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Exact solutions of integrable 2D contour dynamics ☆

Luis Martínez Alonso [a], Elena Medina [b]

[a] *Departamento de Física Teórica II, Universidad Complutense, E28040 Madrid, Spain*
[b] *Departamento de Matemáticas, Universidad de Cádiz, E11510 Puerto Real, Cádiz, Spain*

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Abstract

A class of exact solutions of the dispersionless Toda hierarchy constrained by a string equation is obtained. These solutions represent deformations of analytic curves with a finite number of nonzero harmonic moments. The corresponding τ-functions are determined and the emergence of cusps is studied.

MSC: 58B20

Keywords: Contour dynamics; Toda hierarchy; Conformal maps

1. Introduction

Integrable contour dynamics governed by the dispersionless Toda (dToda) hierarchy is a multifaceted subject. It underlies problems of complex analysis [1, 2], interface dynamics (Laplacian growth) [3], quantum Hall effect [4] and associativity (WDVV) equations [5]. A common ingredient in many of its applications is the presence of random models of normal $N \times N$ matrices [1–4,6,7] with partition functions of the form

$$Z_N = \int \mathrm{d}M\,\mathrm{d}M^{\dagger} \exp\left(-\frac{1}{\hbar}\operatorname{tr} W\left(M, M^{\dagger}\right)\right), \tag{1}$$

where

$$W(z, \bar{z}) = z\bar{z} + v_0 - \sum_{k\geqslant 1}\left(t_k z^k + \bar{t}_k \bar{z}^k\right). \tag{2}$$

In an appropriate large N limit ($\hbar \to 0$, $s := \hbar N$ fixed), the eigenvalues of the matrices are distributed within a planar domain (*support of eigenvalues*) with sharp edges, which depends on the parameters $t := (s = \bar{s},\ t_1, t_2, \ldots)$.

If the support of eigenvalues is a simply-connected bounded domain with boundary given by an analytic curve γ $(z = z(p),\ |p| = 1)$, then $(s, t_1, t_2, \ldots)$ are

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E-mail addresses: luism@fis.ucm.es (L. Martínez Alonso), elena.medina@uca.es (E. Medina).

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harmonic moments of γ and the curve evolves with $(t, \bar{t})$ according to the dToda hierarchy. Moreover, the corresponding τ-function represents the quasiclassical limit of the partition function (1). A particularly interesting feature is that for almost all initial conditions the evolution of γ leads to critical configurations in which cusp-like singularities develop. This behaviour is well-known in Laplacian growth [8] and random matrix theory [9].

In order to obtain solutions of the dToda hierarchy describing contour dynamics one must impose a string equation which leads to a particular type of Riemann–Hilbert problem [10–12]. In this Letter we present a method for finding solutions in the form of Laurent polynomials

$$z = rp + u_0 + \cdots + \frac{u_{K-1}}{p^{K-1}}, \tag{3}$$

which describe dynamics of curves with a finite number of nonzero harmonic moments, namely $t_k = \bar{t}_k = 0$ for $k \geqslant K$. We exhibit examples for arbitrary K and derive their corresponding τ-functions. Furthermore, the emergence of cusps is analytically studied.

2. dToda contour dynamics

Let $z = z(p)$ be an invertible conformal map of the exterior of the unit circle to the exterior of a simply connected domain bounded by a simple analytic curve γ of the form

$$\bar{z} = S(z), \tag{4}$$

where bar stands for complex conjugation ($z(\bar{p}) = z(p^{-1})$ on γ) and the *Schwarz function* $S(z)$ is analytic in some domain containing γ.

