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A SYMMETRIC PIECEWISE-LINEAR CHAOTIC SYSTEM WITH A SINGLE EQUILIBRIUM POINT

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In this paper, we propose a new autonomous electronic oscillator designed with some modifications of the well-known Wien bridge oscillator. In the mathematical model planned for such a circuit, the nonlinearity in the operational amplifier saturation is considered and reference is made to the only equilibrium point at the origin of phase-space. We show how the relation between the bifurcation parameters starts stable oscillations, providing an example for chaotic behavior and bifurcations diagrams. Finally, we conclude with a brief summary of the oscillators operation using a parameters plane.

Keywords: Piecewise-linear system; universal Chua's circuit; chaotic behavior.

1. Introduction

In a recent paper in this journal [Yang & Chua, 2000], it has been shown that the existence of diverse equilibrium points is not essential to generate chaos in piecewise-linear systems. In fact, they give mathematical models with only one equilibrium point capable of exhibiting chaotic behavior.

To show an example of such a system, we propose a simple electronic autonomous oscillator implemented with one operational amplifier, four linear resistors, two capacitors and one inductance, which is, in fact, a modified Wien bridge oscillator [González-López, 1998; González-López *et al.*, 1999a; González-López *et al.*, 2000a].

There are some previous works where some modifications of the Wien bridge oscillator are studied to obtain chaotic behavior. Among others, we can quote the addition of a nonlinear resistor in parallel with another linear [Namajuanas & Tamasevicius, 1995], the coupling of a Chua diode with a Wien bridge oscillator in parallel [Morgull, 1955] or the different circuits shown in [Elwakil & Soliman, 1997] where a nonlinear resistor is combined with various RC circuits. The Wien bridge oscillator modification proposed in this paper has not been studied by any author, nor we do have any previous knowledge about it.

2. A Modified Wien Bridge Oscillator

Starting from a classic Wien bridge oscillator [Millman, 1981; González-López *et al.*, 1999b], if we replace the linear resistor, which is in series with the capacitor, with an inductance L which has resistance R_L, we will have the autonomous electronic oscillator shown in Fig. 1. It can be seen that two voltage dividers produce negative feedback (resistances R_1 and R_2) and positive feedback, through capacitors C and C_1, the inductance L (and associated R_L) and resistor R.

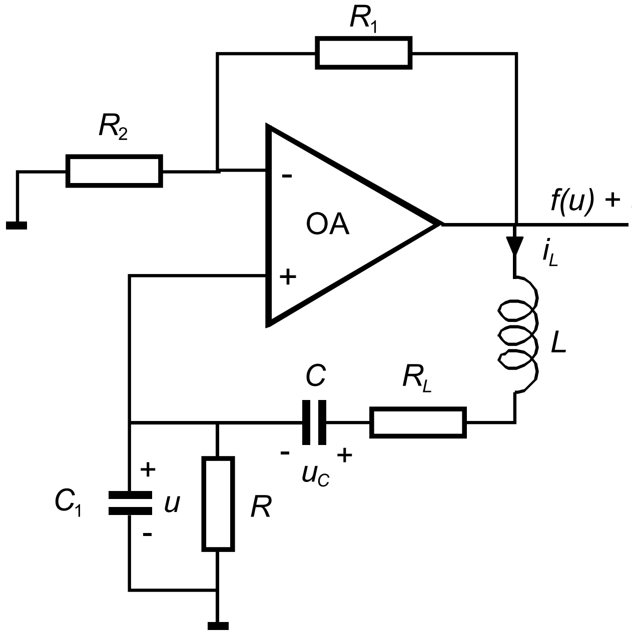

Fig. 1. Modified Wien bridge oscillator.

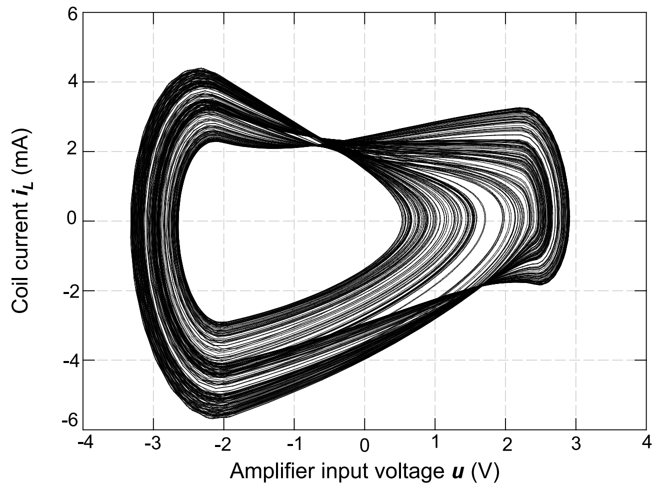

Fig. 3. Phase portrait of a chaotic attractor of the modified Wien bridge oscillator obtained by simulation, $A = 5.8$, $R = 23\,\mathrm{K\Omega}$. The other parameters are indicated in Sec. 2.

