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On variational problems: Characterization of solutions and duality

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Abstract

In this paper we introduce a new class of pseudoinvex functions for variational problems. Using this new concept, we obtain a necessary and sufficient condition for a critical point of the variational problem to be an optimal solution, illustrated with an example. Also, weak, strong and converse duality are established.

Keywords: Variational problem; Pseudoinvexity; Critical point; Duality

1. Introduction

There exists a huge literature on the necessary and sufficient conditions on calculus of variations, see, for instance, [1–4], for the classical results and to more modern theory, see [5,6]. It is the well-known Euler necessary condition. The function that satisfies

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the Euler condition is called a stationary point of the problem or an extremal; generally speaking, the converse is not true, i.e., the functions that satisfy the Euler condition are not necessary solutions of the problem. Consequently, a more fine analysis is needed to decide if the extremal point is in fact the optimal solution. The classical approach is to give higher order optimality conditions or to use some convexity hypotheses, see, for instance, [3,7,8]. In mathematical programming, the Kuhn–Tucker conditions are sufficient for optimality if the functions involved are convex. In the last few years, attempts have been made to weaken the convexity hypotheses and thus to explore the extent of optimality conditions applicability. As it is known, invexity has been introduced in optimization theory by Hanson, see [9], as a substitute for convexity in constrained optimization. Craven and Glover [10] showed that any differentiable scalar function is invex if and only if every stationary point is a global minimizer. For constrained problems, the invexity defined by Hanson is a sufficient condition but not a necessary condition for every Kuhn–Tucker stationary point to be a global minimizer. Martin, see [11], defined a weaker invexity notion, called Kuhn–Tucker invexity or KT-invexity, which is both necessary and sufficient to establish the Kuhn–Tucker optimality conditions in scalar programming problems. Invexity was extended to variational problems by Mond, Chandra and Husain, see [12] (see also [13–17]). In Section 2 we will give the preliminaries. In Section 3 we define the new concepts of L-(KT/FJ)-pseudoinvex functions. In Section 4 we establish a necessary and sufficient condition in order that a critical point of the Variational Problem be an optimal solution, i.e., it is obtained a characterization which has not been obtained to date. We propose an example to illustrate the nature of L-KT-pseudoinvexity, and where the functions involved are not invex. In Section 5 weak, strong and converse duality are established.

2. Preliminaries

Let us introduce the variational problem and definitions. Let $I = [a, b]$ be a real interval, and let $f : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $g : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ be continuously differentiable functions with respect to each of their arguments. For notational convenience $f(t, x(t), \dot{x}(t))$ will be written $f(t, x, \dot{x})$, where $x : I \to \mathbb{R}^n$, with derivative $\dot{x}$. Let us denote the partial derivative of f with respect to t, x, and $\dot{x}$, by f_t, f_x, $f_{\dot{x}}$, respectively. Analogously, we write the partial derivatives of g_t, g_x, $g_{\dot{x}}$, using matrices with m rows instead of one. Let X be denote the space of piecewise smooth functions $x : I \to \mathbb{R}^n$ with the norm

$$\|x\| = \|x\|_\infty + \|Dx\|_\infty,$$

where the differentiation operator D is given by

$$u = Dx \quad \Leftrightarrow \quad x(t) = \alpha + \int_a^t u(s)\, ds,$$

where α is a given boundary value. Therefore, $D = d/dt$ except at discontinuities. Then, we can consider the scalar Constrained Variational Problem:

$$
\begin{aligned}
\text{(CVP)} \quad & \text{Minimize} \quad && F(x) = \int_a^b f(t, x, \dot{x})\, dt \\
& \text{subject to} \quad && x(a) = \alpha, \qquad x(b) = \beta, \\
& && g\big(t, x(t), \dot{x}(t)\big) \leqslant 0, \quad t \in I.
\end{aligned}
$$

We observe that the classical problem of the calculus of variations is a particular case of (CVP), because it is sufficient to put $g \equiv 0$. This last case we refer as (VP). We denote by K be the set of feasible solutions of (CVP), i.e.,

$$K = \big\{x \in X\colon\ x(a) = \alpha,\ x(b) = \beta,\ g\big(t, x(t), \dot{x}(t)\big) \leqslant 0,\ t \in I\big\}.$$

Definition 2.1. $\bar{x} \in K$ is said to be an optimal solution or global minimum of (CVP) if

$$F(\bar{x}) \leqslant F(x)$$

for all $x \in K$ or equivalently,

$$\nexists x \in K\colon \quad F(x) < F(\bar{x}).$$

We write in the following $f_x(t)$ to denote $f_x(t, x(t), \dot{x}(t))$ and $f_{\dot{x}}(t) = f_{\dot{x}}(t, x(t), \dot{x}(t))$.