The map $z(p)$ can be represented by a Laurent series

$$z(p) = rp + \sum_{k=0}^{\infty} \frac{u_k}{p^k}, \tag{5}$$

with a real coefficient r. The coefficients $(r, u_0, u_1, \ldots)$ are functions of the harmonic moments $t = (s = \bar{s}, t_1, t_2, \ldots)$ of the exterior of γ, which in turn can be introduced through the expansion of the Schwarz function

$$S(z) = \sum_{k=1}^{\infty} k t_k z^{k-1} + \frac{s}{z} + \sum_{k=1}^{\infty} \frac{v_k}{z^{k+1}}, \tag{6}$$

with $(v_1, v_2, \ldots)$ being functions dependent on t. As a consequence of (1)–(3), it follows that $z(p, t, \bar{t})$ solves the dToda hierarchy

$$\begin{aligned}
&\partial_{t_k} z = \{H_k, z\}, \qquad \partial_{\bar{t}_k} z = -\{\bar{H}_k, z\},\\
&H_k := (z^k)_{\geqslant 1} + \frac{1}{2}(z^k)_0,\\
&\bar{H}_k := (\bar{z}^k)_{\leqslant -1} + \frac{1}{2}(\bar{z}^k)_0,
\end{aligned} \tag{7}$$

where $\{f, g\} := p(\partial_p f \partial_s g - \partial_p g \partial_s f)$, the function $\bar{z}(p^{-1})$ is defined by the Laurent series

$$\bar{z}(p^{-1}) := \frac{r}{p} + \sum_{k=0}^{\infty} \bar{u}_k p^k, \tag{8}$$

and the symbols $(\ldots)_{\geqslant 1}$ $((\ldots)_{\leqslant -1})$ and $(\ldots)_0$ mean truncated Laurent series with only positive (negative) terms and the constant term, respectively. Furthermore, this solution satisfies the string equation

$$\{z(p), \bar{z}(p^{-1})\} = 1. \tag{9}$$

These properties can be proved through the twistor scheme of Takasaki–Takebe [3]. It uses Orlov–Schulman functions of the dToda hierarchy

$$\begin{aligned}
&m = \sum_{k=1}^{\infty} k t_k z^k + s + \sum_{k=1}^{\infty} \frac{v_k}{z^k},\\
&\bar{m} = \sum_{k=1}^{\infty} k \bar{t}_k \bar{z}^k + s + \sum_{k=1}^{\infty} \frac{\bar{v}_k}{\bar{z}^k},
\end{aligned} \tag{10}$$

and can be summarized as follows:

Theorem 1. *If $(z, m, \bar{z}, \bar{m})$ are functions of $(p, t, \bar{t})$ which admit expansions of the form (5), (8), (10) and satisfy the equations*

$$\bar{z} = \frac{m}{z}, \qquad \bar{m} = m, \tag{11}$$

then $(z, \bar{z})$ is a solution of the dToda hierarchy constrained by the string equation (9).

3. Solutions

Eq. (11) are meaningful only when they are interpreted as a suitable Riemann–Hilbert problem on the complex plane of the variable p. Thus (z, m) must be analytic functions in a neighborhood $D = \{|p| > r\}$

of $p=\infty$ and $(\bar{z},\bar{m})$ must be analytic functions in a neighborhood $D'=\{|p|<r'\}$ of $p=0$. The statement of Theorem 1 holds provided $A:=D\cap D'\neq\emptyset$.

We next prove that Eq. (11) have solutions satisfying (5), (8) and (10) with

$$t_k=0,\quad k>K,\quad t_K\neq 0.$$

In this way we assume

$$m=\sum_{k=1}^{K}kt_k z^k+s+\sum_{k=1}^{\infty}\frac{v_k}{z^k},$$
$$\bar{m}=\sum_{k=1}^{K}k\bar{t}_k\bar{z}^k+s+\sum_{k=1}^{\infty}\frac{\bar{v}_k}{\bar{z}^k}. \tag{12}$$

Given two integers $r_1\leqslant r_2$ we denote by $\mathrm{V}[r_1,r_2]$ the set of Laurent polynomials of the form

$$c_{r_1}p^{r_1}+c_{r_1+1}p^{r_1+1}+\cdots+c_{r_2}p^{r_2}.$$

Let us look for solutions of (11) such that z and $\bar{z}$ are meromorphic functions of p with possible poles at $p=0$ and $p=\infty$ only. Then, as a consequence of the assumptions (5), (8) and (12), from (11) it follows that

$$z\in\mathrm{V}[1-K,1],\qquad \bar{z}\in\mathrm{V}[-1,K-1]. \tag{13}$$

The equation $\bar{m}=m$ is equivalent to the system:

$$\bar{m}_{\geqslant 1}=m_{\geqslant 1}, \tag{14}$$
$$\bar{m}_0=m_0, \tag{15}$$
$$\bar{m}_{\leqslant -1}=m_{\leqslant -1}. \tag{16}$$

