mathbb{K}$, $(a, b) \mapsto a \circ b$, $(a, b) \mapsto a \,\square\, b$ and $(a, b) \mapsto a \,\triangle\, b$ are absolute valued.

Let us consider, for instance, $(a, b) \mapsto a \circ b$. If we write $k := 1 \circ 1$, then the $\mathbb{K}$-linear character of $\circ$ gives $x \circ y = kxy$. As $|kxy| = |x||y|$ we have $|k| = 1$. The same applies to $\square$ and $\triangle$ and so, if $\mathbb{K} = \mathbb{R}$, then $k = \pm 1$ and, taking into account (5), A is of the form $A(\epsilon_1, \epsilon_2, \epsilon_3)$ for some $\epsilon_i \in \pm 1$, $i = 1, 2, 3$. If $\mathbb{K} = \mathbb{C}$, we conclude as in the real case that $A \cong A(p_1, p_2, p_3)$ for some $|p_j| = 1$, $j = 1, 2, 3$.

Let us consider $\mathbb{K} = \mathbb{R}$ and let us study the isomorphism classes of the algebras $A(\epsilon_1, \epsilon_2, \epsilon_3)$. First, we observe that if

$$\phi : A(\epsilon_1, \epsilon_2, \epsilon_3) \to A\big(\epsilon_1', \epsilon_2', \epsilon_3'\big) \tag{6}$$

is an isomorphism, then its restriction to the homogeneous parts are $\mathbb{R}$-linear isometries on $\mathbb{R}$ with the Euclidean inner product, therefore $\phi|_{A_0} = \pm \mathrm{Id}$ and $\phi|_{A_1} = \pm \mathrm{Id}$, so we can assert that ϕ is of the form $\phi(x_0, x_1) = (\pm x_0, \pm x_1)$. Taking into account these possibilities for ϕ it is easy to check that (6) only holds if $\epsilon_i' = \epsilon_i'$ for any $i = 1, 2, 3$.

If $\mathbb{K} = \mathbb{C}$, then arguing as above we have that any isomorphism

$$\phi : A(p_1, p_2, p_3) \to A\big(p_1', p_2', p_3'\big),$$

is of the form $\phi(x_0, x_1) = (vx_0, wx_1)$ with $v, w \in \mathbb{C}$ satisfying $|v| = |w| = 1$. Then, it is not difficult to check that $\phi(x_0, x_1) := (x_0, \sqrt{p_1} x_1)$ is an isomorphism from $A(p_1, p_2, p_3)$ onto $A(1, p_2, p_3)$, and that $A(1, p_2, p_3)$ is not isomorphic to any $A(1, p_2', p_3')$ if some $p_j \neq p_j'$, $j = 2, 3$.

Summarizing we have proved the following:

Theorem 5.3. *Let $A = A_0 \oplus A_1$ be a non-trivial two-dimensional two-graded a.v. algebra over $\mathbb{K}$.*

(1) *If $\mathbb{K} = \mathbb{R}$, then $A \cong A(\epsilon_1, \epsilon_2, \epsilon_3)$ with $\epsilon_i \in \pm 1$, $i = 1, 2, 3$. Moreover, $A(\epsilon_1, \epsilon_2, \epsilon_3) \cong A(\epsilon_1', \epsilon_2', \epsilon_3')$ if and only if $\epsilon_i = \epsilon_i'$ for any $i = 1, 2, 3$.*
(2) *If $\mathbb{K} = \mathbb{C}$, then $A \cong A(1, p_1, p_2)$, with $|p_i| = 1$, $i = 1, 2$. Moreover, $A(1, p_1, p_2) \cong A(1, p_1', p_2')$ if and only if $p_i = p_i'$ for any $i = 1, 2$.*

As any complex finite-dimensional two-graded a.v. algebra has dimension 1 or 2 (see 2.1), we have completed the study of the complex case, therefore we confine ourselves to the real case.