Definition 2.2. $x \in K$ is said to be a Fritz–John critical point if there exists $\tau \in \mathbb{R}$ and $y \in X$ such that

$$\tau f_x(t) + y(t)^T g_x(t) = \frac{d}{dt}\big\{\tau f_{\dot{x}}(t) + y(t)^T g_{\dot{x}}(t)\big\}, \tag{1}$$

$$y(t)^T g(t, x, \dot{x}) = 0, \tag{2}$$

$$\big(\tau, y(t)\big) \geqslant 0, \qquad \big(\tau, y(t)\big) \neq 0, \tag{3}$$

$\forall t \in I$, except at discontinuities.

As is usual in optimization theory, if $\tau \neq 0$, we say that the problem is normal or regular, see [18], and in this case, we say that the critical point is a Kunh–Tucker critical point. Note that if the problem is normal, the condition (3) is reduced to $y(t) \geqslant 0$.

Remark 2.1. We observe that the above definitions are exactly the Euler necessary condition for optimality of (CVP). We recall that some additional hypotheses are necessary to guarantee $\tau \neq 0$, these conditions are called qualification of restrictions (see [5]).

3. Invexity and pseudoinvexity

Mond, Chandra and Husain [12], extended the concept of invexity to continuous functions:

Definition 3.1. The function $f(t, x, \dot{x})$ is said to be invex at $\bar{x} \in X$ with respect to η if for all $x \in X$ there exists a vector function $\eta(t, \bar{x}, x)$, with $\eta(t, x, x) = 0$ such that

$$f(t, x, \dot{x}) - f(t, \bar{x}, \dot{\bar{x}}) \geqslant \bar{f}_x(t)\eta(t, \bar{x}, x) + \bar{f}_{\dot{x}}(t)\frac{d}{dt}\eta(t, \bar{x}, x).$$

Where we denote $\bar{f}_x(t) = f_x(t, \bar{x}, \dot{\bar{x}})$ and $\bar{f}_{\dot{x}}(t, \bar{x}, \dot{\bar{x}})$. If f does not depend explicitly on t, the previous definition essentially reduces to be the definition of invexity given by Hanson, see [9]. We recall the definition of invexity for functionals given in [13,14].

Definition 3.2. The functional $F(x) = \int_a^b f(t, x, \dot{x})\,dt$ is called invex at $\bar{x} \in K$ with respect to η, if for all $x \in X$ there exists a vector function $\eta(t, \bar{x}, x)$ with $\eta(t, x, x) = 0$ such that

$$F(x) - F(\bar{x}) \geqslant \int_a^b \left\{ \bar{f}_x(t)\eta(t, \bar{x}, x) + \bar{f}_{\dot{x}}(t)\,\frac{d}{dt}\eta(t, \bar{x}, x) \right\} dt.$$

Looking for optimality sufficient conditions for (CVP), Mond and Husain [13] resorted to generalize convexity:

Definition 3.3. F is pseudoinvex at $\bar{x} \in X$, with respect to η if for all $x \in X$ there exists a vector function $\eta(t, \bar{x}, x)$, with $\eta(t, x, x) = 0$ such that

$$F(x) - F(\bar{x}) < 0 \quad \Rightarrow \quad \int_a^b \left\{ \bar{f}_x(t)\eta(t, \bar{x}, x) + \bar{f}_{\dot{x}}(t)\,\frac{d}{dt}\eta(t, \bar{x}, x) \right\} dt < 0.$$

Under these generalized invexity conditions, Mond and Husain [13] got sufficient Kuhn–Tucker conditions. Let us consider the problem (CVP) and f, g, $F(x) = \int_a^b f(t, x, \dot{x})\,dt$ and $G(x) = \int_a^b g(t, x, \dot{x})\,dt$.