If we now set

$$m=\bar{m}=\sum_{k=1}^{K}kt_k\big(z^k\big)_{\geqslant 1}+\bar{m}_0+\sum_{k=1}^{K}k\bar{t}_k\big(\bar{z}^k\big)_{\leqslant -1}, \tag{17}$$

with

$$\bar{m}_0=s+\sum_{k=1}^{K}k\bar{t}_k\big(\bar{z}^k\big)_0,$$

it can be easily seen that $\bar{m}$ has the required expansion of the form (12) provided z and $\bar{z}$ satisfy (5) and (8). On the other hand, the expression (17) for m has an expansion of the form (12) if the residue of m/z corresponding to its Laurent expansion in powers of z verifies

$$\mathrm{Res}\left(\frac{m}{z},z\right)=s. \tag{18}$$

Hence the problem reduces to finding z and $\bar{z}$ satisfying (5), (8), (18) and

$$z=\frac{m}{\bar{z}}. \tag{19}$$

In view of (13) we look for z and $\bar{z}$ of the form

$$z=rp+u_0+\cdots+\frac{u_{K-1}}{p^{K-1}},$$
$$\bar{z}=\frac{r}{p}+\bar{u}_0+\cdots+\bar{u}_{K-1}p^{K-1}. \tag{20}$$

Now, in order to prevent z from having poles different from $p=0$ and $p=\infty$ we have to impose

$$m(p_i)=0, \tag{21}$$

where p_i denote the K zeros of

$$r+\bar{u}_0p+\cdots+\bar{u}_{K-1}p^K=0.$$

In this way by using the expression (17) of m, the only variables appearing in (19) are

$$(p,t,\bar{t},r,u_0,\dots,u_{K-1},w_0,\dots,w_{K-1}),\qquad w_i:=\bar{u}_i.$$

Thus, by identifying coefficients of the powers p^i, $i=1-K,\dots,1$, we get $K+1$ equations which together with the K equations (21) determine the $2K+1$ unknowns variables $(r,u_0,\dots,u_{K-1},w_0,\dots,w_{K-1})$ as functions of $(t,\bar{t})$. Moreover, provided r is a real coefficient, the equations (11) are invariant under the transformation

$$\mathrm{T}f(p)=\overline{f\left(\frac{1}{\bar{p}}\right)}.$$

Hence if $(r,u_0,\dots,u_{K-1},w_0,\dots,w_{K-1})$ solves (19) so does

$$(r,\bar{w}_0,\dots,\bar{w}_{K-1},\bar{u}_0,\dots,\bar{u}_{K-1}).$$

Therefore, if both solutions are close enough, they coincide and consequently $w_i=\bar{u}_i$, as required.

To complete our proof we must show that (18) is satisfied too. To do that let us take two circles γ $(|p|=r)$ and γ' $(|p|=r')$ in the complex p-plane and denote by Γ and Γ' their images under the maps $z=z(p)$ and $\bar{z}=\bar{z}(1/p)$, respectively. Notice that due to (5) and (8), the curves Γ and Γ' have positive orientation if γ and γ' have positive and negative orientation, respectively. Then we have

$$\begin{aligned}\operatorname{Res}\left(\frac{m}{z}, z\right) - \operatorname{Res}\left(\frac{\bar m}{\bar z}, \bar z\right) &= \frac{1}{2i\pi}\oint_{\Gamma} \frac{m}{z}\,\mathrm{d}z - \frac{1}{2i\pi}\oint_{\Gamma'} \frac{\bar m}{\bar z}\,\mathrm{d}\bar z \\ &= \frac{1}{2i\pi}\oint_{\gamma} \bar z\partial_p z\,\mathrm{d}p - \frac{1}{2i\pi}\oint_{\gamma'} z\partial_p \bar z\,\mathrm{d}p \\ &= \frac{1}{2i\pi}\oint_{\gamma} \partial_p(\bar z z)\,\mathrm{d}p = 0,\end{aligned}$$

