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Comparison of theoretical complexities of two methods for computing annihilating ideals of polynomials

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Abstract

Let $f_1, \dots, f_p$ be polynomials in $\mathbf{C}[x_1, \dots, x_n]$ and let $D = D_n$ be the n-th Weyl algebra. We provide upper bounds for the complexity of computing the annihilating ideal of $f^s = f_1^{s_1} \cdots f_p^{s_p}$ in $D[s] = D[s_1, \dots, s_p]$. These bounds provide an initial explanation of the differences between the running times of the two methods known to obtain the so-called Bernstein–Sato ideals.

Keywords: Complexity; Poincaré–Birkhoff–Witt algebras; Bernstein–Sato ideals

1. Introduction

Fix two integers $n \geq 1$, $p \geq 1$ and two sets of variables $(x_1, \dots, x_n)$ and $(s_1, \dots, s_p)$. Let us consider $f_1, \dots, f_p \in \mathbf{C}[x] = \mathbf{C}[x_1, \dots, x_n]$ and let $D = D_n$ be the n-th Weyl algebra. A polynomial $b(s) \in \mathbf{C}[s] = \mathbf{C}[s_1, \dots, s_p]$ is said to be a *Bernstein–Sato*

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polynomial associated with $f_1, \ldots, f_p$ if the following functional equation holds for a certain $P(s) \in D[s]$:

$$b(s) f^s = P(s) f^{s+\mathbf{1}},$$

where $f^s = f_1^{s_1} \cdots f_p^{s_p}$ and $\mathbf{1} = (1, \ldots, 1)$. These polynomials form an ideal called the Bernstein–Sato ideal, denoted as $\mathcal{B}_f$ or simply $\mathcal{B}$ if no confusion arises. Analogous functional equations with respect to vectors different from **1** yield other versions of Bernstein–Sato ideals (see for example Bahloul, 2001).

In Lichtin (1988) it is proved that $\mathcal{B}$ is not zero. This fact is a generalization of the classical proof of Bernstein (Bernstein, 1972) in the algebraic setting for the case $p = 1$, in which $\mathcal{B}$ is generated by the so-called *Bernstein–Sato polynomial* denoted as $b_f(s)$. The analytical case was covered in Björk (1973) for $p = 1$ and Sabbah (1987a) and Sabbah (1987b) for $p > 1$ (an interesting new proof using the Gröbner fan has been given in Bahloul (2005)). The roots of $b_f(s)$ encode important algebro-geometrical data (see Malgrange (1974), Hamm (1975) or Budur-Saito (2003), to mention but a few) and a complete understanding of all roots for a general f is open. The case $p > 1$ seems to be much more complex and there are conjectures on the primary decomposition of $\mathcal{B}$, on the conditions over f for $\mathcal{B}$ to be principal, etc. (see for example Maynadier, 1996).

Until Oaku (1997) there were no algorithms for finding the Bernstein–Sato polynomial. Since then, alternative methods have been proposed for obtaining $\mathcal{B}$ in the general case (see Oaku and Takayama (1999), Bahloul (2001) and Briançon and Maisonobe (2002)). These methods have a feature in common: their first step is the computation of the *annihilating ideal* of f^s in $D[s]$, $Ann_{D[s]} f^s$. In Castro-Ucha (2004) some experimental evidence was given in favor of the Briançon–Maisonobe (BM) method for computing $Ann_{D[s]} f^s$, with respect to the Oaku–Takayama (OT) method, but no clues about which facts support this superiority were provided.

Our work is a first step towards comparing the two methods theoretically. We give upper bounds for the complexity of computing $Ann_{D[s]} f^s$, the previous requirement for both algorithms. To obtain these bounds we use the techniques and results of Grigoriev (1990) on the complexity of solving systems of linear equations over rings of differential operators. These extend the classical polynomial case treated in Seidenberg (1974). In particular, we show that Grigoriev's construction cannot be directly generalized to the algebra proposed by Briançon and Maisonobe. We prove that the complexity of computing $Ann_{D[s]} f^s$ using the BM method is that of the calculation of a Gröbner basis in the n-th Weyl algebra with some extra p commutative variables, so $2n + p$ variables at most. On the other hand, in the case of the OT method the calculation of such a basis is made in a $(n + p)$-th Weyl algebra with some extra $2p$ variables, so $2n + 4p$ variables altogether.