Definition 3.4. The pair (F, G) is said to be L-FJ-pseudoinvex at $\bar{x} \in X$, if for all $x \in X$, $\bar{\tau} \in \mathbb{R}$ and $\bar{y} \in X$, which verify (2) and (3), there exists a differentiable vector function $\eta(t, x, \bar{x}, \bar{\tau}, \bar{y})$, with $\eta(a, x, \bar{x}, \bar{\tau}, \bar{y}) = 0 = \eta(b, x, \bar{x}, \bar{\tau}, \bar{y})$, such that if $F(x) - F(\bar{x}) < 0$, then

$$\int_a^b \Big\{ \big(\bar{\tau}\bar{f}_x(t) + \bar{y}(t)^T \bar{g}_x(t)\big)\eta(t, x, \bar{x}, \bar{\tau}, \bar{y}) + \big(\bar{\tau}\bar{f}_{\dot{x}}(t) + \bar{y}(t)^T \bar{g}_{\dot{x}}(t)\big)\,\frac{d}{dt}\eta(t, x, \bar{x}, \bar{\tau}, \bar{y}) \Big\}\,dt < 0$$

or equivalently,

$$\int_a^b \Big\{ \big(\bar{\tau}\bar{f}_x(t) + \bar{y}(t)^T \bar{g}_x(t)\big)\eta(t, x, \bar{x}, \bar{\tau}, \bar{y}) + \big(\bar{\tau}\bar{f}_{\dot{x}}(t) + \bar{y}(t)^T \bar{g}_{\dot{x}}(t)\big)\,\frac{d}{dt}\eta(t, x, \bar{x}, \bar{\tau}, \bar{y}) \Big\}\,dt \geqslant 0$$

implies $F(x) - F(\bar{x}) \geqslant 0$.

If the problem is normal, i.e, $\bar{\tau} \neq 0$, and taking $\tau = 1$, we say that the pair (F, G) is L-KT-pseudoinvex at $\bar{x} \in X$.

If the previous definitions are reduced on the set of feasible solutions of (CVP), K, we establish the following concepts:

Definition 3.5. The Constrained Variational Problem (CVP) is said to be L-FJ-pseudoinvex if it is verified by Definition 3.4 for all $x, \bar{x} \in K$. If the problem is normal, we say that (CVP) is L-KT-pseudoinvex.

Some of the relationships between these definitions are as follows, the proof is easy.

Proposition 3.1. *If (F, G) is L-KT-pseudoinvex at $\bar{x} \in X$, then F is pseudoinvex at $\bar{x}$.*

In relation to the concept of pseudoinvexity on the suggestion of [13], we propose the following result:

Proposition 3.2. *Let $\bar{x} \in K$. If for all $\bar{y} \in X$ such that $(\bar{x}, \bar{y})$ verifies* (2) *and* (3), *with $\tau = 1$, the Lagrangian function $\phi(x, \bar{y}) = \int_a^b \{f(t, x, \dot{x}) + \bar{y}(t)^T g(t, x, \dot{x})\}\, dt$, with $x \in K$, is pseudoinvex at $\bar{x}$, then* (*CVP*) *is L-KT-pseudoinvex.*

Proof. Let $\bar{y}$ be such that $(\bar{x}, \bar{y})$ verifies (2) and (3), with $\tau = 1$, and let us suppose $x \in K$ such that $F(x) - F(\bar{x}) < 0$, i.e.,

$$\int_a^b f(t, x, \dot{x})\, dt < \int_a^b f(t, \bar{x}, \dot{\bar{x}})\, dt.$$

Since x is feasible and (3), $\bar{y}(t)^T g(t, x, \dot{x}) \leqslant 0$, $\forall t \in I$; and moreover from (2), it follows

$$\int_a^b \big(f(t, x, \dot{x}) + \bar{y}(t)^T g(t, x, \dot{x})\big)\, dt < \int_a^b \big(f(t, \bar{x}, \dot{\bar{x}}) + \bar{y}(t)^T g(t, \bar{x}, \dot{\bar{x}})\big)\, dt.$$

