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Distributions and Models

Distribution of a Sum of Weighted Central Chi-Square Variables

ANTONIA CASTAÑO-MARTÍNEZ[1] AND FERNANDO LÓPEZ-BLÁZQUEZ[2]

[1]Departamento de Estadística e Investigación Operativa, Universidad de Cádiz, Spain
[2]Departamento de Estadística e Investigación Operativa, Universidad de Sevilla, Spain

We derive a Laguerre expansion for the inverse Laplace transform, based on the estimation problem in the gamma distribution. This procedure is used to obtain the density and distribution functions of a sum of positive weighted central chi-square variables as a series in Laguerre polynomials. The formulas so obtained will depend on certain parameters which adequately chosen will give some expressions already known in the literature and some new ones. Finally, we obtain bounds for the truncation error in the numerical approximations.

Keywords Gamma distribution; Inverse Laplace transform; Laguerre expansion; Truncation error; Unbiased estimation.

1. Introduction

The statistics employed in many test and estimation procedures are expressible as quadratic forms in normal variables. The sample variance is one of the most common examples. We are interested in definite central quadratic forms. It is well-known that we can make an orthogonal transformation reducing this type of quadratic form to its canonical form, i.e., to a positive linear combination of independent central chi-square variables.

In addition, the distribution of a positive linear combination of central chi-square random variables with any degrees of freedom arises in the null asymptotic theory of goodness-of-fit tests, see Rao and Scott (1984). Other applications are found in connection with ballistics problems, in biology, in communications theory, etc. (see Jensen and Solomon, 1972).

Address correspondence to Antonia Castaño-Martínez, Departamento de Estadística e Investigación Operativa, Universidad de Cádiz, Spain; E-mail: antonia.castano@uca.es

The problem of obtaining the distribution of a positive linear combination of central chi-square random variables has been addressed by many authors. A comprehensive survey and literature review of the subject was given by Johnson et al. (1994). Various representations of this distribution have been given including certain mixtures of chi-square distributions (see Robbins, 1948; Robbins and Pitman, 1949), Laguerre series expansions (see Bhattacharyya, 1945; Gurland, 1955, 1956; Kotz et al., 1967; Gideon and Gurland, 1976), and power series expansions (Pachares, 1955). "Though all three types of expansions yield correct convergent representations, it has been found that the Laguerre series representation is computationally the most convenient and effective throughout the range of interesting values" (see Mathai and Provost, 1992, p. 117 and this expansion was used by Johnson and Kotz, 1968 to tabulate this distribution).

We consider $Q_n = \sum_{i=1}^{n} \alpha_i X_i$, where the α's are known positive constants and the X's are independent chi-square variables with ν_i degrees of freedom, respectively. Our aim is to obtain the density and distribution functions of Q_n as a Laguerre series expansion. We also derive bounds on the truncation error in the given expansions and compare our results with those given by Kotz et al. (1967).

The method that we present is based on the inverse Laplace transform. The method of inversion that we propose in Sec. 2 is based on the property of uniqueness of minimum variance unbiased estimators (MVUE) in the gamma distribution. Then, in Sec. 3, we apply this method for the obtention of the density and distribution functions of Q_n. In Sec. 4, we present some necessary results in order to study, and in Sec. 5, the truncation error of the proposed expansions are presented. Moreover we give empirical results on the truncation errors.

2. The Inversion of Laplace Transforms

The main difficulty in applying Laplace-transform techniques is the determination of the original function from its transform. For a review of analytical methods, see Spiegel (1991).

In many cases, analytical methods fail and numerical methods must be used. The best known numerical methods for the inversion of the Laplace transform are based on the numerical integration of the Bromwich integral or on the expansion of the original function in a series of orthogonal functions, particularly orthogonal exponential functions and Laguerre polynomials. In Piessens and Branders (1971) we found a discussion about the reasons for which Laguerre expansions are preferable to expansions in orthogonal exponential functions.

In this section, we obtain a method to invert Laplace transforms based on unbiased estimation in the Gamma distribution. Therefore, first we will treat the unbiased estimation problem in this distribution.