5.3.   Let us study two-graded a.v. algebras of dimension 4. The first case to consider would be that in which A has the trivial grading, that is, A is simply a four-dimensional a.v. algebra. These have been considered in [28, Proposition 2.1, p. 170]. In this reference, it is proved that up to isomorphism, any four-dimensional a.v. (real) algebra is one the algebras $H(a, b)$, $\overset{*}{H}(a, b)$, $^*H(a, b)$ or $H^*(a, b)$ for some $a, b \in \mathbb{H} = \{x \in \mathbb{H}: |x| = 1\}$. These algebras are called *principal isotopes* of $\mathbb{H}$ and their products are, respectively: $axyb$, $a\bar{x}\bar{y}b$, $\bar{x}ayb$, and $axb\bar{y}$, where the juxtaposition is the product in $\mathbb{H}$ and $x \mapsto \bar{x}$ its Cayley involution. Moreover $H(a, b) \cong H(a', b')$ (respectively $\overset{*}{H}(a, b) \cong \overset{*}{H}(a', b')$ or $^*H(a, b) \cong {}^*H(a', b')$ or $H^*(a, b) \cong H^*(a', b')$) if and only if there is a $p \in S^3 \subset \mathbb{H}$ such that $a' = \epsilon pap^{-1}$ and $b' = \delta pbp^{-1}$, with $\epsilon, \delta \in \{-1, 1\}$. It is possible to give a further refinement of this description. If we consider the standard basis $\{1, i, j, k\}$ of $\mathbb{H}$, with $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$, then $\mathbb{H} = \mathbb{R}1 \oplus W$ where $W = 1^\perp = \text{span}(\{i, j, k\})$. Taking S^3, this is obviously a compact connected Lie group, and it is easy to prove directly that one of its maximal tori is the subgroup $\{\exp(\theta i): \theta \in \mathbb{R}\} \cong S^1$ (see also [7, (3.7) Theorem, p. 173]). It is also a well-known result that in a compact connected Lie group, any element is conjugate to some element in a pre-fixed maximal torus (see [7, (1.7) Main Lemma, p. 159], or [3]). In particular for any $a \in S^3$ there is some $q \in S^3$ such that $qaq^{-1} = \exp(\theta i)$ for some $\theta \in [0, 2\pi)$. Thus $H(a, b) \cong H(\exp(\theta i), b')$ where $b' = qbq^{-1}$. If $a = 1$ we have $H(1, b) \cong H(1, qbq^{-1})$, and for a suitable q we can write $qbq^{-1} = \exp(\phi i)$ for some $\phi \in [0, 2\pi)$, then $H(1, b) \cong H(1, \exp(\phi i))$. If $a \neq \pm 1$, we can now consider an arbitrary element $q_1 = \exp(si)$ for some $s \in \mathbb{R}$. We also have an isomorphism $H(\exp(\theta i), b') \cong H(q_1 \exp(\theta i)q_1^{-1}, q_1 b' q_1^{-1})$. But $q_1 \exp(\theta i)q_1^{-1} = \exp(\theta i)$ for any such q_1. As a consequence we still have a certain degrees of freedom to simplify b' by conjugating it with some q_1 chosen as above. So if $b' = b_0' + b_1' i + w'$ for some $w' \in \text{span}(\{j, k\})$, then

$$q_1 b' q_1^{-1} = b_0' + b_1' i + q_1 w' q_1^{-1} = b_0' + b_1' i + q_1^2 w'$$

but $q_1^2 w' = \exp(2si)w'$ can have any value in $\text{span}(\{j, k\})$ with the same norm as w'. For instance, $q_1^2 w' = |w'|j$ for a suitable q_1. Therefore $q_1 b' q_1^{-1} = b_0' + b_1' i + |w'|j$ and $b_0'^2 + b_1'^2 + |w'|^2 = 1$. Thus we may write $b_0' = \rho \cos\phi$, $b_1' = \rho \sin\phi$, and $|w'| =$

$\sqrt{1-\rho^2}$ for some $\phi, \rho \in \mathbb{R}$, $0 \leqslant \rho \leqslant 1$. In this way $q_1 b' q_1^{-1} = \rho \exp(\phi i) + \sqrt{1-\rho^2} j$ and $H(\exp(\theta i), b') \cong H(\exp(\theta i), \rho \exp(\phi i) + \sqrt{1-\rho^2} j)$. Summarizing

$$H(a, b) \cong H\left(\exp(\theta i), \rho \exp(\phi i) + \sqrt{1-\rho^2} j\right).$$

If we define $H(\theta, \phi, \rho) := H(\exp(\theta i), \rho \exp(\phi i) + \sqrt{1-\rho^2} j)$, we conclude that any four-dimensional absolute valued algebra is isomorphic to some of the algebras $H(\theta, \phi, \rho)$, $\overset{*}{H}(\theta, \phi, \rho)$, $^*H(\theta, \phi, \rho)$, or $H^*(\theta, \phi, \rho)$ where

$$W(\theta, \phi, \rho) := W\left(\exp(\theta i), \rho \exp(\phi i) + \sqrt{1-\rho^2} j\right)$$

for $W \in \{H, \overset{*}{H}, {}^*H, H^*\}$. The dependence of $W(a, b)$ (W as before) on the quaternions a and b is thus replaced by the dependence of $W(\theta, \phi, \rho)$ on the three real numbers θ, ϕ and ρ. This will also give some advantages from the viewpoint of the isomorphism conditions. Let us now bound the possible values of the three parameters θ, ϕ and ρ. We already know that $\rho \in [0, 1]$. Taking into account the relation: $j \exp(\theta i) j^{-1} = \exp(-\theta i)$, we can limit θ to be in the interval $[0, \pi]$, and since $j \exp(\theta i) j = \exp((\pi - \theta) i)$, we can take $\theta \in [0, \pi/2]$. On the other hand, the relation $-i(\rho \exp(\phi i) + \sqrt{1-\rho^2} j) i = -(\rho \exp((\phi + \pi) i) + \sqrt{1-\rho^2} j)$ implies that we can take $\phi \in [0, \pi)$. Thus, we can claim:

Theorem 5.4. *Any absolute valued four-dimensional algebra is isomorphic to one of the algebras $W(\theta, \phi, \rho)$ with $W \in \{H, \overset{*}{H}, {}^*H, H^*\}$, $\theta \in [0, \pi/2]$, $\phi \in [0, \pi)$ and $\rho \in [0, 1]$. If $\theta = 0$ then $\rho = 1$ and, with the above conditions for the parameters, $W(\theta, \phi, \rho) \cong W(\theta', \phi', \rho')$ if and only if $\theta = \theta'$, $\phi = \phi'$ and $\rho = \rho'$ except in the cases*:

(1) $\theta = 0$, $\rho = 1$. *We have $W(0, \phi, 1) \cong W(0, \phi', 1)$ if and only if $\phi' = \phi$ or $\phi' = \pi - \phi$.*
(2) $\theta = \pi/2$. *In this case $W(\pi/2, \phi, \rho) \cong W(\pi/2, \phi', \rho')$ if and only if $\rho = \rho'$, $\phi' = \phi$ or $\phi' = \pi - \phi$.*
(3) $\rho = 0$, *in which case $W(\theta, \phi, 0) \cong W(\theta, \phi', 0)$ for arbitrary ϕ and ϕ'.*

Proof. The only thing we have to prove is the isomorphism condition. Suppose $W(\theta, \phi, \rho) \cong W(\theta', \phi', \rho')$, with $\theta, \theta', \phi, \phi' \in [0, \pi/2]$ and $\rho, \rho' \in [0, 1]$. Then, there exists a $q \in \mathbb{H}$, $|q| = 1$, such that

$$\exp(\theta' i) = \epsilon q \exp(\theta i) q^{-1},$$

$$\rho' \exp(\phi' i) + \sqrt{1-\rho'^2} j = \delta q\left(\rho \exp(\phi i) + \sqrt{1-\rho^2} j\right) q^{-1},$$

with $\epsilon, \delta \in \{-1, 1\}$. Thus by writing $q = q_1 + q_2$ with $q_1 \in \text{span}(\{1, i\})$, $q_2 \in \text{span}(\{j, k\})$, we conclude

$$\exp(\theta' i) q_r = \epsilon q_r \exp(\theta i) \quad \text{for } r = 1, 2, \tag{7}$$

$$\rho' \exp(\phi' i) q_1 + \sqrt{1-\rho'^2} j q_2 = \delta q_1 \rho \exp(\phi i) + \delta q_2 \sqrt{1-\rho^2} j, \tag{8}$$

$$\rho' \exp(\phi' i) q_2 + \sqrt{1 - \rho'^2} j q_1 = \delta q_2 \rho \exp(\phi i) + \delta q_1 \sqrt{1 - \rho^2} j. \tag{9}$$

Since $q = q_1 + q_2 \neq 0$, we shall distinguish three cases:

(a) If $q_1 \neq 0$ and $q_2 \neq 0$, then as q_1 commutes with $\exp(\theta i)$, we have by Eq. (7) $\exp(\theta i) = \epsilon \exp(\theta' i)$. For $\epsilon = 1$ this implies $\theta = \theta'$. For $\epsilon = -1$, we get $\theta' = \theta + \pi$ which is impossible given the restrictions on the parameters. Then necessarily $\epsilon = 1$ and by (7) $\exp(\theta' i) q_2 = q_2 \exp(\theta i) = \exp(-\theta i) q_2$. We conclude $\theta = \theta' = 0$ and so $\rho = \rho' = 1$. Taking now into account Eqs. (8) and (9), we have $\exp(\phi' i) q_r = \delta q_r \exp(\phi i)$, $r = 1, 2$. Arguing as above we also get that necessarily $\delta = 1$ and $\phi = \phi' = 0$.

(b) If $q_1 \neq 0$ and $q_2 = 0$, we obtain as in (a), $\epsilon = 1$ and $\theta = \theta'$. By taking norms in Eq. (8) we obtain $\rho = \rho'$.