Since $\phi(\cdot, \bar{y})$ is pseudoinvex at $\bar{x}$, there exists a differentiable function $\eta(t, x, \bar{x})$ such that

$$\int_a^b \left\{\big(\bar{f}_x(t) + \bar{y}(t)^T \bar{g}_x(t)\big)\eta(t, x, \bar{x}) + \big(\bar{f}_{\dot{x}}(t) + \bar{y}(t)^T \bar{g}_{\dot{x}}(t)\big)\frac{d}{dt}\eta(t, x, \bar{x})\right\} dt < 0,$$

and therefore, (CVP) is L-KT-pseudoinvex at $\bar{x}$ with respect to $\eta(t, x, \bar{x}, \bar{y}) = \eta(t, x, \bar{x})$. □

4. Necessary and sufficient optimality conditions

In this section, first we prove the sufficiency of the Kuhn–Tucker optimality conditions, under L-KT-pseudoinvexity assumptions on (CVP). Analogous results are true for the L-FJ-pseudoinvexity.

Theorem 4.1. *If* (*CVP*) *is L-KT-pseudoinvex, then all Kuhn–Tucker critical points are optimal solutions.*

Proof. Let $\bar{x}$ be a Kuhn–Tucker critical point, $\bar{x} \in K$, i.e., there exists $\bar{y} \in X$ such that $(\bar{x}, \bar{y})$ verifies (1)–(3), with $\tau = 1$. Let $x \in K$. Since (CVP) is L-KT-pseudoinvex, there exists $\eta(t, x, \bar{x}, \bar{y}) = \eta(t, x, \bar{x}, 1, \bar{y})$ differentiable, with $\eta(a, x, \bar{x}, \bar{y}) = 0 = \eta(b, x, \bar{x}, \bar{y})$ which verifies Definition 3.4. It follows

$$\int_a^b \bigl\{\bigl(\bar{f}_x(t) + \bar{y}(t)^T \bar{g}_x(t)\bigr)\eta(t, x, \bar{x}, \bar{y}) + \bigl(\bar{f}_{\dot{x}}(t) + \bar{y}(t)^T \bar{g}_{\dot{x}}(t)\bigr)\frac{d}{dt}\eta(t, x, \bar{x}, \bar{y})\bigr\}\,dt$$

$$= \int_a^b \biggl\{\bigl(\bar{f}_x(t) + \bar{y}(t)^T \bar{g}_x(t)\bigr)\eta(t, x, \bar{x}, \bar{y}) - \frac{d}{dt}\bigl(\bar{f}_{\dot{x}}(t) + \bar{y}(t)^T \bar{g}_{\dot{x}}(t)\bigr)\eta(t, x, \bar{x}, \bar{y})\biggr\}\,dt + \bigl(\bar{f}_x(t) + \bar{y}(t)^T \bar{g}(t, x, \bar{x}, \bar{y})\bigr)\Big|_{t=a}^{t=b} \quad \text{(by integration by parts)}$$

$$= \int_a^b \biggl\{\bar{f}_x(t) + \bar{y}(t)^T \bar{g}_x(t) - \frac{d}{dt}\bigl(\bar{f}_{\dot{x}}(t) + \bar{y}(t)^T \bar{g}_{\dot{x}}(t)\bigr)\biggr\}\eta(t, x, \bar{x}, \bar{y})\,dt = 0 \quad \bigl(\text{by (1)}\bigr).$$

Since (CVP) is L-KT-pseudoinvex, it follows

$$F(x) - F(\bar{x}) \geqslant 0 \quad \forall x \in K.$$

Therefore, $\bar{x}$ is an optimal solution of (CVP). □

We have proved that L-KT-pseudoinvexity is a sufficient condition, and now, we are going to prove that it is a necessary condition.

Theorem 4.2. *If all Kuhn–Tucker critical points are optimal solutions for* (*CVP*)*, then* (*CVP*) *is L-KT-pseudoinvex.*

Proof. Let $x, \bar{x} \in K$, $(\bar{x}, \bar{y})$ verifies (2) and (3), with $\tau = 1$, such that $F(x) - F(\bar{x}) < 0$. We have to find $\eta(t, x, \bar{x}, \bar{y}) \equiv \eta(t, x, \bar{x}, 1, \bar{y})$ differentiable, with $\eta(a, x, \bar{x}, \bar{y}) = 0 = \eta(b, x, \bar{x}, \bar{y})$, such that

$$P\bigl(\eta(\cdot, x, \bar{x}, \bar{y})\bigr) = \int_a^b \biggl\{\bigl(\bar{f}_x(t) + \bar{y}(t)^T \bar{g}_x(t)\bigr)\eta(t, x, \bar{x}, \bar{y}) + \bigl(\bar{f}_{\dot{x}}(t) + \bar{y}(t)^T \bar{g}_{\dot{x}}(t)\bigr)\frac{d}{dt}\eta(t, x, \bar{x}, \bar{y})\biggr\}\,dt < 0.$$