2.1. *Estimation in the Gamma Distribution*

Let Y be a random variable following a $Ga(p, \lambda)$ distribution with $p > 0$ known and $\lambda > 0$ unknown parameter and density function $g(y, \lambda)$, given by

$$g(y, \lambda) = \frac{\lambda^p}{\Gamma(p)} y^{p-1} e^{-\lambda y}, \quad y > 0.$$

This random variable belongs to the natural exponential family with quadratic variance function, and so it is possible to use the mean, $\mu = p/\lambda$, as the parameter of this distribution, see Morris (1982).

For each $\mu > 0$, consider the space of functions

$$\mathscr{L}^2_\mu = \left\{T : \int T^2(s) f(s,\mu)ds < \infty\right\}$$

with $f(s,\mu) = g(s, p/\mu)$.

As usual in the theory of $\mathscr{L}^2$ spaces, we will consider two functions, $T_1, T_2 \in \mathscr{L}^2_\mu$ to be equivalent if $T_1(x) = T_2(x)$ a.e. With the inner product $\langle T_1, T_2\rangle_\mu = E_\mu[T_1(x)T_2(x)]$, the space $(\mathscr{L}^2_\mu, \langle\cdot,\cdot\rangle_\mu)$ is a Hilbert space for any $\mu > 0$.

Provided that the variance function is quadratic, the functions

$$P_j(x,\mu) = (-1)^j \left(\frac{\mu}{p}\right)^j j! \mathrm{L}_j^{(p-1)}\left(\frac{px}{\mu}\right), \quad j \geq 0, \tag{1}$$

where $\mathrm{L}_j^{(\alpha)}(x) = \sum_{m=0}^{j} \binom{j+\alpha}{j-m} \frac{(-x)^m}{m!}$, $\alpha > 0$, is the jth generalized Laguerre polynomial, form a complete system of monic orthogonal polynomials with respect to the gamma density.

The polynomials defined in (1) satisfy the following properties, see Morris (1982):

(P1) The recurrence relation:

$$P_{k+1} = \left(P_1 - \frac{2k\mu}{p}\right)P_k - k\left\{1 + \frac{k-1}{p}\right\}\frac{\mu^2}{p}P_{k-1}, \quad k \geq 1,$$
$$P_0 = 1, \quad P_1 = x - \mu.$$

(P2) $\frac{d^j}{d\mu^j} f(x,\mu) = \left(\frac{p}{\mu}\right)^j P_j(x,\mu) f(x,\mu)$

(P3) $\langle P_k, P_j\rangle_\mu = \delta_{k,j} j! \left(\frac{\mu}{p}\right)^{2j} (p)_j$, where $\delta_{k,j}$ is the Kronecker's delta and $(a)_j = a(a+1)\cdots(a+j-1)$ if $j \geq 1$ and $(a)_0 = 1$.

Definition 1. A function $h(\mu)$ is said to be MVU-estimable if there exists a function $T \in \mathscr{L}^2_\mu$, $\forall\mu > 0$, satisfying: $E[T(X)] = h(\mu)$, for all $\mu > 0$.

In such a case, T is the minimum variance unbiased estimator, MVUE, of $h(\mu)$. The set of all the MVU-estimable functions will be denoted by $\mathscr{U}$.

We will obtain an expression for T, in the next theorem.

Theorem 1. *Let h be a MVU-estimable function, then its MVU-estimator admits the following expression*:

$$T(x) = \sum_{j=0}^{\infty} \frac{(-\mu)^j h^{(j)}(\mu)}{(p)_j} \mathrm{L}_j^{(p-1)}\left(\frac{px}{\mu}\right), \quad \forall\mu > 0, \ (a.e.) \tag{2}$$

with $h^{(j)}(\mu) = \frac{d^j}{d\mu^j} h(\mu)$.

Proof. This is a particular case of the formula (3.6) of Morris (1983). □

Remark 1. One expression for the variance of T is easily obtained by the orthogonality property of Laguerre polynomials

$$\mathrm{Var}[T(x)] = \sum_{k=1}^{\infty} \frac{(\mu)^{2k}(h^{(k)}(\mu))^2}{k!(p)_k}. \tag{3}$$

From the a.s.-uniqueness of the MVU estimators, it follows that the choice of $\mu > 0$ in the right-hand side (rhs) of (2) is arbitrary. Adequate choices of this parameter may yield formulas computationally efficient as we will see later.