It is an open problem whether the bound proposed in this work is reached à la Mayr-Meyer (1982), that is to say, whether an example with this worst complexity can be explicitly obtained. Such an example would mean a complete answer to the question of what the complexity of computing $Ann_{D[s]} f^s$ is, proposed by Professor N. Takayama.

2. Preliminaries

In this section we just recall briefly some details of the Briançon–Maisonobe and Oaku–Takayama methods.

2.1. Briançon–Maisonobe method

In this case the computations are made in the non-commutative algebra[1]

$$R = D[s, t] = D[s_1, \dots, s_p, t_1, \dots, t_p],$$

an extension of the n-th Weyl algebra D in which the new variables s, t satisfy the relations $[s_i, t_j] = \delta_{ij} t_i$. It is a *Poincaré–Birkhoff–Witt (PBW) algebra*:

Definition 1. A PBW algebra R over a ring k is an associative algebra generated by finitely many elements $x_1, \dots, x_n$ verifying the relations

$$Q = \{x_j x_i = q_{ji} x_i x_j + p_{ji}, 1 \le i < j \le n\},$$

where each p_{ji} is a finite k-linear combination of *standard* terms $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, each $q_{ji} \in k$ verifying the two following conditions:

(1) There is an *admissible*[2] ordering $\prec$ on $\mathbf{N}^n$ such that $\exp(p_{ji}) \prec \exp(x_j x_i)$ for every $1 \le i < j \le n$.
(2) The standard terms $\mathbf{x}^\alpha$, with $\alpha \in \mathbf{N}^n$, form a k-basis of R as a vector space.

It is possible to compute Gröbner bases in PBW algebras. The book Bueso et al. (2003) is a good introduction to the subject of effective calculus in this fairly general family.

The following algorithm computes $\mathcal{B}$, starting from

$$I := Ann_R(f^s) = \left\langle s_j + f_j t_j, \partial_i + \sum_j \frac{\partial f_j}{\partial x_i} t_j, 1 \le i \le n, 1 \le j \le p \right\rangle.$$

Algorithm 1. (1) Obtain $J = Ann_{D_n[s]} f^s = \langle G_1 \cap D_n[s] \rangle$ where G_1 is a Gröbner basis of I with respect to any term ordering where the variables t_j are greater than the others (that is, an *elimination ordering for the* t_j.)
(2) $\mathcal{B} = (\langle G_2 \rangle + \langle f_1, \dots, f_p \rangle) \cap \mathbf{C}[s]\rangle$, where G_2 is a Gröbner basis of J with respect to any term ordering with x_i, ∂_j greater than s_l, for all i, j, l.

2.2. Oaku–Takayama method

All the computations are made in Weyl algebras. More precisely, we start from

$$I' = \left\langle t_j - f_j, \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_i} \partial_{t_j} + \partial_i,\ i = 1, \dots, n,\ j = 1, \dots, p \right\rangle.$$

[1] It is, in fact, the ring introduced in classical works by Malgrange and Kashiwara for $p = 1$.
[2] Here admissible means a total ordering among the elements of $\mathbf{N}^n$ with $\mathbf{0}$ as the smallest element.

Algorithm 2. (1) Obtain $J' = I' \bigcap \mathbf{C}[t_1\partial_{t_1}, \ldots, t_n\partial_{t_n}]\langle x, \partial_x\rangle$.

(2) $J = Ann_{D_n[s]}(f^s) = J''$, where J'' denotes the ideal generated by the generators of J' after replacing each $t_i\partial_{t_i}$ by $-s_i - 1$.

(3) $\mathcal{B} = (\langle G_2\rangle + \langle f_1, \ldots, f_p\rangle) \cap \mathbf{C}[s]\rangle$, where G_2 is a Gröbner basis of J with respect to any term ordering with x_i, ∂_j greater than s_l, for all $i, j, l\ldots$

Remark 2. The second step above is, as in Algorithm 1, the elimination of all the variables but $(s_1, \ldots, s_p)$. Often the bottleneck for obtaining the Bernstein–Sato ideal is this step. As far as we know, the example for $p = 2$ with $f_1 = x^2 + y^3$, $f_2 = x^3 + y^2$ is intractable for available computer algebra systems.