(1) If $\rho = \rho' \neq 0$, then $\exp(\phi' i) = \delta \exp(\phi i)$ and for $\delta = 1$ we conclude $\phi' = \phi$. For $\delta = -1$ we have $\phi' = \phi + \pi$ which is impossible if $\phi, \phi' \in [0, \pi)$.
(2) If $\rho = \rho' = 0$, $W(\theta, \phi, 0) \cong W(\theta, \phi', 0)$ for any ϕ, ϕ'. So we can choose $\phi = 0$ and we have exception (3) in the theorem.

(c) If $q_1 = 0$ and $q_2 \neq 0$, then $q = q_2 \in \text{span}(\{j, k\})$. From Eq. (7), $\exp(\theta' i) q_2 = \epsilon q_2 \exp(\theta i) = \epsilon \exp(-\theta i) q_2$. This implies $\exp(\theta' i) = \epsilon \exp(-\theta i)$. Hence, for $\epsilon = 1$ we have $\theta' = -\theta$, and for $\theta, \theta' \in [0, \pi/2]$ this is only possible if $\theta = \theta' = 0$ and so $\rho = \rho' = 1$. By arguing now as in (b) with Eq. (9), we obtain that $\phi = \phi' = 0$ in case $\delta = 1$, and that $\phi + \phi' = \pi$ in case $\delta = -1$. For $\epsilon = -1$ we get $\exp(\theta' i) = -\exp(-\theta i) = \exp((\pi - \theta) i)$ implying $\theta = \theta' = \pi/2$. By taking norms in Eq. (9), we obtain $\rho = \rho'$.

(1) If $\rho = \rho' \neq 0$, Eq. (3) now gives $\exp(\phi' i) = \delta \exp(-\phi i)$, and by arguing as above we get either $\phi = \phi' = 0$ if $\delta = 1$, or $\phi + \phi' = \pi$ if $\delta = -1$.
(2) If $\rho = \rho' = 0$, we have $W(\pi/2, \phi, 0) \cong W(\pi/2, \phi', 0)$ for any ϕ and ϕ'. Thus we can take $\phi' = 0$ and we obtain exception (3) in the theorem. $\quad \square$

Finally, let us consider the problem of the classification of non-trivial four-dimensional two-graded a.v. real algebras. Let us denote $S^1 := \{x \in \mathbb{R}^2 : |x| = 1\}$, $\mathcal{O}(2)$ the group of all isometries in $\mathbb{R}^2$, $\mathcal{O}^+(2) := \{f \in \mathcal{O}(2) : \det(f) = 1\}$ and $\mathcal{O}^-(2) := \{f \in \mathcal{O}(2) : \det(f) = -1\}$. In this study we are going to develop the techniques introduced in 5.2.

As in 5.2, the case $A_1 = 0$ is the one we have just described. So, we take $A_1 \neq 0$. As A_0 is an a.v. algebra and $\dim A_0 = \dim A_1$ (see 2.1), we have $\dim A_0 = \dim A_1 = 2$ and so by applying [29, Lemma 2] $A_0 \cong \mathbb{C}, *\mathbb{C}, \overset{*}{\mathbb{C}},$ or $\mathbb{C}^*$, where the products in $*\mathbb{C}, \overset{*}{\mathbb{C}},$ and $\mathbb{C}^*$ are respectively $x \cdot y := \bar{x} y$, $x \cdot y := \bar{x} \bar{y}$ and $x \cdot y := x \bar{y}$. Moreover, $A_1 \cong \mathbb{C}$ as vector spaces, so we can write $A = \mathbb{C} \times \mathbb{C}$ and, by Lemma 5.1, the product in A can be expressed by

$$(x, y)(u, v) = \big(\sigma_n(x) \sigma_m(u) + y \circ v, \, x \,\square\, v + y \,\triangle\, u\big) \tag{10}$$

where any σ_n, σ_m is either the identity map or the complex conjugation map, and where the products $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$, $(a, b) \mapsto a \circ b$, $(a, b) \mapsto a \,\square\, b$ and $(a, b) \mapsto a \,\triangle\, b$ are absolute valued. As in 2.3, σ_1 denote the identity map in $\mathbb{C}$ and σ_{-1} the complex conjugation map.

Lemma 5.5. *Any map* $\circ, \square, \triangle : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ *given in* (10) *is either $\mathbb{C}$-linear or $\mathbb{C}$-conjugate-linear in any variable.*

Proof. Let us consider, for instance, $(a, b) \mapsto a \circ b$. Fix $x \in S^1$ and write $v := L(x)(1)$, ($L(x)$ denotes the left product operator, see 1.3). Since $L(x) \in \mathcal{O}(2)$, we have two possibilities: In the first one, $L(x) \in \mathcal{O}^+(2)$, hence by applying elemental results in linear algebra, $L(x)(i) = iv$ and then $L(x)(z) = zv$ for any $z \in \mathbb{C}$. In the second possibility, $L(x) \in \mathcal{O}^-(2)$, then $L(x)(i) = -iv$ and we have $L(x)(z) = \bar{z}v$.