From the unbiasedness condition, $E_\mu[T(X)] = h(\mu)$, it is easy to obtain an alternative expression for the unbiased estimator based on the inverse Laplace transform (denoted by $\mathscr{L}^{-1}$):

$$T(x) = \frac{\Gamma(p)}{x^{p-1}} \mathscr{L}^{-1}\left(\left(\frac{p}{\mu}\right)^{-p} h(\mu)\right)(x), \quad x > 0. \tag{4}$$

2.2. *The Inversion of Laplace Transforms*

From the uniqueness of the MVU estimators, we can obtain the following result which gives us an expression for inverse Laplace transforms.

Theorem 2. *Let $G(\lambda)$, $\lambda > 0$, be a function such that for certain $p > 0$, $h(\mu) = (p/\mu)^p G(p/\mu)$ is MVU-estimable function, then*:

$$\mathscr{L}^{-1}(G(\lambda))(x) = \frac{x^{p-1}}{\Gamma(p)} \sum_{j=0}^{\infty} \frac{(-\mu_0)^j h^{(j)}(\mu_0)}{(p)_j} \mathrm{L}_j^{(p-1)}\left(\frac{px}{\mu_0}\right), \quad (a.e.) \tag{5}$$

for any $\mu_0 > 0$.

Proof. It is immediate from (2) and (4). □

Note that the choice of μ_0 is irrelevant. For instance, if we consider $\mu_0 = p = a + 1$ we obtain the expression given by Piessens and Branders (1971).

3. Computation of the Distribution of Q_n

Our aim in this section is to obtain the density and distribution functions of Q_n as applications of Theorem 2.

Let f be the density of Q_n, then it can be easily shown that its Laplace transform is:

$$\mathscr{L}(f(x))(\lambda) = \prod_{i=1}^{n} (1 + 2\alpha_i \lambda)^{-\nu_i/2} = G(\lambda).$$

We consider the following transformation to get a better approximation:

$$H(\lambda) = G\left(\frac{\lambda - 1}{2\beta}\right) = \beta^{\nu/2} \prod_{i=1}^{n} (\beta + \alpha_i(\lambda - 1))^{-\nu_i/2},$$

with $\beta > 0$. So, using standard properties of Laplace transforms:

$$f(x) = \mathscr{L}^{-1}(G(\lambda))(x) = \mathscr{L}^{-1}(H(1+2\beta\lambda))(x)$$
$$= \frac{e^{-\frac{x}{2\beta}}}{2\beta}\mathscr{L}^{-1}(H(\lambda))\left(\frac{x}{2\beta}\right). \tag{6}$$

We now apply Theorem 2 to invert H.

Letting $h(\mu) = (p/\mu)^p H(p/\mu)$, with $p = v/2$ and $v = \sum_{i=1}^{n} v_i$,

$$h(\mu) = (\beta v/2)^{v/2}\prod_{i=1}^{n}(\beta\mu + \alpha_i(v/2-\mu))^{-v_i/2},$$

and

$$h^{(k)}(\mu) = (-1)^k(k-1)!\sum_{j=0}^{k-1}\frac{(-1)^j h^{(j)}(\mu)}{j!}\sum_{i=1}^{n}\frac{v_i}{2}\left(\frac{\beta-\alpha_i}{\beta\mu+\alpha_i(v/2-\mu)}\right)^{k-j}.$$

