REV. MAT. IBEROAMERICANA 20 (2004), 673-736

Multiplicative Square Functions

María José González and Artur Nicolau

Abstract

We study regularity properties of a positive measure in the euclidean space in terms of two square functions which are the multiplicative analogues of the usual martingale square function and of the Lusin area function of a harmonic function. The size of these square functions is related to the rate at which the measure doubles at small scales and determines several regularity properties of the measure. We consider the non-tangential maximal function of the logarithm of the densities of the measure in the dyadic setting, and of the logarithm of the harmonic extension of the measure, in the continuous setting. We relate the size of these maximal functions to the size of the corresponding square functions. Fatou type results, L^p estimates and versions of the Law of the Iterated Logarithm are proved. As applications we introduce a hyperbolic version of the Lusin Area function of an analytic mapping from the unit disc into itself, and use it to characterize inner functions. Another application to the theory of quasiconformal mappings is given showing that our methods can also be applied to prove a result by Din'kyn's on the smoothness of quasiconformal mappings of the disc.

1. Introduction

A positive measure μ in $\mathbb{R}^N$ is called doubling if there exists a positive constant $C = C(\mu)$ such that

$$\mu(2Q) \leq C\mu(Q)\,,$$

for any cube $Q \subset \mathbb{R}^N$, where $2Q$ denotes the cube which the same center as Q and twice the sidelength. Lebesgue measure dm is doubling and, as a

2000 Mathematics Subject Classification: 42B25, 31A20.
Keywords: Doubling measure, martingale, positive harmonic functions, square functions, hyperbolic derivative, quasiconformal mappings.

matter of fact, it is the only doubling measure for which the constant C can be taken $C = 2^N$. However, classical constructions ([7, 13]) provide examples of doubling measures which are singular with respect to Lebesgue measure. See also [2], [12], [24], [33].

Our main purpose is to describe regularity properties of a measure, such as being singular, in terms of two different square functions. The first square function is related to the rate at which the measure doubles at small scales, while the second one is connected to the logarithm of the harmonic extension of the measure. An important result in this direction, stated below in a different way, was proved by L. Carleson ([13]) for $N = 1$, when studying regularity properties of quasiconformal mappings. See also [27, p. 5], and [16] for the extension to $\mathbb{R}^N$.

Theorem A. *(a) Let μ be a doubling measure in $\mathbb{R}^N$. Consider*

$$w(t) = \sup \left| \frac{\mu(Q)2^N}{\mu(2Q)} - 1 \right| ,$$

where the supremum is taken over all cubes Q in $\mathbb{R}^N$ of sidelength smaller than t. Assume

$$\int_0 w^2(t) \frac{dt}{t} < \infty .$$

Then, the measure μ is absolutely continuous and its density f satisfies

$$\int_Q \exp(Cf^2) dm < \infty ,$$

for any cube $Q \subset \mathbb{R}^N$ and any $C > 0$.

(b) Let $w\colon [0,1] \to [0,\infty]$ be an increasing function with $w(0) = 0$. Assume that for some $0 < \varepsilon < 1$, the function $w(t)/t^{1-\varepsilon}$ is non-decreasing. Assume also

$$\int_0 w^2(t) \frac{dt}{t} = \infty .$$

Then, there exists a singular doubling measure μ on $\mathbb{R}^N$ such that

$$\left| \frac{\mu(Q)2^N}{\mu(2Q)} - 1 \right| \leq w(l(Q)) ,$$

for any cube $Q \subset \mathbb{R}^N$.

So, the quadratic condition

$$\int_0 w^2(t) \frac{dt}{t} = \infty$$

governs the existence of singular measures that double at rate $w(t)$. However, this is a global result because the function $w(t)$ controls the error done by the measure when doubling at all cubes of sidelength t.

We are interested in local versions of this result. In the additive settir local results of this sort were proved by Stein and Zygmund [35, p. 26 They showed that a real valued function f on $\mathbb{R}^N$ has an ordinary derivat at almost every point of a set E if and only if both

$$f(x+t)+f(x-t)-2f(x)=O(|t|)\,,\quad t\to 0\,,$$

$$\int_{|t|\le\delta}|f(x+t)+f(x-t)-2f(x)|^2\frac{dt}{|t|^{N+2}}<\infty$$

for almost every $x\in E$.