In the first possibility $L(x)(zz') = zz'v = zL(x)(z')$, that is, $L(x)$ is $\mathbb{C}$-linear. In the second possibility we similarly obtain $L(x)$ is conjugate-linear. This clearly extends to any $x \in \mathbb{C}$ by a connection argument. We argue in a similar way for the right product operators and so $(a, b) \mapsto a \circ b$ is either $\mathbb{C}$-linear or $\mathbb{C}$-conjugate-linear in any variable. The same applies for $\square$ and $\triangle$. $\square$

By Lemma 5.5, if we denote $1 \circ 1 = v$, ($v \in S^1$), we have $x \circ y = v\sigma_i(x)\sigma_j(y)$ where any σ_l is either the identity map or the complex conjugation map. The same applies to $\square$ and $\triangle$ and so we can write the product in A as

$$(x, y)(u, v) = \big(\sigma_n(x)\sigma_m(u) + v_1\sigma_i(y)\sigma_j(v), \, v_2\sigma_p(x)\sigma_q(v) + v_3\sigma_r(y)\sigma_s(u)\big) \quad (11)$$

where $n, m, i, j, p, q, r, s \in \pm 1$ and any $v_k \in S^1$.

For each multiindex $\underline{n} = (\sigma_n, \sigma_m, \sigma_i, \sigma_j, \sigma_p, \sigma_q, \sigma_r, \sigma_s)$, denote by $A_{\underline{n}}(v_1, v_2, v_3)$ the algebra A with the product in (11).

Let us observe that if

$$\phi : A_{\underline{n}}(v_1, v_2, v_3) \to A_{\underline{n}'}(v_1', v_2', v_3') \quad (12)$$

is an isomorphism, where $\underline{n}' = (\sigma_n', \sigma_m', \sigma_i', \sigma_j', \sigma_p', \sigma_q', \sigma_r', \sigma_s')$, then its restriction to the homogeneous parts

$$\phi|_{(A_{\underline{n}}(v_1, v_2, v_3))_0} \quad \text{and} \quad \phi|_{(A_{\underline{n}}(v_1, v_2, v_3))_1},$$

are linear isometries on $\mathbb{C}$ with the Euclidean inner product, thus either $\phi|_{A_0}(z) := kz$ with $|k| = 1$ or $\phi|_{A_0}(z) := t\bar{z}$ with $|t| = 1$. The same applies to $\phi|_{A_1}$ and we can assert the following:

Lemma 5.6. *Any isomorphism ϕ between four-dimensional two-graded a.v. algebras is of one of the following types:*

(1) $\phi(x_0, x_1) = (kx_0, tx_1)$,
(2) $\phi(x_0, x_1) = (kx_0, t\overline{x_1})$,
(3) $\phi(x_0, x_1) = (k\overline{x_0}, tx_1)$,
(4) $\phi(x_0, x_1) = (k\overline{x_0}, t\overline{x_1})$,

with $|k| = |t| = 1$.

The following easy lemma will be useful to simplify computations.

Lemma 5.7. *If $v\sigma_i(x)\sigma_j(y) = w\sigma_m(x)\sigma_n(y)$ for any $x, y \in \mathbb{C}$, where v, w are non-zero fixed elements in $\mathbb{C}$. Then $\sigma_i = \sigma_m$ and $\sigma_j = \sigma_n$.*

In the next theorem we determine the values of $\sigma_n', \sigma_m', \sigma_i', \sigma_j', \sigma_p', \sigma_q', \sigma_r', \sigma_s', v_1', v_2', v_3'$ in (12), from $\sigma_n, \sigma_m, \sigma_i, \sigma_j, \sigma_p, \sigma_q, \sigma_r, \sigma_s, v_1, v_2$ and v_3. That is, we give the isomorphism classes of four-dimensional two-graded a.v. algebras. Let us observe that in this theorem we are going to consider for i, j only the cases $i = j = 1$ and $i = 1, j = -1$, since the remaining cases ($i = j = -1$ and $i = -1, j = 1$) are reduced to these ones as consequence of 2.3.