The computation of

$$I' \cap \mathbf{C}[t_1\partial_{t_1}, \ldots, t_n\partial_{t_n}]\langle x, \partial_x\rangle$$

uses $2n + 4p$ variables, as new variables u_j, v_j for $1 \le j \le p$ are introduced. More precisely, the main calculation is an elimination of these new variables for the ideal

$$\left\langle t_j - u_j f_j, \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_i} u_j \partial_{t_j} + \partial_i, 1 - u_j v_j, \;\; 1 \le i \le n, 1 \le j \le p, \right\rangle.$$

3. Complexity

In Grigoriev (1990) a bound for the degree of the solutions of a general system of linear equations over the Weyl algebra is given, with a procedure somewhat similar to that of Seidenberg (1974). In this section we shall see how much of the work of Grigoriev is applicable to our PBW algebra R of Section 2.1.

The construction has two different steps. In the first, the given system is reduced to another system in a diagonal form. In the second, it is shown how to normalize the new system in order to eliminate, successively, the variables.

We need a technical lemma to reduce the system to a diagonal form. This lemma comes from Grigoriev's paper (see Grigoriev, 1990, Lemma 1), but we will write it in a more general way. Here deg means the *total degree* of a term, that is, the sum of the exponents of all of its variables.

Lemma 3. *Let A be a $(m-1) \times m$ matrix with entries in a Poincaré–Birkhoff–Witt algebra S with a basis of p elements. If* $\deg(a_{ij}) \le d$, *there exists a nonzero vector* $f = (f_1, \ldots, f_m) \in S^m$ *such that* $Af = 0$ *and* $\deg(f) \le 2p(m-1)d = N$.

Proof. Consider the linear space $T \subset S^m$ of vectors $c = (c_1, \ldots, c_m) \in S^m$ such that $\deg(c) \le N$. We have $\dim(T) = \binom{N+p}{p} m$. For any vector $c \in T$ it is clear that $\deg(Ac) \le N + d$. If we consider now the vector space γ of vectors $e = (e_1, \ldots, e_{m-1}) \in$

S^{m-1} such that $\deg(e) \le N+d$, we have $\dim(\gamma) = \binom{N+d+p}{p}(m-1)$. We prove that $\dim(\gamma) < \dim(T)$:

$$\binom{N+d+p}{p} \Big/ \binom{N+p}{p} = \frac{N+d+p}{N+p}\frac{N+d+p-1}{N+p-1}\cdots\frac{N+d+1}{N+1} \le \left(\frac{N+d+1}{N+1}\right)^p .$$

It is enough to see that $(\frac{N+d+1}{N+1})^p < 1 + \frac{1}{m-1}$. This inequality follows from

$$\left(1+\frac{1}{m-1}\right)^{\frac{1}{p}} > 1 + \frac{1}{p(m-1)} + \frac{1}{2}\frac{1}{p}\left(\frac{1}{p}-1\right)\left(\frac{1}{m-1}\right)^2 > 1 + \frac{1}{2p(m-1)} > 1 + \frac{d}{N+1}.$$

If we work in a noetherian domain (not necessarily commutative), we can always define the *rank* of a finite module as in Stafford (1978). Given a square matrix in a Poincaré–Birkhoff–Witt algebra we say that it is *non-singular* if it has maximal rank. In this case we can obtain a left quasi-inverse with the previous lemma:

Lemma 4. *Given a $m \times m$ non-singular matrix B over a PBW algebra S as in Lemma 3, it has a left quasi-inverse matrix G over S, such that* $\deg(G) \le N$.

Proof. There is no vector $b \neq 0$ in $\mathcal{R}^m$ such that $bB = 0$. If we consider the matrix $B^{(i)}$ obtained from B by deleting its i-th column, using Lemma 3 we obtain a vector $g_i \neq 0$ such that $g_i B^{(i)} = 0$ and $\deg(g_i) \le N$, so the matrix G which has g_i as its i-th row, for $i = 1, \ldots, m$, is a left quasi-inverse of B.

Lemma 5. *Given a system of linear equations over a PBW algebra defined by an $m \times s$ matrix A of rank r with its elements* $\deg(a_{ij}) \le d$, *we can always construct a matrix C that defines an equivalent system, and such that*

$$CA = \begin{pmatrix} C_1 & 0 \\ C_2 & E \end{pmatrix} A = \left(\begin{array}{ccc|c} a_1 & & 0 & \\ & \ddots & & \star \\ 0 & & a_r & \\ \hline & 0 & & 0 \end{array}\right) \tag{1}$$

where E is the identity matrix.