And thus, suppose $P(\eta(\cdot, x, \bar{x}, \bar{y})) < 0$ has no solution $\eta(t, x, \bar{x}, \bar{y})$, and then $P(\eta(\cdot, x, \bar{x}, \bar{y})) > 0$ has no solution too, since we could consider $-\eta(t, x, \bar{x}, \bar{y})$. Therefore,

$$P\big(\eta(\cdot,x,\bar{x},\bar{y})\big)=\int_a^b\Big\{\big(\bar{f}_x(t)+\bar{y}(t)^T\bar{g}_x(t)\big)\eta(t,x,\bar{x},\bar{y})$$
$$+\big(\bar{f}_{\dot{x}}(t)+\bar{y}(t)^T\bar{g}_{\dot{x}}(t)\big)\frac{d}{dt}\eta(t,x,\bar{x},\bar{y})\Big\}\,dt=0$$

$\forall\eta(t,x,\bar{x},\bar{y})$ differentiable, with $\eta(a,x,\bar{x},\bar{y})=0=\eta(b,x,\bar{x},\bar{y})$. From generalized de Dubois–Raymond Lemma (see [1, p. 307]), we have

$$\bar{f}_{\dot{x}}(t)+\bar{y}(t)^T\bar{g}_{\dot{x}}(t)$$

is piecewise smooth and

$$\bar{f}_x(t)+\bar{y}(t)^T\bar{g}_x(t)=\frac{d}{dt}\big\{\bar{f}_{\dot{x}}(t)+\bar{y}(t)^T\bar{g}_{\dot{x}}(t)\big\},$$

and therefore, $(\bar{x},\bar{y})$ verifies (1)–(3), with $\tau=1$, i.e., $\bar{x}$ is a Kuhn–Tucker critical point, and then $\bar{x}$ is an optimal solution for (CVP), which stands in contradiction to $F(x)-F(\bar{x})<0$. So, there exists $\eta(t,x,\bar{x},\bar{y})$ differentiable, with $\eta(a,x,\bar{x},\bar{y})=0=\eta(b,x,\bar{x},\bar{y})$, such that $P(\eta(\cdot,x,\bar{x},\bar{y}))<0$, and then, (CVP) is L-KT-pseudoinvex. □

Therefore, we have proved that L-KT-pseudoinvexity of (CVP) is both sufficient and necessary condition in order that a Kuhn–Tucker critical point is an optimal solution of (CVP). In the same way, we can prove that L-FJ-pseudoinvexity of (CVP) is both necessary and sufficient condition for that its critical points to be optimal solutions.

Theorem 4.3. *If all Fritz–John critical points are optimal solutions for* (*CVP*)*, then* (*CVP*) *is L-FJ-pseudoinvex.*

Theorem 4.4. *If* (*CVP*) *is L-FJ-pseudoinvex the all Fritz–John critical points are optimal solutions.*

In the following, we consider an example to illustrate L-KT-pseudoinvexity.

Example. In this way, we present an example of L-KT-pseudoinvex variational problem. Besides, we prove that invexity of some of the functions involved in this variational problem is not verified, what is required in Mond, Chandra and Husain [12].

$$\text{(P1)}\quad \text{Minimize}\quad \int_0^1\big(1-\dot{x}(t)\big)^2\,dt$$
$$\text{subject to}\quad x(0)=1,\qquad x(1)=2,$$
$$1-x^2(t)\leqslant 0,\qquad \dot{x}(t)-10\leqslant 0,$$

where $I=[0,1]$, $f:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $g=(g_1,g_2):[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}^2$, $f(t,x,\dot{x})=(1-\dot{x}(t))^2$, $g_1(t,x,\dot{x})=1-x^2(t)$, $g_2(t,x,\dot{x})=\dot{x}(t)-10$. Let K be the set of feasible solutions of (P1), then