Both our results and methods are better explained in the dyadic setti Let Q_0 be the unit cube in $\mathbb{R}^N$. Let us consider the dyadic decomposit of Q_0, that is, for $k=1,2,\ldots$, we consider the collection F_k of the pairwise disjoint subcubes of Q_0 of the form

$$Q=[j_1 2^{-k},(j_1+1)2^{-k})\times\cdots\times[j_N 2^{-k},(j_N+1)2^{-k})\,,$$

where $j_1,\ldots,j_N$ are integers between 0 and 2^k-1. So

$$m(Q)=|Q|=2^{-kN}.$$

A positive measure μ on Q_0 is called dyadic doubling if there exists a cons $C=C(\mu)>0$ such that

$$\mu(\widetilde{Q})\le C\mu(Q)\,,$$

for any dyadic subcube Q of Q_0, where $\widetilde{Q}$ is the smallest dyadic subcube which properly contains Q. These measures were completely characte in terms of their Haar coefficients by Fefferman, Kenig and Pipher (Results on null sets of (dyadic) doubling measures can be found in [39], [40].

A dyadic martingale on Q_0 is a sequence of functions $\{S_n\}$ fro to $\mathbb{R}$ such that S_n is F_n-measurable and the conditional expectation of on F_n is S_n, $n=0,1,2,\ldots$. These two conditions may be rephrase S_n is constant (which we denote $S_n(\widetilde{Q})$) in each cube $\widetilde{Q}$ of F_n and

$$S_n(\widetilde{Q})=\frac{1}{2^N}\sum S_{n+1}(Q)\,.$$

where the sum is taken over all cubes Q of F_{n+1} contained in $\widetilde{Q}$. Not $S_n(x)$ represents the value of the martingale on the cube $Q_n(x)$, where denotes the unique dyadic cube of sidelenght 2^{-n} containing x.

The (truncated) maximal function of the martingale $\{S_n\}$ is

$$S_m^*(x) = \sup_{0 \le n \le m} |S_n(x)|, \qquad m = 0, 1, 2, \dots$$

If $m = \infty$, we simply write $S^*(x)$. The (truncated) martingale square function of $\{S_n\}$ is defined as

$$\langle S \rangle_m(x) = \left(\sum_{k=1}^{m} \left\| (S_k - S_{k-1}) \chi_{Q_{k-1}(x)} \right\|_\infty^2 \right)^{1/2}, \qquad x \in Q_0 .$$

In dimension one, this is the more traditional square function

$$\langle S \rangle_m(x) = \left(\sum_{k=1}^{m} (S_k(x) - S_{k-1}(x))^2 \right)^{1/2} .$$

If $m = \infty$, we simply write $\langle S \rangle(x)$. It is well known that many properties of the martingale $\{S_n\}$ can be described using its square function. Burckholder and Gundy proved that the sets

$$\{x \in Q_0 \colon \lim_{n \to \infty} S_n(x) \text{ exists}\} \quad \text{and} \quad \{x \in Q_0 \colon \langle S \rangle(x) < \infty\}$$

can only differ on a set of Lebesgue measure 0. Also, for $0 < p < \infty$, the maximal function

$$S^*(x) = \sup_n |S_n(x)|$$

is in $L^p(Q_0)$ if and only if $\langle S \rangle \in L^p(Q_0)$. See [9] and [10]. These results give comparisons between S^* and $\langle S \rangle$ on the set where they are finite. In its complement, the following Law of the Iterated Logarithm holds:

$$\limsup_{n \to \infty} \frac{|S_n(x)|}{\langle S \rangle_n(x) \sqrt{2 \log \log \langle S \rangle_n(x)}} \le 1 ,$$

at almost every point $x \in \{x \in Q_0 \colon \sup_n |S_n(x)| = \infty\}$ ([35, 6]). An important result in this direction is the following good-λ inequality established by Chang, Wilson and Wolff,

$$\begin{aligned} &|\{x \in Q_0 \colon S^*(x) > 2\lambda, \quad \langle S \rangle(x) < \varepsilon\lambda\}| \le \\ &\qquad \le C \exp\left(-\frac{1}{2} \left(\frac{1-\varepsilon}{\varepsilon} \right)^2 \right) |\{x \in Q_0 \colon S^*(x) > \lambda\}| . \end{aligned} \tag{1.1}$$

Here ε is any number in $(0, 1)$ and C is a universal constant ([14]).