Proof. C_1 is the left quasi-inverse of the submatrix of A of maximal rank r (after reordering the rows or columns of A if necessary). C_2 is constructed with the requirement on the left lower corner to be zero. The right lower corner is zero by the definition of rank.

Thanks to this lemma, we can assume that our system is equivalent to a system in diagonal form:

$$a_k V_k + \sum_{r+1\le l\le s} a_{k,l} V_l = b_k, \quad 1 \le k \le r, \quad \deg(a_k), \deg(a_{k,l}), \deg(b_k) \le 2pmd.$$

Once the system is in diagonal form, we need to normalize it. To do this, we construct some syzygies, applying Lemma 3 to the submatrix of the first r columns and the column $l > r$. There always exist $h^{(l)}, h_1^{(l)}, \ldots, h_r^{(l)}$ such that

$$a_k h_k^{(l)} + a_{k,l} h^{(l)} = 0, \qquad 1 \le k \le r \qquad \deg(h^{(l)}), \deg(h_i^{(l)}) \le 4p^2 m^2 d.$$

The result that gives the normalization in the Weyl algebra is the following one:

Lemma 6 (Grigoriev (1990), *Lemma 4*). *Given $g_1, \ldots, g_t \in D$ a family of elements, there is a nonsingular linear transformation of $2n$-dimensional space with basis $x_1, \ldots, x_n, \partial_1, \ldots, \partial_n$ under which*

$$x_i \to \Gamma_{x_i} = \sum_{j=1}^{n} \gamma_{i,j}^{(1,1)} x_j + \sum_{j=1}^{n} \gamma_{i,j}^{(1,2)} \partial_j;$$
$$\partial_i \to \Gamma_{\partial_i} = \sum_{j=1}^{n} \gamma_{i,j}^{(2,1)} x_j + \sum_{j=1}^{n} \gamma_{i,j}^{(2,2)} \partial_j$$

such that the following relations hold:

$$\Gamma_{x_i}\Gamma_{\partial_i} = \Gamma_{\partial_i}\Gamma_{x_i} - 1; \qquad \Gamma_{x_i}\Gamma_{x_j} = \Gamma_{x_j}\Gamma_{x_i};$$
$$\Gamma_{\partial_i}\Gamma_{\partial_j} = \Gamma_{\partial_j}\Gamma_{\partial_i}; \qquad \Gamma_{\partial_i}\Gamma_{x_j} = \Gamma_{x_j}\Gamma_{\partial_i}, \qquad i \ne j,$$

and if we denote by Γ_{g_i} the transformed of g_i with the indicated linear transformation, we have $\Gamma_{g_i} = \partial_n^{\deg(g_i)} + \widetilde{\Gamma_{g_i}}$.

Remark 7. The main fact in the proof of Lemma 6 is that the matrices of the linear transformations defined by the relations in the Weyl algebra are a transitive group. Let us see why we cannot ensure the existence of such a normalization lemma for every PBW algebra.

If we consider the PBW algebra defined by Briançon and Maisonobe for $p = 1$, that is

$$R = \mathbf{C}[s, t, x_1, \ldots, x_n, \partial_1, \ldots, \partial_n],$$

a general linear transformation such as the one appearing in Lemma 6 has the form

$$\begin{aligned}
s &\to \Gamma_s = \alpha_1 s + \beta_1 t + \textstyle\sum_{j=1}^{n} \gamma_j^{(s,1)} x_j + \sum_{j=1}^{n} \gamma_j^{(s,2)} \partial_j \\
t &\to \Gamma_t = \alpha_2 s + \beta_2 t + \textstyle\sum_{j=1}^{n} \gamma_j^{(t,1)} x_j + \sum_{j=1}^{n} \gamma_j^{(t,2)} \partial_j \\
x_i &\to \Gamma_{x_i} = \alpha_i^{(1)} s + \beta_i^{(1)} t + \textstyle\sum_{j=1}^{n} \gamma_{i,j}^{(1,1)} x_j + \sum_{j=1}^{n} \gamma_{i,j}^{(1,2)} \partial_j \\
\partial_i &\to \Gamma_{\partial_i} = \alpha_i^{(2)} s + \beta_i^{(2)} t + \textstyle\sum_{j=1}^{n} \gamma_{i,j}^{(2,1)} x_j + \sum_{j=1}^{n} \gamma_{i,j}^{(2,2)} \partial_j.
\end{aligned}$$