(i) Let us see that all Kuhn–Tucker critical points are optimal solutions of (P1).
Let $x \in K$. If x is a Kuhn–Tucker critical point, then there exists a piecewise smooth function $y = (y_1, y_2)$ such that

$$-2x(t)y_1(t) = 2\ddot{x}(t) + \dot{y}_2(t), \qquad y_1(t)\bigl(1 - x^2(t)\bigr) = 0,$$
$$y_2(t)\bigl(\dot{x}(t) - 10\bigr) = 0.$$

Since $y_1(t) \geqslant 0$ and $y_2(t) \geqslant 0$, these conditions are reduced to

$$\bigl(x(t), y_1(t), y_2(t)\bigr) = (ct + d, 0, 0), \quad t \in [0, 1] \quad \text{or}$$
$$\bigl(x(t), y_1(t), y_2(t)\bigr) = (10t + e, 0, k), \quad t \in [0, 1],$$

with $c, d, e, k \in \mathbb{R}$, $k > 0$. And since y_1, y_2 are continuous and $x \in K$, by simple calculations it is obtained that the Kuhn–Tucker critical point is $\bar{x}(t) = t + 1$, with $y_1(t) = 0 = y_2(t)$, $t \in [0, 1]$. Otherwise, and graphically, we have that the optimal solution of (P1) is $\bar{x}(t) = t + 1$.

(ii) The problem (P1) is L-KT-pseudoinvex.
By applying Definition 3.5 to (P1), given $(x, \bar{x}, y_1, y_2)$ which verifies (2) and (3), there exists η such that

$$\int_0^1 \Bigl(\bigl(1 - \dot{x}(t)\bigr)^2 - \bigl(1 - \dot{\bar{x}}(t)\bigr)^2\Bigr)\,dt < 0$$
$$\Rightarrow \quad \int_0^1 \Bigl(y_1(t)\bigl(-2\bar{x}(t)\bigr)\eta + \bigl(2\bigl(\dot{\bar{x}}(t) - 1\bigr) + y_2(t)\bigr)\dot{\eta}\Bigr)\,dt < 0.$$

To prove L-KT-pseudoinvexity, suppose it is not verified, i.e., if

$$\int_0^1 \Bigl(\bigl(1 - \dot{x}(t)\bigr)^2 - \bigl(1 - \dot{\bar{x}}(t)\bigr)^2\Bigr)\,dt < 0$$

then

$$\int_0^1 \Bigl(y_1(t)\bigl(-2\bar{x}(t)\bigr)\eta + \bigl(2\bigl(\dot{\bar{x}}(t) - 1\bigr) + y_2(t)\bigr)\dot{\eta}\Bigr)\,dt = 0$$

for all η, what is equivalent to differential equations in (i), and it is only verified by $\bar{x}(t) = t + 1$. And

$$F(x) - F(\bar{x}) = \int_0^1 \Bigl(\bigl(1 - \dot{x}(t)\bigr)^2 - \bigl(1 - \dot{\bar{x}}(t)\bigr)^2\Bigr)\,dt = \int_0^1 \bigl(1 - \dot{x}(t)\bigr)^2\,dt \geqslant 0,$$

for all $x \in K$. Then, it is not verified $F(x) - F(\bar{x}) < 0$, and therefore (P1) is L-KT-pseudoinvex.

Moreover, $g_1(t, x, \dot{x}) = 1 - x^2(t)$ is not invex, because from Definition 3.1, given x, $\bar{x} \in K$, g_1 is invex if there exists η such that

$$\bar{x}^2(t) - x^2(t) \geqslant 2\big(\bar{x}(t) - 1\big)\eta, \quad t \in [0, 1].$$

But this condition is not verified. For example, take

$$\bar{x}(t) = \begin{cases} 1, & t \in [0, 1/2], \\ 2t, & t \in [1/2, 1], \end{cases} \quad \text{and} \quad x(t) = t + 1, \quad t \in [0, 1].$$

Consequently, g is not invex.

So, and firstly, (P1) verifies that all Kuhn–Tucker critical point are optimal solutions. Secondly, by Theorem 4.1, (P1) is a L-KT-pseudoinvex variational problem, which is showed in (ii). And finally, invexity is not verified for (P1) and, however, some authors (see [12]) require it for optimality results.