Given a positive measure μ on Q_0, one may consider the dyadic martingale

$$S_k = \sum_{Q \in F_k} \frac{\mu(Q)}{|Q|} \chi_Q \,, \quad k = 1, 2, \ldots .$$

Thus, many properties of the martingale $\{S_k\}$ and hence of the measure can be described in terms of its quadratic variation,

$$(S)^2(x) = \sum_{k=1}^{\infty} \max \left\{ \frac{\mu(Q)}{|Q|} - \frac{\mu(Q_{k-1}(x))}{|Q_{k-1}(x)|} \right\}^2 ,$$

where the maximum is taken over all cubes $Q \in F_k$ contained in $Q_{k-1}($ For instance, it is easy to show that, for $1 < p < \infty$, μ has a den (with respect to Lebesgue measure) in $L^p(Q_0)$ if and only if the squ function of the martingale is in $L^p(Q_0)$. Because of the additive nature these expressions, the martingale $\{S_k\}$ is very well suited to study addi properties of the measure μ. See [2], [16], [27], [26]. When dealing v multiplicative properties of a measure, such as doubling conditions, i natural to consider, somehow, the logarithm of S_n. This point of view already been used by other authors. For instance, we mention the wor Fefferman, Kenig and Phiper [17] in connection with the theory of wei (see also [11], [37]) and the work of P. MacManus ([29]) on quasiconfor mappings.

Next we introduce a square function of a positive measure whicl in some sense, a multiplicative analogue of the square function which a from the martingale. Given a dyadic doubling measure μ on ons the dyadic square function

$$A^2(\mu)(x) = \sum_{k=1}^{\infty} d_k^2(x) \,, \qquad x \in Q_0$$

where

$$d_k(x) = \max \left\{ \left| 1 - \frac{2^N \mu(Q)}{\mu(Q_{k-1}(x))} \right| : Q \in F_k \,,\ Q \subset Q_{k-1}(x) \right\} .$$

So, the function $A^2(\mu)(x)$ measures the error done by μ when doublin all dyadic cubes containing $x \in Q_0$. In dimension 1, this is

$$A^2(\mu)(x) = \sum_{k=1}^{\infty} \left(1 - \frac{2\mu(I_k(x))}{\mu(I_{k-1}(x))} \right)^2 ,$$

where $I_j(x)$, $j = 1, 2, \ldots$, is the unique dyadic interval of length 2^{-j} w contains $x \in I_0$. Our first result is the following.

Theorem 1.1. *Let μ be a dyadic doubling measure in Q_0 and $A^2(\mu)$ its dyadic square function. Let $Q_n(x)$ be the unique dyadic cube in F_n which contains $x \in Q_0$. Then,*

(a) The sets

$$\left\{x \in Q_0 \colon \lim_{n\to\infty} \frac{\mu(Q_n(x))}{|Q_n(x)|} > 0\right\} \quad \text{and} \quad \{x \in Q_0 \colon A^2(\mu)(x) < \infty\}$$

can only differ in a set of Lebesgue measure 0.

(b) Consider the maximal function

$$M(\log \mu)(x) = \sup_n \left|\log \frac{\mu(Q_n(x))}{|Q_n(x)|}\right|, \quad x \in Q_0 \quad .$$

Then, for $0 < p < \infty$, $M(\log \mu) \in L^p(Q_0)$ if and only if $A^2(\mu) \in L^p(Q_0)$.

(c) Let f be the density of μ respect Lebesgue measure. Then, for $1 < p < \infty$, $\log f \in L^p(Q_0)$ if and only if $A^2(\mu) \in L^p(Q_0)$.

(d) There exists a constant $C = C(N)$, only depending on the dimension, such that $\exp(CA^2(\mu)) \in L^1(Q_0)$ implies that μ and Lebesgue measure are mutually absolutely continuous.

(e) There exists a constant $c = c(N)$, only depending on the dimension, such that if $A^2(\mu)(x) < 1$ for any $x \in Q_0$, then $\exp(cM^2(\log \mu)) \in L^1(Q_0)$.

The statements in (a), (b) and (c) are the analogues, in our setting, of the classical results by Burkholder and Gundy cited above, while (d) and (e) are consequences of certain good-λ inequalities relating the growth of $M(\log u)$ and $A^2(\mu)$ and which are the analogues of the subgaussian estimates of Chang, Wilson and Wolff. As a consequence of (a) we obtain

Corollary 1.2. *Let μ be a dyadic doubling measure in Q_0 and let $A^2(\mu)$ be its dyadic square function. Then, μ is singular with respect to Lebesgue measure if and only if*

$$A^2(\mu)(x) = \infty,$$

at almost (dm) every point $x \in Q_0$.