and it has to verify the following relations:

$$\begin{array}{llll} (1)\ \Gamma_s\Gamma_t = \Gamma_t\Gamma_s + \Gamma_t; & (2)\ \Gamma_s\Gamma_{x_i} = \Gamma_{x_i}\Gamma_s; & (3)\ \Gamma_s\Gamma_{\partial_i} = \Gamma_{\partial_i}\Gamma_s; \\ (4)\ \Gamma_t\Gamma_{x_i} = \Gamma_{x_i}\Gamma_t; & (5)\ \Gamma_t\Gamma_{\partial_i} = \Gamma_{\partial_i}\Gamma_t; & (6)\ \Gamma_{x_i}\Gamma_{\partial_i} = \Gamma_{\partial_i}\Gamma_{x_i} - 1; \\ (7)\ \Gamma_{x_i}\Gamma_{x_j} = \Gamma_{x_j}\Gamma_{x_i}; & (8)\ \Gamma_{\partial_i}\Gamma_{\partial_j} = \Gamma_{\partial_j}\Gamma_{\partial_i}; & (9)\ \Gamma_{x_i}\Gamma_{\partial_j} = \Gamma_{\partial_j}\Gamma_{x_i}. \end{array}$$

From relation (1), we obtain $\alpha_2 = \gamma_j^{(t,1)} = \gamma_j^{(t,2)} = 0$ for all j, so $\Gamma_t = \beta_2 t$. The transformation must be nonsingular, so we must have $\beta_2 \neq 0$, and again using (1) we deduce that $\alpha_1 = 1$. Using (4), we obtain that $\alpha_i^{(1)} = 0$ for all i. This, together with (5), implies that $\alpha_i^{(2)} = 0$ for all i.

From relation (2) (Γ_s commutes with Γ_{x_i}) we have $\beta_i^{(1)} = 0$, and relation (3) gives $\beta_i^{(2)} = 0$. Due to relations (6) to (9) (between Γ_{x_i} and Γ_{∂_j}) we have that the submatrix

$$\begin{pmatrix} \gamma_{i,j}^{(1,1)} & \gamma_{i,j}^{(1,2)} \\ \gamma_{i,j}^{(2,1)} & \gamma_{i,j}^{(2,2)} \end{pmatrix}$$

verifies the relations of Lemma 6, and in addition, from the relations with Γ_s, it verifies

$$\sum \gamma_i^{(s,1)}\gamma_{i,i}^{(1,2)} = \sum \gamma_i^{(s,2)}\gamma_{i,i}^{(1,1)} \qquad \sum \gamma_i^{(s,1)}\gamma_{i,i}^{(2,2)} = \sum \gamma_i^{(s,2)}\gamma_{i,i}^{(2,1)}.$$

So it is clear that we cannot normalize with respect to the variables in R. Thus we can not repeat the second step of the process towards a general PBW algebra in the way that it appears in Grigoriev (1990).

It is an open problem to obtain a general bound for the solutions of a general linear system over any PBW algebra or, at least, to give such a bound for R. We give up on this general problem at this point: with the aim of obtaining a bound for the complexity of the annihilating ideal of f^s, we will treat only the particular case of one equation of the type produced by the definition of the ideal I in Section 2.1 or I' in Section 2.2. In both cases we want to measure the complexity of computing Gröbner bases (in different rings) and we will do this by considering the equivalent problem of computing the *syzygies* of the generators of our respective ideals.

Remark 8. In the OT algorithm the calculations are computed in a Weyl algebra of $2n+4p$ variables, or more precisely in a commutative polynomial ring with $n + 3p$, (x, u, v, t) commutative variables extended with $n + p$, (∂_x, ∂_t) "differential" variables. Let us denote this algebra by A. The complexity of computing the annihilating ideal of f^s is bounded by the complexity of computing a Gröbner basis in A.