where we have taken into account that the integrands are analytic functions of p in $\mathbb{C} - \{0\}$ and that γ and the opposite curve of γ' are homotopic with respect to $\mathbb{C} - \{0\}$. Therefore, as we have already proved that $\bar m$ has an expansion of the form (12), we deduce

$$\operatorname{Res}\left(\frac{m}{z}, z\right) = \operatorname{Res}\left(\frac{\bar m}{\bar z}, \bar z\right) = s,$$

so that (18) follows.

Let us illustrate the method with the case $K = 2$. The polynomial $p\bar z$ has two zeros at the points

$$p_1 = \frac{-\bar u_0 + \sqrt{\bar u_0^2 - 4r\bar u_1}}{2\bar u_1}, \qquad p_2 = \frac{-\bar u_0 - \sqrt{\bar u_0^2 - 4r\bar u_1}}{2\bar u_1},$$

and from (21) we get two equations which lead to

$$\begin{aligned}&-2r^2t_2\bar u_0^3 + 4r^3t_2\bar u_0\bar u_1 + rt_1\bar u_0^2\bar u_1 \\ &\quad + 4rt_2u_0\bar u_0^2\bar u_1 - r^2t_1\bar u_1^2 - 4r^2t_2u_0\bar u_1^2 \\ &\quad - s\bar u_0\bar u_1^2 - \bar t_1\bar u_0^2\bar u_1^2 - 2\bar t_2\bar u_0^3\bar u_1^2 + r\bar t_1\bar u_1^3 = 0, \qquad (22)\end{aligned}$$

$$\begin{aligned}&-2r^3t_2\bar u_0^2 + 2r^4t_2\bar u_1 + r^2t_1\bar u_0\bar u_1 \\ &\quad + 4r^2t_2u_0\bar u_0\bar u_1 - rs\bar u_1^2 - r\bar t_1\bar u_0\bar u_1^2 \\ &\quad - 2r\bar t_2\bar u_0^2\bar u_1^2 - 2r^2\bar t_2\bar u_1^3 = 0. \qquad (23)\end{aligned}$$

Identification of the powers of p in (19) implies

$$\begin{aligned}&p\colon\quad -2r^2t_2 + r\bar u_1 = 0, \\ &p^0\colon\quad 2r^2t_2\bar u_0 - rt_1\bar u_1 - 4rt_2u_0\bar u_1 + u_0\bar u_1^2 = 0, \\ &p^{-1}\colon\quad -2r^2t_2\bar u_0^2 + 2r^3t_2\bar u_1 + rt_1\bar u_0\bar u_1 \\ &\qquad + 4rt_2u_0\bar u_0\bar u_1 - s\bar u_1^2 - \bar t_1\bar u_0\bar u_1^2 \\ &\qquad - 2\bar t_2\bar u_0^2\bar u_1^2 - 4r\bar t_2\bar u_1^3 + u_1\bar u_1^3 = 0. \qquad (24)\end{aligned}$$

Then by solving Eqs. (22)–(24) we get the solution:

$$z = \frac{p\sqrt{s}}{\sqrt{1-4t_2\bar t_2}} + \frac{2\sqrt{s}\bar t_2}{p\sqrt{1-4t_2\bar t_2}} - \frac{\bar t_1 + 2t_1\bar t_2}{-1+4t_2\bar t_2}, \qquad (25)$$

which corresponds to the conformal map describing an *ellipse growing from a circle* [6].