Then, if we let $c_k = (-\mu_0)^k h^{(k)}(\mu_0)/k!$, we have:

$$\mathscr{L}^{-1}(H(\lambda))(x) = \frac{x^{(v/2)-1}}{\Gamma(v/2)}\sum_{k=0}^{\infty}\frac{k!c_k}{(v/2)_k}\mathrm{L}_k^{(\frac{v}{2}-1)}\left(\frac{vx}{2\mu_0}\right), \quad \forall\mu_0 > 0,$$

and from (6)

$$f(x) = \frac{e^{-\frac{x}{2\beta}}}{(2\beta)^{v/2}}\frac{x^{(v/2)-1}}{\Gamma(v/2)}\sum_{k=0}^{\infty}\frac{k!c_k}{(v/2)_k}\mathrm{L}_k^{(\frac{v}{2}-1)}\left(\frac{vx}{4\beta\mu_0}\right), \quad \forall\mu_0 > 0, \tag{7}$$

with

$$\begin{aligned} c_k &= \frac{1}{k}\sum_{j=0}^{k-1}c_j d_{k-j}, \quad k \geq 1, \\ c_0 &= \left(\frac{v}{2\mu_0}\right)^{v/2}\prod_{i=1}^{n}\left(1+\frac{\alpha_i}{\beta}\left(\frac{v}{2\mu_0}-1\right)\right)^{-v_i/2}, \end{aligned} \tag{8}$$

$$d_j = \frac{1}{2}\sum_{i=1}^{n}v_i\left(\frac{1-\frac{\alpha_i}{\beta}}{1+\frac{\alpha_i}{\beta}\left(\frac{v}{2\mu_0}-1\right)}\right)^j, \quad j \geq 1. \tag{9}$$

If we let $\mu_0 = v/2$ in (7), we obtain the expansion given by Kotz et al. (1967). However, we can consider other choices of the parameter in order to improve the speed of convergence of the series in the rhs of (7).

Similarly, we obtain the following expression for the distribution function:

$$F(x) = \frac{e^{-\frac{x}{2\beta}}}{(2\beta)^{1+v/2}}\frac{x^{(v/2)}}{\Gamma(v/2+1)}\sum_{k=0}^{\infty}\frac{k!m_k}{(v/2+1)_k}\mathrm{L}_k^{(\frac{v}{2})}\left(\frac{(v+2)x}{4\beta\mu_0}\right), \quad \mu_0 > 0, \tag{10}$$

with $p = v/2 + 1$, and

$$\begin{aligned} m_k &= \frac{1}{k}\sum_{j=0}^{k-1} m_j l_{k-j}, \quad k \geq 1, \\ m_0 &= \left(\frac{p}{\mu_0}\right)^{v/2} \frac{2\beta p}{p-\mu_0} \prod_{i=1}^{n} \left(1 + \frac{\alpha_i}{\beta}(p/\mu_0 - 1)\right)^{-v_i/2}, \end{aligned} \tag{11}$$

$$l_j = \left(\frac{-1}{p/\mu_0 - 1}\right)^j + \frac{1}{2}\sum_{i=1}^{n} v_i \left(\frac{1-\alpha_i/\beta}{1+\alpha_i/\beta\,(p/\mu_0 - 1)}\right)^j, \quad j \geq 1. \tag{12}$$

In this case, we offer an alternative expression for the distribution function without knowing the density function. Most of the authors in the literature obtain an expression for the distribution function from the density function, as was done for example by Kotz et al. (1967).

An important advantage of this method is that we propose a set of equivalent expansions depending on a parameter, and so adequate choices of this parameter give some of the already known expressions as well as computationally efficient expressions.

On the other hand, as our objective is to evaluate these formulas we study the errors produced when the infinite series given in (7) and (10) are truncated. So, in the following section we derive some bounds for the truncation error.

4. Preliminary Results

The following results give us bounds for the Laguerre polynomials as well as for the coefficients expressed in (8) and (11), respectively. They are necessary to bound the truncation error.

Lemma 1. *A classical global uniform* (*w.r.t. n, x and* α) *estimate given by Szegö* (1979) *for the Laguerre polynomials is*:

$$\begin{aligned} \left|\mathrm{L}_k^{(\alpha)}(x)\right| &\leq \frac{(\alpha+1)_k}{k!}\exp\left(\frac{x}{2}\right), \quad \alpha \geq 0, \\ \left|\mathrm{L}_k^{(\alpha)}(x)\right| &\leq \left(2 - \frac{(\alpha+1)_k}{k!}\right)\exp\left(\frac{x}{2}\right), \quad -1 < \alpha < 0. \end{aligned} \tag{13}$$

Lemma 2. *Let* c_k *as given in* (8), *then*:

$$|c_k| \leq |c_0| \frac{(v/2)_k}{k!}\xi^k, \quad \forall k \geq 0,$$

with $\xi = \max_i \left|\frac{1-\alpha_i/\beta}{1-\alpha_i/\beta+\alpha_i/\beta(p/\mu_0)}\right|$.