Recall that the complexity in the Weyl algebra is given by the following theorem:

Theorem 9 (*Theorem 6, Grigoriev (1990)*). *Given a solvable system in the Weyl algebra* D_n,

$$\sum_{1\le l\le s} u_{k,l} V_l = w_k, \qquad 1 \le k \le m$$

with $\deg(u_{k,l}), \deg(w_k) \le d$. *There exists a solution with* $\deg(V_l) < (md)^{2^{O(n)}}$.

As we said before, in the Briançon–Maisonobe ring R we cannot construct a similar algorithm to bound the degree of a solution for a system in general. But in our very special case, our problem is equivalent to computing the solutions of the equation

$$(s_1 + f_1 t_1)V_1 + \cdots + (s_p + f_p t_p)V_p + \left(\partial_1 + \sum_j \frac{\partial f_j}{\partial x_1} t_j\right) V_{p+1} + \cdots$$
$$+ \left(\partial_n + \sum_j \frac{\partial f_j}{\partial x_n} t_j\right) V_{p+n} = 0.$$

To simplify notation we write the preceding equation as $\sum_l Q_l V_l = 0$.

Theorem 10. *Given $f = (f_1, \ldots, f_p)$, the computation of the annihilating ideal of f^s in the Briançon–Maisonobe algebra $R = D[s_1, \ldots, s_p, t_1, \ldots, t_p]$ can be reduced to the computation of the syzygies of the generators $\partial_i + \sum_j \frac{\partial f_j}{\partial x_i} t_j$ in the Weyl algebra $D[t_1, \ldots, t_p]$.*

Proof. Trying to repeat Grigoriev's ideas, the first step is the reduction of the system to one in diagonal form. Due to the fact that we have only one equation, this step is done. Then, we need to compute $h_1^{(l)}, h^{(l)}$ for $2 \leq l \leq n + p$ such that

$$\begin{array}{rcll}
(s_1 + f_1 t_1)h_1^{(2)} & + & (s_2 + f_2 t_2)h^{(2)} & = 0 \\
 & \vdots & & \\
(s_1 + f_1 t_1)h_1^{(p)} & + & (s_p + f_p t_p)h^{(p)} & = 0 \\
(s_1 + f_1 t_1)h_1^{(p+1)} & + & (\partial_1 + \sum_j \frac{\partial f_j}{\partial x_1} t_j)h^{(p+1)} & = 0 \\
 & \vdots & & \\
(s_1 + f_1 t_1)h_1^{(p+n)} & + & (\partial_n + \sum_j \frac{\partial f_j}{\partial x_n} t_j)h^{(p+n)} & = 0.
\end{array}$$

It is easy to see that

$$[s_i + f_i t_i, s_j + f_j t_j] = 0$$
$$\left[s_i + f_i t_i, \partial_j + \sum_l \frac{\partial f_l}{\partial x_j} t_l\right] = s_i \left(\sum_l \frac{\partial f_l}{\partial x_j} t_l\right) + f_i t_i \partial_j - \partial_j f_i t_i - \left(\sum_l \frac{\partial f_l}{\partial x_j} t_l\right) s_i$$
$$= t_i s_i \frac{\partial f_i}{\partial x_j} + t_i \frac{\partial f_i}{\partial x_j} + \sum_{l \neq i} t_l s_i \frac{\partial f_l}{\partial x_j} + t_i f_i \partial_j - t_i f_i \partial_j - t_i \frac{\partial f_i}{\partial x_j} - \sum_l \frac{\partial f_l}{\partial x_j} t_l s_i = 0$$

and we obtain $h^{(l)} = s_1 + f_1 t_1$ for all $l \geq 2$.

These are the elements we need to normalize, and they are almost in normal form with respect to the variable s_1. This form is required to make the division of the solutions V_l, $l \geq 2$, by $h^{(l)}$ with respect to a lexicographical ordering with leading term s_1. We obtain a remainder $\bar{V}_l$ such that $\deg_{s_1}(\bar{V}_l) < \deg_{s_1}(h^{(l)}) = 1$, so s_1 does not appear in $\bar{V}_l$.