Theorem 5.8. *Let A be a four-dimensional two-graded a.v. algebra with non-trivial odd part, then A is isomorphic to one of the type*

$$A_{\underline{n}}(v_1, v_2, v_3).$$

Moreover, the isomorphism classes in the family of these algebras are described by

$$A_{\underline{n}}(v_1, v_2, v_3) \cong A_{\underline{n}}(\alpha_1, \gamma_1, \delta_1) \cong A_{\underline{n}}(\alpha_2, \gamma_2, \delta_2) \cong A_{\underline{m}}(\alpha_3, \gamma_3, \delta_3) \cong A_{\underline{m}}(\alpha_4, \gamma_4, \delta_4),$$

where $\underline{m} = (\sigma_n, \sigma_m, \sigma_{-i}, \sigma_{-j}, \sigma_{-p}, \sigma_q, \sigma_r, \sigma_{-s})$ and

$$\alpha_1 = \begin{cases} k\bar{t}^2 v_1, & \text{if } i = j = 1, \\ k v_1, & \text{if } i = 1, \ j = -1, \end{cases} \qquad \alpha_2 = \begin{cases} k\bar{t}^2 \overline{v_1}, & \text{if } i = j = 1, \\ k \overline{v_1}, & \text{if } i = 1, \ j = -1, \end{cases}$$

$$\alpha_3 = \begin{cases} k t^2 v_1, & \text{if } i = j = 1, \\ k v_1, & \text{if } i = 1, \ j = -1, \end{cases} \qquad \alpha_4 = \begin{cases} k t^2 \overline{v_1}, & \text{if } i = j = 1, \\ k \overline{v_1}, & \text{if } i = 1, \ j = -1, \end{cases}$$

$$\gamma_1 = \begin{cases} \bar{k} v_2, & \text{if } p = q = 1, \\ t^2 \bar{k} v_2, & \text{if } p = 1, \ q = -1, \\ k v_2 & \text{if } p = -1, \ q = 1, \\ t^2 k v_2, & \text{if } p = q = -1, \end{cases} \qquad \gamma_2 = \begin{cases} \bar{k} v_2, & \text{if } p = q = 1, \\ t^2 \bar{k} v_2, & \text{if } p = 1, \ q = -1, \\ k \overline{v_2}, & \text{if } p = -1, \ q = 1, \\ t^2 k \overline{v_2}, & \text{if } p = q = -1, \end{cases}$$

$$\gamma_3 = \begin{cases} k \overline{v_2}, & \text{if } p = q = 1, \\ k t^2 \overline{v_2}, & \text{if } p = 1, \ q = -1, \\ \bar{k} v_2, & \text{if } p = -1, \ q = 1, \\ t^2 \bar{k} v_2, & \text{if } p = q = -1, \end{cases} \qquad \gamma_4 = \begin{cases} k v_2, & \text{if } p = q = 1, \\ t^2 k v_2, & \text{if } p = 1, \ q = -1, \\ \bar{k} v_2, & \text{if } p = -1, \ q = 1, \\ t^2 \bar{k} v_2, & \text{if } p = q = -1, \end{cases}$$

$$\delta_1 = \begin{cases} \bar{k} v_3, & \text{if } r = s = 1, \\ k v_3, & \text{if } r = 1, \ s = -1, \\ t^2 \bar{k} v_3, & \text{if } r = -1, \ s = 1, \\ t^2 k v_3, & \text{if } r = s = -1, \end{cases} \qquad \delta_2 = \begin{cases} \bar{k} v_3, & \text{if } r = s = 1, \\ k v_3, & \text{if } r = 1, \ s = -1, \\ t^2 \bar{k} \overline{v_3}, & \text{if } r = -1, \ s = 1, \\ t^2 k \overline{v_3}, & \text{if } r = s = -1, \end{cases}$$

$$\delta_3 = \begin{cases} k\overline{v_3}, & \text{if } r = s = 1, \\ \overline{kv_3}, & \text{if } r = 1,\ s = -1, \\ t^2 k\overline{v_3}, & \text{if } r = -1,\ s = 1, \\ t^2 \overline{kv_3}, & \text{if } r = s = -1, \end{cases} \qquad \delta_4 = \begin{cases} kv_3, & \text{if } r = s = 1, \\ \overline{k}v_3, & \text{if } r = 1,\ s = -1, \\ t^2 k v_3, & \text{if } r = -1,\ s = 1, \\ t^2 \overline{k} v_3, & \text{if } r = s = -1, \end{cases}$$

for some $t \in S^1$, where $k = 1$ if $n = m = 1$ or $n \neq m$, and $k = \sqrt[3]{1}$ if $n = m = -1$.