Proof. It is immediate by induction. □

Similarly, the following lemma is proved by induction.

Lemma 3. *Let* m_k *given in* (11), *then*

$$|m_k| \leq |m_0| \frac{(v/2+1)_k}{k!} \varepsilon^k, \quad \forall k \geq 0,$$

where $\varepsilon = \max\left(\left|\frac{-1}{p/\mu_0 - 1}\right|, \gamma\right)$, $\gamma = \max_i \left|\frac{1-\alpha_i/\beta}{1-\alpha_i/\beta+\alpha_i/\beta(p/\mu_0)}\right|$, *and* $p = v/2+1$.

Remark 2. If $\mu_0 < p/2$, then $0 < \varepsilon < 1$.

5. Truncation Error and Numerical Results

Given the bounds obtained for the Laguerre polynomials and their coefficients, we study the truncation error associated with the proposed expansions.

Consider the truncation error for the density function as:

$$\mathscr{E}_N(f, x, \mu_0, \beta) = \left| \frac{e^{-\frac{x}{2\beta}}}{(2\beta)^{v/2}} \frac{x^{(v/2)-1}}{\Gamma(v/2)} \sum_{k=N+1}^{\infty} \frac{k! c_k}{(v/2)_k} \mathrm{L}_k^{(\frac{v}{2}-1)}\left(\frac{vx}{4\beta\mu_0}\right) \right|.$$

Kotz et al. (1967) proposed the following bound:

$$\mathscr{E}_N(f, x, \mu_0, \beta) \leq \frac{e^{-\frac{x}{4\beta}}}{(2\beta)^{v/2}} \frac{x^{(v/2)-1}}{\Gamma(v/2)} \left(1-\phi^{1/2}\right)^{-v/2-1} \phi^{(N+1)/2}, \tag{14}$$

with $\phi = \max_i \left|1 - \frac{\alpha_i}{\beta}\right|$.

We obtain a better bound from Lemma 2 and Lemma 1 for $v \geq 2$,

$$\mathscr{E}_N(f, x, \mu_0, \beta) \leq \frac{e^{-\frac{x}{2\beta}}}{(2\beta)^{v/2}} \frac{x^{(v/2)-1}|c_0|}{\Gamma(v/2)} \exp\left(\frac{vx}{8\beta\mu_0}\right) \sum_{k=N+1}^{\infty} a_k, \tag{15}$$

with

$$a_k = \frac{\xi^k (v/2)_k}{k!}.$$

The bound (15) is well defined since the series of general term a_k is absolutely convergent if $0 < \xi < 1$, see Remark 2. As a consequence we have that the expansion

Table 1

Bounds for the truncation error of the density of $Q_3 = 0.6\chi_1^2 + 0.3\chi_1^2 + 0.1\chi_1^2$

x	Kotz et al.'s bound (14)	$\mu_0 = p = v/2$ (15)	$\mu_0 = p/10 = v/20$ (15)
0.1	1.756687	0.009402	$0.2093 \cdot 10^{-13}$
0.7	3.027730	0.016575	$0.1707 \cdot 10^{-11}$
2	2.02211	0.010823	$0.4858 \cdot 10^{-8}$
3	1.212397	$0.6489 \cdot 10^{-2}$	$0.1804 \cdot 10^{-5}$
4	0.685336	$0.1508 \cdot 10^{-3*}$	$0.7799 \cdot 10^{-10*}$
5	0.375101	$0.8255 \cdot 10^{-4*}$	$0.2643 \cdot 10^{-7*}$

Note: The values indicated with * have been calculated with $N = 30$.