So $V_l = h^{(l)}\bar{\bar{V}}_l + \bar{V}_l$, and adding the relation $Q_1 h_1^{(l)} + Q_l h^{(l)} = 0$ multiplied by $-\bar{\bar{V}}_l$ to our initial equation, we obtain

$$Q_1\bar{V}_1 + Q_2\bar{V}_2 + \cdots + Q_{n+p}\bar{V}_{n+p} = 0$$

with Q_i, $\bar{V}_i$ without s_1 for $i \geq 2$, so $\bar{V}_1 = 0$, where $\bar{V}_1 = V_1 - h_1^{(2)}\bar{\bar{V}}_2 - \cdots - h_1^{(n+p)}\bar{\bar{V}}_{n+p}$. We have then the new equation

$$Q_2\bar{V}_2 + \cdots + Q_{n+p}\bar{V}_{n+p} = 0$$

in a Briançon–Maisonobe algebra $\mathbf{C}[s_2, \ldots, s_p, t_1, \ldots, t_p, x, \partial]$.

Repeating the process for $Q_2, \ldots, Q_p$, we reduce our problem to solving

$$\left(\partial_1 + \sum_j \frac{\partial f_j}{\partial x_1} t_j\right) V_{p+1} + \cdots + \left(\partial_n + \sum_j \frac{\partial f_j}{\partial x_n} t_j\right) V_{p+n} = 0$$

in the Weyl algebra $D[t_1, \ldots, t_p]$.

Remark 11. As a consequence of Theorem 10, the bound for the complexity of computing the annihilating ideal of f^s in R is bounded by the complexity of computing a Gröbner basis in a Weyl algebra with $3p$ *variables fewer* that the one required by the OT method. Although the complexity of computing these objects in any case is known to be double exponential (with respect to the number of variables and the total degree of the generators of the ideal) by Theorem 9, it is clear that the reduction of $3p$ variables in the BM method is a significant advantage, both theoretically and in practice, as is shown in examples (see Castro-Ucha, 2004).

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4. Appendix. Experimental data

In Tables 1–3 we give some examples for which the superiority of the Briançon–Maisonobe method is clear. They have been tested[3] using SINGULAR::PLURAL 2.1 (see Greuel et al. (2003)) on a PC Pentium IV, 1 Gb RAM and 3.06 GHz running under Windows XP.

SINGULAR::PLURAL 2.1 is a system for non-commutative general purposes, so the calculations in our algebras are not supposed to be optimal. We present the data only for the sake of comparing the two methods in the same system. In the case of the Briançon and Maisonobe (2002) method we have used a pure lexicographical ordering, while for the Oaku and Takayama (1999) method we have used typical elimination ordering. These are the orderings with the best results for each case.

[3] The CPU times must be considered as approximations: as is explained in the SINGULAR::PLURAL 2.1 Manual, the command `timer` is not absolutely reliable due to the shortcomings of the Windows operating system.

Table 1
CPU times for the computation of $Ann\, f^s$

f	Briançon–Maisonobe method	Oaku–Takayama method
$x^3 + xy^2 + z^2$	<0.01 s	0.39 s
$x^4 + y^3 + z^2$	<0.01 s	0.39 s
$yx^3 + y^3 + z^2$	0.06 s	3.97 s
$x^3 + y^2 + z^2$	<0.01 s	0.02 s
$x^5 + y^2 + z^2$	<0.01 s	4.66 s
$x^7 + y^2 + z^2$	<0.01 s	298.56 s
$x^4 + y^5 + xy^4$	0.56 s	E (>12 h)

Table 2
CPU times for the computation of $Ann\, f_1^{s_1} f_2^{s_2}$

f_1	f_2	Briançon–Maisonobe method	Oaku–Takayama method
$x^3 + y^2$	$x^2 + y^3$	0.7 2s	6363.97 s
$x^5 + y^3$	$x^3 + y^5$	3.53 s	E (>6 h)
$x^7 + y^5$	$x^5 + y^7$	11.84 s	E (>6 h)
$x^3 + y^2$	$xz + y$	<0.01 s	9.73 s
$x^5 + y^2$	$xz + y$	<0.01 s	1568.59 s
$x^{11} + y^5$	$xz + y$	3 s	E (>6 h)

Table 3
CPU times for the computation of $Ann\, f_1^{s_1} \cdots f_p^{s_p}$

f_1	f_2	f_3	Briançon–Maisonobe method	Oaku–Takayama method
$x + y$	$x - y$	$x^2 + y$	<0.01	29.46 s
$x + y$	$x^2 + y$	$x + y^2$	2.64 s	E
$x + y$	$x^2 + y$	$x^2 + y^3$	116.24 s	E
$x + y$	$x^2 + y$	$x^3 + y^2$	1728.41 s	E

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