3.1. Solutions for $K \geqslant 3$

Exact solutions associated to arbitrary values of K can be found from the previous scheme. However, in order to avoid complicated expressions, we set

$$t_1 = t_2 = \cdots = t_{K-1} = \bar t_1 = \bar t_2 = \cdots = \bar t_{K-1} = 0.$$

and look for particular solutions satisfying

$$u_1 = u_2 = \cdots = u_{K-2} = \bar u_1 = \bar u_2 = \cdots = \bar u_{K-2} = 0,$$

or equivalently

$$z = rp + \frac{u_{K-1}}{p^{K-1}}, \qquad \bar z = \frac{r}{p} + \bar u_{K-1}p^{K-1}. \qquad (26)$$

Under the previous assumptions and from (17) we have that

$$m = Kt_Kr^Kp^K + s + K^2\bar t_Kr^{K-1}\bar u_{K-1} + \frac{K\bar t_Kr^K}{p^K}. \qquad (27)$$

Thus, we see that (21) leads us to a unique equation since from (27) it follows that m depends on p through p^K. Furthermore, if p_i satisfies $\bar z(p_i) = 0$, then

$$p_i^K = -\frac{r}{\bar u_{K-1}}.$$

Therefore, (21) becomes

$$s - \frac{Kr^{K+1}t_K}{\bar u_{K-1}} + (K-1)Kr^{K-1}\bar t_K\bar u_{K-1} = 0. \qquad (28)$$

On the other hand, it is easy to see that

$$\frac{m}{\bar z} = \frac{Kr^Kt_K}{\bar u_{K-1}}p + \left(K^2r^{K-1}\bar t_K + \frac{s\bar u_{K-1} - Kr^{K+1}t_K}{\bar u_{K-1}^2}\right)\frac{1}{p^{K-1}},$$

consequently, by equating coefficients and taking (26) into account, we find that (19) leads to two equations

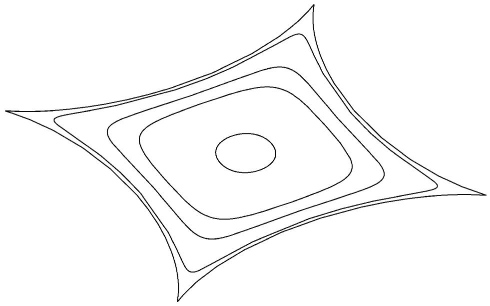

Fig. 1. Solution corresponding to $K = 4$.

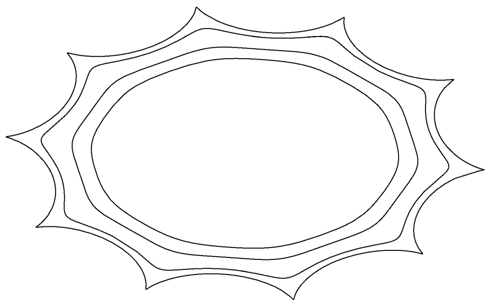

Fig. 2. Solution corresponding to $K = 10$.

only. More precisely,

$$p{:}\quad r = \frac{K r^K t_K}{\bar{u}_{K-1}},$$
$$p^{-(K-1)}{:}\quad u_{K-1} = K^2 r^{K-1} \bar{t}_K + \frac{s\bar{u}_{K-1} - K r^{K+1} t_K}{\bar{u}_{K-1}^2}. \tag{29}$$

Then, we get three equations for the three unknowns r, u_{K-1}, $\bar{u}_{K-1}$, which proves that there exits a solution of the form (26). In fact, by solving (28), (29) we find that

$$z = rp + \frac{K \bar{t}_K r^{K-1}}{p^{K-1}}, \tag{30}$$

with r satisfying the implicit equation

$$K^2(K-1) t_K \bar{t}_K r^{2(K-1)} - r^2 + s = 0. \tag{31}$$

Figs. 1 and 2 show examples of the evolution of the curve $z(p)$, $|p| = 1$, as s grows and t_K is kept fixed.