To implement the circuit, we choose the operational amplifier μA741C, powered by a voltage source with $\pm 15\,\mathrm{V}$. R_1 has $100\,\mathrm{K\Omega}$ and R_2 is a variable linear resistor of $20\,\mathrm{K\Omega}$ which makes possible to change the A gain of the noninverter amplifier.

For the remaining elements we have chosen the values: $C = 100\,\mathrm{nF}$, $C_1 = 470\,\mathrm{nF}$, $L = 100\,\mathrm{mH}$ and R is a linear potentiometer just like R_2. The R_L resistor was tested in $90\,\Omega$.

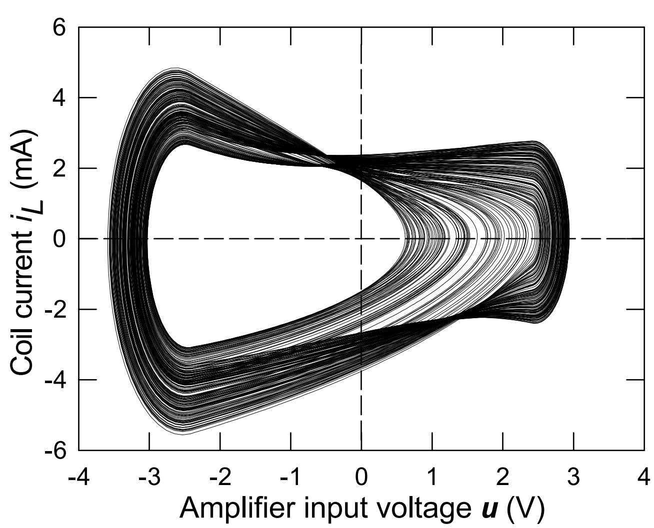

Fig. 2. Phase portrait experimentally obtained in which a chaotic attractor of the modified Wien bridge oscillator is shown. The voltage u is the horizontal axis, and the vertical axis is the current i_L. The parameters are indicated in Sec. 2.

In Fig. 2 an experimental phase portrait is shown, where the horizontal axis is the voltage u at the amplifier input and the vertical axis represents the current i_L. This experimental result agreed with the simulation model shown in Fig. 3.

3. Mathematical Model

The circuit can be modeled by an autonomous system with three state variables, namely

$$\begin{aligned} L\frac{di_L}{dt} &= f(u) - R_L i_L - u_C \\ C\frac{du_C}{dt} &= i_L \\ C_1\frac{du}{dt} &= i_L - \frac{u}{R} \end{aligned} \tag{1}$$

where $f(u)$ is, as can be seen below, a nonlinear function which represents the difference between the input and output amplifier voltages. The graph is shown in Fig. 4. We remark that this function is a three-segment piecewise-linear function [Chua & Kang, 1977] which corresponds to the amplifier saturations and active operation region. Also assuming the same value for the saturation of positive and negative voltage, we can write it as

$$f(u) = \frac{A}{2}\left(\left|u + \frac{V_s}{A}\right| - \left|u - \frac{V_s}{A}\right|\right) - u \tag{2}$$

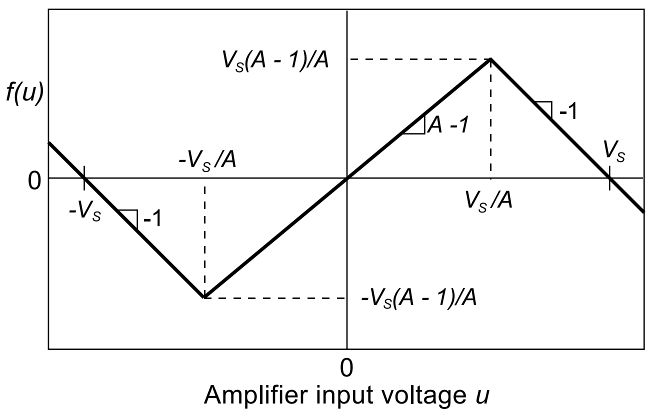

Fig. 4. Three-segment piecewise-linear function $f(u)$. The positive and negative saturation voltages are equal in absolute value.

V_s is the output voltage of the amplifier in saturation and A is the gain of the amplifier:

$$A = 1 + \frac{R_1}{R_2} \tag{3}$$

If it is assumed that resistance value R is finite and nonzero the system has only one equilibrium point in the origin.