Table 2
Bounds for the truncation error of the distribution of Q_3

x	Kotz et al.'s bound (17)	$\mu_0 = p/4$ (16)	$\mu_0 = p/10$ (16)
0.1	0.045657	$0.2022 \cdot 10^{-7}$	$0.2352 \cdot 10^{-13}$
0.7	0.550849	$0.8825 \cdot 10^{-6}$	$0.1343 \cdot 10^{-10}$
2	1.051131	0.000027	$0.1092 \cdot 10^{-6}$
3	0.945330	0.000209	0.000060
4	0.712494	$0.1640 \cdot 10^{-6*}$	$0.5046 \cdot 10^{-8*}$
5	0.487456	$0.9566 \cdot 10^{-6*}$	$0.2138 \cdot 10^{-5*}$

Note: The values indicated with * have been calculated with $N = 30$.

converges uniformly in any finite interval, for all μ and β conveniently chosen. However, it is possible to get uniform convergence for all $x > 0$ if $\mu_0 > p/2$.

In Table 1 we compare the bounds (14), given by Kotz et al., and (15) for $Q_3 = 0.6\chi_1^2 + 0.3\chi_1^2 + 0.1\chi_1^2$, see Imhof (1961), $\beta = (0.6 + 0.1)/2$ and $N = 20$:

Similarly, we obtain the following bound for the truncation error of the distribution function

$$\mathscr{E}_N(F, x, \mu_0, \beta) \leq \frac{e^{-\frac{x}{2\beta}}}{(2\beta)^{1+v/2}} \frac{x^{(v/2)}|m_0|}{\Gamma(v/2+1)} \exp\left(\frac{(v+2)x}{8\beta\mu_0}\right) \sum_{k=N+1}^{\infty} b_k, \tag{16}$$

with

$$b_k = \frac{\xi^k (v/2+1)_k}{k!}.$$

Kotz et al. (1967) obtained the bound:

$$\mathscr{E}_N \leq \frac{v}{N+1} \frac{x^{v/2}}{2^{1+v/2}\beta^{v/2}\Gamma(v/2+1)} \exp\left(\frac{-x}{4\beta}\right), \quad \phi^{(N+2)/2}\left(1-\phi^{1/2}\right)^{-v/2-2}, \tag{17}$$

with $\phi = \max_i(|1 - \alpha_i/\beta|)$. We compare both expressions (16), (17) for $Q_3 = 0.6\chi_1^2 + 0.3\chi_1^2 + 0.1\chi_1^2$, $N = 20$, and $p = v/2 + 1$, in Table 2.

Table 3
Approximations for the distribution function of Q_5

x	Approx. (10)	Imhof	Trunc. error bounds (16)
5	0.094143	0.094149	$0.4759 \cdot 10^{-12}$
10	0.291739	0.291731	$0.1658 \cdot 10^{-10}$
20	0.624755	0.624757	$0.3561 \cdot 10^{-8}$
30	0.807274	0.807275	$0.3724 \cdot 10^{-6}$
40	0.899140	0.899138	$0.2901 \cdot 10^{-4}$
50	0.945864	0.945865	$0.3078 \cdot 10^{-7*}$

Note: The column labelled Imhof has been calculated using the result given in Imhof (1961), p. 463, with $\epsilon = 0.0001$. The value indicated with * has been calculated with $N = 40$.

To end this section we present Table 3, in which we present another example with unbalanced coefficients and more terms, given by $Q_5 = 10\chi_1^2 + 4\chi_1^2 + 3\chi_1^2 + 2\chi_1^2 + \chi_1^2$. We compare our results for $\mu_0 = p/10$, $\beta = 11/2$, and $N = 30$, with the results obtained by the Imhof's formula.

6. Conclusions

We have derived a Laguerre expansion for the inverse Laplace transform. This expansion has been used to evaluate the density and distribution function of the sum of weighted central chi-square variables. The formulas so obtained depend on certain parameters. Adequate choices of these parameters give some expressions already known in the literature and some new ones. We have also obtained bounds for the truncation errors in the numerical approximations. Some numerical results show that our bounds are sharper than those given by Kotz et al. (1967).

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