Proof. Let us suppose ϕ is as in the first possibility of Lemma 5.6, that is,

$$\phi(x_0, x_1) = (kx_0, tx_1) \quad \text{with } |k| = |t| = 1.$$

We have for any

$$(x, y), (u, v) \in A_{\underline{n}}(v_1, v_2, v_3)$$

that $\phi((x, y)(u, v)) = \phi(x, y)\phi(u, v)$, so:

$$\begin{aligned} &\big(k\sigma_n(x)\sigma_m(u) + kv_1\sigma_i(y)\sigma_j(v),\, tv_2\sigma_p(x)\sigma_q(v) + tv_3\sigma_r(y)\sigma_s(u)\big) \\ &= \big(\sigma'_n(k)\sigma'_m(k)\sigma'_n(x)\sigma'_m(u) + v'_1\sigma'_i(t)\sigma'_j(t)\sigma'_i(y)\sigma'_j(v), \\ &\qquad v'_2\sigma'_p(k)\sigma'_q(t)\sigma'_p(x)\sigma'_q(v) + v'_3\sigma'_r(t)\sigma'_s(k)\sigma'_r(y)\sigma'_s(u)\big). \end{aligned} \tag{13}$$

Taking $y = v = 0$ in (13), we conclude

$$\big(k\sigma_n(x)\sigma_m(u), 0\big) = \big(\sigma'_n(k)\sigma'_m(k)\sigma'_n(x)\sigma'_m(u), 0\big).$$

Lemma 5.7 gives

$$\sigma'_n = \sigma_n, \qquad \sigma'_m = \sigma_m.$$

Then, we also have $k = \sigma_n(k)\sigma_m(k)$, and we obtain the following three possibilities:

(3) If $n = m = 1$, then $k = k^2$ and so $k = 1$.
(2) If $n = m = -1$, then $k = \overline{k}^2$, and from here $k = \sqrt[3]{1}$.
(3) If $n = 1,\ m = -1$ or $n = -1,\ m = 1$, then $k = 1$.

Taking in (13), $x = u = 0$ we have

$$\big(kv_1\sigma_i(y)\sigma_j(v), 0\big) = \big(v'_1\sigma'_i(t)\sigma'_j(t)\sigma'_i(y)\sigma'_j(v), 0\big).$$

Lemma 5.7 now gives

$$\sigma'_i = \sigma_i, \qquad \sigma'_j = \sigma_j.$$

Now, we also have $kv_1 = v'_1\sigma_i(t)\sigma_j(t)$, and as above we obtain the following possibilities:

(1) If $i = j = 1$, then $kv_1 = v_1' t^2$, that is $v_1' = k\bar{t}^2 v_1$.
(2) If $i = j = -1$, then $kv_1 = v_1' \bar{t}^2$, that is $v_1' = kt^2 v_1$.
(3) If $i = 1$, $j = -1$ or $i = -1$, $j = 1$, then $v_1' = kv_1$.

Taking now in (13), $y = u = 0$ we obtain as in the previous cases the following possibilities:

(1) If $p = q = 1$, then $v_2' = \bar{k}v_2$.
(2) If $p = q = -1$, then $v_2' = kt^2 v_2$.
(3) If $p = 1$, $q = -1$, then $v_2' = \bar{k}t^2 v_2$.
(4) If $p = -1$, $q = 1$, then $v_2' = kv_2$.

And finally, by taking in (13), $x = v = 0$ we have

(1) If $r = s = 1$, then $v_3' = \bar{k}v_3$.
(2) If $r = s = -1$, then $v_3' = kt^2 v_3$.
(3) If $r = 1$, $s = -1$, then $v_3' = kv_3$.
(4) If $r = -1$, $s = 1$, then $v_3' = \bar{k}t^2 v_3$.

A similar study for the remaining possibilities for ϕ in Lemma 5.6 completes the proof. $\quad\square$

6. Some open questions

Of course the problem of the classification of two-graded a.v. algebras in dimensions greater than 4 is a problem which seems to be difficult to accomplish even for the simpler case of eight-dimensional a.v. algebras, that is, with trivial grading. However some partial results probably could be given by imposing additional restrictions on the algebras under study. Many other interesting question can be posed in relation with these algebras. For instance:

(1) We have stated the fact that any finite-dimensional a.v. triple system is the odd part of a suitable two-graded a.v. algebra. Is this also true without assuming the finite-dimensional hypothesis on the triple system?
(2) The study of the automorphisms and derivations of these algebras seems also to be a natural question.
(3) Given the diversity of algebras we find in the class of two-graded a.v. algebras, one could think about some interesting subclass of them. For instance the alternative two-graded a.v. algebras or more generally the flexible ones. A classification of these would be also an interesting result with applications to a.v. triple systems.
(4) Under which conditions does an infinite-dimensional two-graded a.v. algebra turn out to be finite-dimensional? This suggests the study of algebraic two-graded a.v. algebras.

Acknowledgment

References

[1] K. Abdel-Khalek, The ring division self duality, hep-th/9710177, 1997.

[2] K. Abdel-Khalek, Beyond octonions, math-ph/0002023, 2000.

[3] J.F. Adams, Lectures on exceptional Lie groups, in: Z. Mahmud, M. Mimura (Eds.), Chicago Lectures in Math. Ser., vol. xiv, 1996.

[4] A.A. Albert, Absolute valued real algebras, Ann. of Math. 48 (1947) 495–501.