5. Duality

We establish duality between (CVP) and the next dual problem (CVD1), which is a modified Mond–Weir type dual problem formulated by Bector, Chandra and Husain, see [18].

$$\begin{aligned} \text{(CVD1)} \quad & \text{Maximize} \quad \int_a^b f(t, u, \dot{u})\, dt \\ & \text{subject to} \quad u(a) = \alpha, \qquad u(b) = \beta, \\ & f_u(t) + y(t)^T g_u(t) = \frac{d}{dt}\big\{ f_{\dot{u}}(t) + y(t)^T g_{\dot{u}}(t) \big\}, \quad t \in I, \\ & y(t)^T g(t, u, \dot{u}) = 0, \\ & y(t) \geqslant 0, \quad t \in I. \end{aligned}$$

We recall that $f_u(t) = f_u(t, u, \dot{u})$ and $f_{\dot{u}}(t) = f_{\dot{u}}(t, u, \dot{u})$. Analogously for g. Let H be the feasible set of (CVD1).

Theorem 5.1 (Weak duality). *Let $x \in K$ and $(u, y) \in H$. If (F, G) is L-KT-pseudoinvex at u, then $\int_a^b f(t, x, \dot{x}) dt \geqslant \int_a^b f(t, u, \dot{u})\, dt$.*

Proof. Suppose $\int_a^b f(t, x, \dot{x})\, dt \geqslant \int_a^b f(t, u, \dot{u})\, dt$ is not verified, i.e., $F(x) - F(u) < 0$. Since (u, y) verifies the third and fourth restrictions in (CVD1), and since (F,G) is L-KT-pseudoinvex, there exists a differentiable function $\eta(t) = \eta(t, x, u, y)$, with $\eta(a, x, u, y) = 0 = \eta(b, x, u, y)$ such that

$$\int_a^b \left\{ \big(f_u(t) + y(t)^T g_u(t)\big)\eta(t) + \big(f_{\dot{u}}(t) + y(t)^T g_{\dot{u}}(t)\big) \frac{d}{dt}\eta(t) \right\} dt < 0. \tag{4}$$

On the other hand,

$$\int_a^b \left\{ \left(f_u(t) + y(t)^T g_u(t)\right)\eta(t) + \left(f_{\dot u}(t) + y(t)^T g_{\dot u}(t)\right) \frac{d}{dt}\eta(t) \right\} dt$$

$$= \int_a^b \left\{ \left(f_u(t) + y(t)^T g_u(t)\right)\eta(t) - \left(\frac{d}{dt}\left(f_{\dot u}(t) + y(t)^T g_{\dot u}(t)\right)\right)\eta(t) \right\} dt$$

$$+ \left(f_{\dot u}(t) + y(t)^T g_{\dot u}(t)\right)\eta(t)\Big|_{t=a}^{t=b} \quad \text{(by integration by parts)}$$

$$= \int_a^b \left\{ f_u(t) + y(t)^T g_u(t) - \left(\frac{d}{dt}\left(f_{\dot u}(t) + y(t)^T g_{\dot u}(t)\right)\right) \right\}\eta(t)\, dt = 0,$$

where we use the second restriction in (CVD1). This is a contradiction with (4). Therefore,

$$\int_a^b f(t, x, \dot x)\, dt \geqslant \int_a^b f(t, u, \dot u)\, dt. \qquad \square$$

As a consequence of previous theorem, if (F, G) is L-KT-pseudoinvex, then

$$\int_a^b f(t, x, \dot x)\, dt \geqslant \int_a^b f(t, u, \dot u)\, dt, \quad \forall x \in K,\ \forall (u, y) \in H.$$

Once weak duality has been established, strong and converse duality follows:

Theorem 5.2 (Strong duality). *Let $\bar x$ be an optimal normal solution of (CVP). If (F,G) is L-KT-pseudoinvex, then there exists $\bar y \in X$ such that $(\bar x, \bar y)$ is an optimal solution of (CVD1), and their objective function values are equal at these points.*