3.2. τ-functions

In [1] it was proved that there is a dToda τ-function associated to each analytic curve $z = z(p)$, $|p| = 1$, given by

$$2\log\tau = -\frac{1}{2}s^2 + s v_0 - \frac{1}{2}\sum_{k\geqslant 1}(t_k v_k + \bar{t}_k \bar{v}_k), \tag{32}$$

where v_k are the coefficients of the expansion (6), and v_0 is determined by

$$\frac{\partial v_0}{\partial s} = \log r^2, \qquad v_0 = \frac{\partial \log\tau}{\partial s}. \tag{33}$$

For the class of solutions (30) we have

$$2\log\tau = -\frac{1}{2}s^2 + s v_0 - \frac{1}{2}(t_K v_K + \bar{t}_K \bar{v}_K), \tag{34}$$

and from (11) and (30) it follows that

$$\begin{aligned} v_K &= \frac{1}{2i\pi}\oint_\Gamma \bar{z} z^K \,\mathrm{d}z \\ &= \frac{1}{2i\pi}\oint_\Gamma \bar{z}(p) z(p)^K \left(r - (K-1)\frac{u_{K-1}}{p^K}\right)\mathrm{d}p \\ &= \frac{(r^2 - s)(Ks - (K-2)r^2)}{2K(K-1)s}. \end{aligned} \tag{35}$$

On the other hand, by differentiating (34) with respect to s and by taking into account (33) one finds

$$\begin{aligned} v_0 = {} & -s + s\log r^2 \\ & + (K-2)(K-1)K|t_K|^2 \\ & \times \frac{(Ks - (K-2)r^2) r^{2K}}{2((K-1)^2 K^2 |t_K|^2 r^{2K} - r^4)} \\ & + \frac{(K-2)(s - r^2)}{2(K-1)K} \\ & \times \left(K + \frac{(K-2)r^4}{(K-1)^2 K^2 |t_K|^2 r^{2K} - r^4}\right). \end{aligned} \tag{36}$$

Thus, (34)–(36) and (31) characterize the τ-function of the curves determined by (30).

3.3. Cusps

The pictures of the curves associated with (30), (31) show the presence of cusps at some value of s for each fixed value of t_K. Indeed, by using the parametric equation $p = e^{i\theta}$ $(0 \leqslant \theta \leqslant 2\pi)$ for the unit circle, we have that cusps on the curve $z = z(p)$ appear at points where $z_\theta = 0, z_{\theta\theta} \neq 0$ and $z_{\theta\theta\theta}/z_{\theta\theta}$ has a nonzero imaginary part. Therefore, a necessary condition for

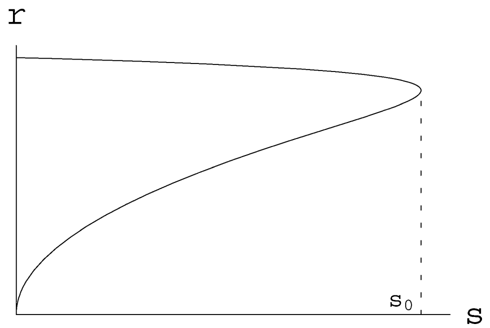

Fig. 3. The positive branches of $r(s)$ for $K = 10$.

$p = p(\theta)$ is

$$\frac{\partial z}{\partial p}(p) = 0, \quad |p| = 1.$$

Thus from (30) we deduce

$$p^K = K(K-1)\bar{t}_K r^{K-2},$$

which together with the condition $|p| = 1$ requires that

$$r = \big(K(K-1)|t_K|\big)^{-\frac{1}{K-2}}, \tag{37}$$

at some value $s = s(t_K)$. But according to (31) one finds that this happens at the value s_0 given by

$$s_0 = \frac{K-2}{K-1}\big(K(K-1)|t_K|\big)^{-\frac{2}{K-2}}, \tag{38}$$

which is the point at which the profile of both positive branches of r, as functions of s, develop an infinite slop (see Fig. 3).

Therefore, there are K cusps given by the roots

$$z_j = \frac{K}{K-1}\left(\frac{r^2 - s_0}{K t_K}\right)^{1/K}, \tag{39}$$

which emerge when s reaches the extreme value s_0 of the domain of existence of the two positive branches of r as a function of s.

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