[5] A.A. Albert, Absolute valued algebraic algebras, Bull. Amer. Math. Soc. 55 (1949) 763–768.

[6] J.C. Baez, The octonions, Bull. Amer. Math. Soc. (N.S.) 39 (2001) 145–205.

[7] T. Bröcker, T. tom Dieck, Representations of Compact Lie Groups, Springer, New York, 1985.

[8] A.J. Calderón, C. Martín, On L^*-triples and Jordan H^*-pairs, in: Ring Theory and Algebraic Geometry, Dekker, 2001, pp. 87–94.

[9] A.J. Calderón, C. Martín, Hilbert space methods in the theory of Lie triple systems, in: Recent Progress in Functional Analysis, in: North-Holland Math. Stud., 2001, pp. 309–319.

[10] A.J. Calderón, C. Martín, Absolute valued triple systems, Internat. Math. J. 32 (6) (2004) 2443–2455.

[11] A.J. Calderón, C. Martín, A geometric approach to four-dimensional absolute valued triple systems, in: Algebras, Rings and Their Representations, in: Lecture Notes in Pure and Appl. Math., Dekker, in press.

[12] A. Castellón, J.A. Cuenca, Associative H^*-triple systems, in: Workshop on Non-associative Algebraic Models, Nova Sci. Publ., New York, 1992, pp. 45–67.

[13] A. Castellón, J.A. Cuenca, The centroid and metacentroid of an H^*-triple system, Bull. Soc. Math. Belg. 45 (1993) 85–93.

[14] A. Castellón, J.A. Cuenca, C. Martín, Special Jordan H^*-triple systems, Comm. Algebra 28 (2000) 4699–4706.

[15] J.A. Cuenca, A. Rodríguez, Absolute values on H^*-algebras, Comm. Algebra 23 (1995) 1709–1740.

[16] F.R. Cohen, On Whitehead squares, Cayley–Dickson algebras and rational functions, Bol. Soc. Mat. Mexicana 37 (1992) 55–62.

[17] J.R. Faulkner, Dynkin diagrams for Lie triple systems, J. Algebra 62 (1980) 384–392.

[18] N.C. Hopkins, Some structure theory for a class of triple systems, Trans. Amer. Math. Soc. 1 (1985) 203–212.

[19] K. El-Amin, M.I. Ramírez, A. Rodríguez, Absolute-valued algebraic algebras are finite-dimensional, J. Algebra 195 (1997) 295–307.

[20] J. Lohmus, E. Paal, L. Sorgsepp, Non-associative Algebras in Physics, Hadronic Press, Palm Harbor, FL, 1994.

[21] M.L. El-Mallah, Absolute valued algebras containing a central idempotent, J. Algebra 128 (1990) 180–187.

[22] M.L. El-Mallah, Absolute valued algebras containing a central element, Italian J. Pure Appl. Math. 3 (1998) 103–105.

[23] M.L. El-Mallah, Absolute valued algebras satisfying $(x, x, x^2) = 0$, Arch. Math. 77 (2001) 378–382.

[24] R. Guillermo Moreno, The zero divisors of the Cayley–Dickson algebras over the real numbers, preprint available at q-alg/9710013.

[25] K. McCrimmon, Quadratic forms permitting triple composition, Trans. Amer. Math. Soc. 275 (1983) 107–130.

[26] S. Okubo, Pseudo-quaternion and pseudo-octonion algebras, Hadronic J. 1 (1978) 1250–1278.

[27] M. Postnikov, Lecons de Géométrie. Groupes et Algèbres de Lie, Editions Mir, 1985.

[28] M.I. Ramírez, On four-dimensional absolute-valued algebras, in: Proceedings of the International Conference on Jordan Structures, Univ. Málaga, Málaga, 1999, pp. 169–173.

[29] A. Rodríguez, Absolute valued algebras of degree two, in: Non-Associative Algebra and Its Applications, Kluwer Acad. Publ., 1994, pp. 350–357.

[30] A. Rodríguez, Absolute-valued algebras and absolute-valuable Banach spaces, in: Proceedings of the First International Course of Mathematical Analysis in Andalucía, 2004.

[31] R. Shaw, Ternary composition algebras. I. Structure theorems: Definite and neutral signatures, Proc. Soc. London Ser. A 431 (1990) 1–9.

[32] R. Shaw, Ternary composition algebras. II. Automorphism groups and subgroups, Proc. Soc. London Ser. A 431 (1990) 21–36.

[33] K. Urbanik, F.B. Wright, Absolute valued algebras, Proc. Amer. Math. Soc. 11 (1960) 861–866.

[34] F.B. Wright, Absolute valued algebras, Proc. Natl. Acad. Sci. USA 39 (1953) 330–332.