Proof. Since $\bar x$ is an optimal normal solution of (CVP), from Valentine necessary condition, see [8] (see also [4]), there exists $\bar y \in X$ such that $(\bar x, \bar y)$ verifies

$$f_x(t) + \bar y(t)^T g_x(t) = \frac{d}{dt}\left\{ f_{\dot x}(t) + \bar y(t)^T g_{\dot x}(t) \right\},$$

$$\bar y(t)^T g(t, \bar x, \dot{\bar x}) = 0,$$

$$\bar y(t) \geqslant 0, \quad t \in I.$$

Therefore, $(\bar x, \bar y) \in H$. From Theorem 5.1, $(\bar x, \bar y)$ is an optimal solution of (CVD1), and obviously, the objective function values of (CVD1) and (CVP) are equal. $\square$

We now consider the converse dual problem, that is, of finding conditions under which the existence of optimal solution to problem (CVD1) implies the existence of an optimal solution to problem (CVP).

Theorem 5.3 (Converse duality). *Let $(\bar{u}, \bar{y})$ be an optimal solution of (CVD1). If $\bar{u} \in K$ and (F, G) is L-KT-pseudoinvex, then $\bar{u}$ is an optimal solution of (CVP) and their objective function values are equal at these points.*

Proof. Since (F, G) is L-KT-pseudoinvex, and from Theorem 5.1, it follows that

$$\int_a^b f(t, x, \dot{x})\, dt \geqslant \int_a^b f(t, \bar{u}, \dot{\bar{u}})\, dt,$$

$\forall x \in K$. And since $\bar{u} \in K$, it follows that $\bar{u}$ is an optimal solution of (CVP), and their objective function values are equal at this point. □

We continue our duality study with the dual problem (CVD2), as follows:

$$\begin{array}{lll} \text{(CVD2)} & \text{Maximize} & \displaystyle\int_a^b f(t, u, \dot{u})\, dt \\ & \text{subject to} & u(a) = \alpha, \qquad u(b) = \beta, \\ & & \tau f_u(t) + y(t)^T g_u(t) = \dfrac{d}{dt}\big\{\tau f_{\dot{u}}(t) + y(t)^T g_{\dot{u}}(t)\big\}, \quad t \in I, \\ & & y(t)^T g(t, u, \dot{u}) = 0, \\ & & \big(\tau, y(t)\big) \geqslant 0, \quad t \in I. \end{array}$$

Again, let H be the feasible set of (CVD2). Proceeding in the same way as in the proofs of Theorems 5.1–5.3, but under L-FJ-pseudoinvexity, we state the following duality results between (CVP) and (CVD2).

Theorem 5.4 (Weak duality). *Let $x \in K$, $(u, \tau, y) \in H$. If (F, G) is L-FJ-pseudoinvex at u, then $\int_a^b f(t, x, \dot{x})\, dt \geqslant \int_a^b f(t, u, \dot{u})\, dt$.*

As a consequence of this theorem, if (F, G) is L-FJ-pseudoinvex, then

$$\int_a^b f(t, x, \dot{x})\, dt \geqslant \int_a^b f(t, u, \dot{u})\, dt, \quad \forall x \in K,\ \forall (u, \tau, y) \in H.$$

Theorem 5.5 (Strong duality). *Let $\bar{x}$ be an optimal solution of (CVP). If (F, G) is L-FJ-pseudoinvex, then there exists $\bar{\tau} \in \mathbb{R}$, $\bar{y} \in X$ such that $(\bar{x}, \bar{\tau}, \bar{y})$ is an optimal solution of (CVD2), and their objective function values are equal at this point.*

Theorem 5.6 (Converse duality). *Let $(\bar{u}, \bar{\tau}, \bar{y})$ be an optimal solution of (CVD2). If $\bar{u} \in K$ and (F, G) is L-FJ-pseudoinvex, then $\bar{u}$ is an optimal solution of (CVP) and their objective function values are equal.*

6. Conclusion

In this paper, we have studied the properties of the functions of a constrained variational problem, such that from a critical point it follows an optimal solution. These properties are L-KT-pseudoinvexity and L-FJ-pseudoinvexity, and we have proved that these are necessary and sufficient conditions for a critical point to be an optimal solution of the variational problem: a characterization. The nature of these results has been illustrated with an example of L-KT-pseudoinvex variational problem, in which invexity conditions are not verified. Also, we have proved that the problems (CVP) and (CVD1) are a dual pair subject to L-KT-pseudoinvexity conditions; and (CVP) and (CVD2), under L-FJ-pseudoinvexity conditions.

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