By means of linearization in the equilibrium point it is obtained:

$$\begin{pmatrix} \dot{i}_L \\ \dot{u}_C \\ \dot{u} \end{pmatrix} = \begin{pmatrix} -\frac{R_L}{L} & -\frac{1}{L} & \frac{A-1}{L} \\ \frac{1}{C} & 0 & 0 \\ \frac{1}{C_1} & 0 & -\frac{1}{C_1 R} \end{pmatrix} \begin{pmatrix} i_L \\ u_C \\ u \end{pmatrix} \tag{4}$$

The corresponding characteristic equation is:

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0 \tag{5}$$

At that equilibrium point, a bifurcation takes place characterized by the sudden appearance of stable periodic oscillations, similar to that of Hopf's bifurcation in systems, whose nonlinear functions change progressively with slope [Freire *et al.*, 1999; Freire *et al.*, 1997]. In this case, the coefficients of (5) must fulfill the conditions:

$$\begin{gathered} a_0 - a_1 a_2 = 0 \\ \omega^2 = a_1 > 0 \end{gathered} \tag{6}$$

From the previous expressions it is deduced that the pulsation of the initial oscillations in the linearized system is:

$$w = \sqrt{\frac{1}{CC_1RR_L + LC}} \tag{7}$$

The Hopf-like bifurcation takes place for the following relation between the parameters:

$$A = 1 + \frac{R_L}{R} + \frac{C_1}{C} - \frac{C_1 L}{CC_1RR_L + LC}, \tag{8}$$

If in the previous expression we take $R \to \infty$,

$$A = \frac{C_1}{C} + 1 \tag{9}$$

When oscillations appear, linearization at the origin has a real and negative eigenvalue. The other two eigenvalues are real positive or complex with positive real portion, so it always presents a saddle point.

The system can produce single periodic oscillations or show a wide complex chaotic behavior [González-López, 1998; González-López *et al.*, 2000a], which can be verified by Lyapunov exponents determinations and with the presence of homoclinic connections [González-López, 1998] (Shilnikov's chaos [Silva, 1993]).

4. Some Simulation Results

To compute the orbits of this system an integration method has been used by linear sections, which consists in determining the symbolic solution of the corresponding linear differential equation for each one of them, taking in each section, as initial condition, the values of the common point with the previous section, calculated before.

For the same values of the parameters, and depending on the initial conditions, three different periodic attractors or three chaotic attractors can coexist, or both of them. The stable behaviors can coexist with numerous periodic unstable orbits and some intermittent behaviors have been detected in time, a periodic and a chaotic attractor appear alternatively for the same initial conditions.

For different results that we subsequently present, phase portrait and bifurcations diagram, only the gain A and resistance R parameters have been changed, the others stay unchanged with the values set in Sec. 2. The saturation voltages have been supposed $\pm 14\,\text{V}$ in all cases.

4.1. *Strange attractor*

Among the diverse morphology of strange attractor presented in this system, we have selected the one shown in Fig. 3, with a clearly asymmetrical look, mentioned in Sec. 2. In this case, the gain $A = 5.8$ and resistance $R = 23\,\text{K}\Omega$.

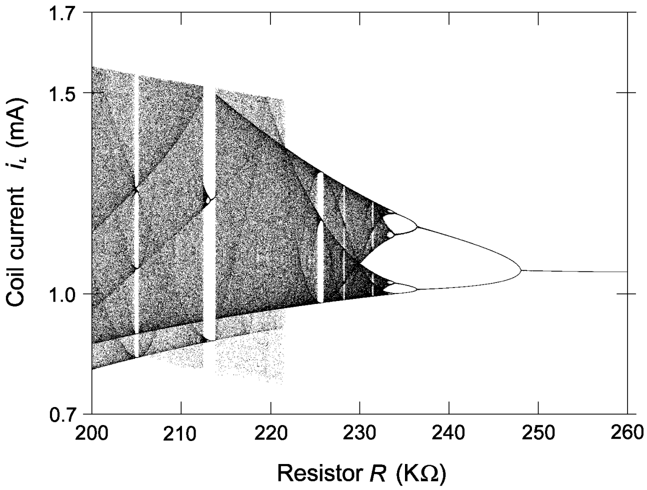

Fig. 5. Bifurcations diagram obtained in changing the parameter R (abscissa) by means of simulation. The ordinate is the coil current in a Poincaré section with the plane $u = -V_s/A$ leaving saturation, the gain $A = 5.7$. The other parameters are indicated in text.

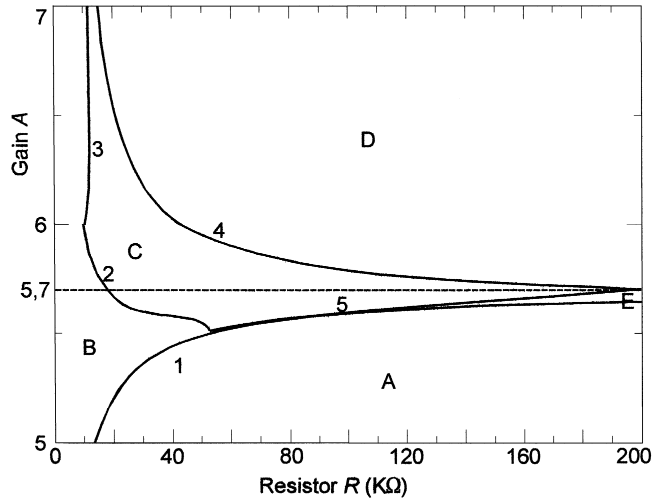

Fig. 6. Different behavior zones of the modified Wien bridge oscillator and the lines that delimit these zones, according to the parameter R (abscissa) and A (ordinate). Line 1 is the representation of Eq. (8). The rest of the lines have been obtained across simulation.

4.2. *Bifurcation diagram*

The bifurcation diagram shown in Fig. 5, has been obtained for a fixed value from the gain A, having changed the R parameter, represented in x axis. The y axis is the current i_L in the inductance L into a Poincaré section which corresponding to the plane $u = -V_s/A$ when passing from saturation to a linear zone.

The graph has been obtained for a gain $A = 5.70$, that corresponds to the relationship (9), possibly the most interesting operation zone of the oscillator. R values vary between $200\,\mathrm{K}\Omega$ and $260\,\mathrm{K}\Omega$.

The diagram shows a Feigenbaum's cascade and periodic zones can be appreciated inside the chaotic ones.

5. Summary of Operation

To make a summary of the different ways of operation of the modified Wien bridge oscillator and the range of values comprised by its parameters, a plot is obtained as shown in Fig. 6, which is a parameters plane where R is taken as the resistance on the x axis and the gain for small signal A on the y axis.

Line 1 has been calculated using Eq. (8) and the others have been determined using simulation. In the plane, the following zones can been distinguished:

Zone A. The system is asymptotically stable, so oscillations are not produced and the oscillator does not operate as such. It is delimited by the beginning line of stable oscillations, indicated as 1 in such a plot. This line is asymptotic to the gain $A = 5,70$ for $R \to \infty$.

Zone B. Periodic oscillations are produced whose period is fundamental. It is limited by several lines:

Line of $R = 0$. When this resistor is in short circuit, the positive feedback is cancelled out and the oscillator cannot operate (asymptotically stable).

Line 1. As it has already been indicated, line 1 corresponds to the bifurcation which produces the sharp appearance of stable oscillations and limits the operation zone of periodic oscillations of fundamental period for small values of the gain. Next to line 1 the oscillator gives an almost sinusoidal waveform.

Line 2. In this line a periodic split is produced, obtaining a waveform with a period almost double that of the fundamental. After the periodic split, it gradually enters into a chaotic situation.

Line 3. In this line a sharp transition from the periodic form (of fundamental period) proceeds to chaos.

Zone C. In this part, chaos prevails where small intervals of numerous periodic behaviors appear. It is limited, in addition to lines 2 and 3 commented above, by the following:

Line 4. Is the frontier where periodic relaxation forms begin. It is tangent to the $A = 5.70$ horizontal line gain. In the proximities of this line we have fundamental period periodic movements.

Line 5. Is the frontier with a fundamental period periodic behavior zone. A part of this line coincides with line 1 of the bifurcation, therefore, in this case, a strange chaotic situation proceeds directly to the oscillations cease. The behavior is same as in line 4 with respect to the gain $A = 5,70$.

Zone D. This part is mainly made up from periodic behaviors with relax oscillations. Near line 4 small chaotic zones appear between those periodic oscillations. The pulse number by these waveforms period grows for such a line.

Zone E. Corresponds to fundamental period periodic behavior. Actually, it is a prolongation of zone B and is limited by lines 1 and 5, already commented before.

6. Conclusions

This modified Wien bridge oscillator circuit presents an impressive behavior variety with only one equilibrium point, coexisting with as far as three periodic and/or chaotic attractors, time intermittence phenomena, homoclinic connections, diversity of unstable limit cycle, etc. And this seems to offer a lot of possibilities for future studies.

On the other hand, the good approximation obtained between the experimentally determined values and the simulation results, seem to validate, for this oscillator, the mathematic model used, despite its simplicity.

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Obituary: During the revision process of this manuscript, our friend and coauthor Rafael González López suddenly died from a heart attack. The authors wish to acknowledge his dedication in technical teaching and research. In memoriam.

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