The main idea in the proof is to consider the logarithmic transformation of the martingale $\{S_n\}$. More concretely, given a dyadic doubling measure μ on Q_0, we consider the dyadic martingale

$$S_n = \sum_{Q \in F_n} \frac{\mu(Q)}{|Q|} \chi_Q \,.$$

It is easy to see that there exists a unique (except for adding constan process X_n verifying the following two conditions:

(a) X_n is F_{n-1}-measurable, that is, X_n is constant on any cube of F_{n-1};

(b) $T_n = \log S_n + X_n$ is a dyadic martingale.

It turns out that the martingale $\{T_n\}$ is the inverse of the exponential tra formation of $\{S_n\}$ (see [27, p. 27], [6, p. 47]). Actually the process X_n c be given explicitly by

$$X_n = \sum_{Q \in F_n} X_n(Q)\chi_Q ,$$

where

$$X_n(Q) = -\sum \log \frac{\text{g.m.}(Q_j)}{\text{a.m.}(Q_j)} . \tag{1.2}$$

Here the sum is taken over all dyadic cubes $\{Q_j\}$ which properly contain Also, if $Q \in F_l$ and $Q_1, \dots, Q_{2^N}$ are the dyadic cubes in F_{l+1} contained in then g.m. (Q) (a.m. (Q)) denotes the geometric (arithmetic) mean of density of μ over $Q_1, \dots, Q_{2^N}$, that is

$$\text{g.m.}(Q) = \prod_{i=1}^{2^N} \left(\frac{\mu(Q_i)}{|Q_i|} \right)^{1/2^N} , \qquad \text{a.m.}(Q) = \frac{\mu(Q)}{|Q|} = \frac{1}{2^N} \sum_{i=1}^{2^N} \frac{\mu(Q_i)}{|Q_i|} .$$

Now, the main observations are that, under the doubling hypothesis, quadratic variation $\langle T \rangle^2(x)$ is pointwise comparable to $A^2(\mu)(x)$, and X_n is pointwise comparable to the truncated area function $A_n^2(\mu)(x)$, that i

$$X_n(x) \simeq \sum_{k=1}^{n} d_k^2(x) = A_n^2(\mu)(x) .$$

Let

$$T^*(x) = \sup_n |T_n(x)| = \sup_n \left| \log \frac{\mu(Q_n(x))}{|Q_n(x)|} + X_n(x) \right| .$$

Now, since $\langle T \rangle$ is comparable to $A(\mu)$, the usual stopping time argum which go back to Burkholder and Gundy (see [9, 10]) used to compare distribution functions of $T^*(x)$ and $\langle T \rangle(x)$ can be used to compare $M(\log$ and $A^2(\mu)$ and obtain (a) and (b) in Theorem 1.1. However the proof of and (c) uses the deeper subgaussian estimates proved by Chang, Wi and Wolff mentioned above ([14]). These subgaussian estimates are very closely related to the law of the iterated logarithm. In our cas get that the logarithm of the density of μ behaves very closely to $X_n(x$ the set $\{x \in Q_0 \colon \lim_{n\to\infty} \frac{\mu(Q_n(x))}{|Q_n(x)|} = 0\}$, which coincides, except for a s Lebesgue measure 0, with $\{x \in Q_0 \colon X(x) = \infty\} = \{x \in Q_0 \colon A(\mu)(x) =$

Theorem 1.3. *Let μ be a dyadic doubling measure on Q_0 and let X_n be given by* (1.2). *Then there exists a constant C depending on the dimension and on the doubling constant of μ, such that at almost every point $x \in \{x \in Q_0\colon \lim_{n\to\infty} \frac{\mu(Q_n(x))}{|Q_n(x)|} = 0\}$, one has*

$$C^{-1} \le \limsup_{n\to\infty} \frac{\left|\log \frac{\mu(Q_n(x))}{|Q_n(x)|} + X_n(x)\right|}{\sqrt{X_n(x)\log\log X_n(x)}} \le C\,.$$

Since, as mentioned before, $X_n(x)$ is comparable to $A_n^2(\mu)(x)$, we deduce

Corollary 1.4. *Let μ be a dyadic doubling measure on Q_0, let $A^2(\mu)$ be its square function and let X_n be given by* (1.2). *Then at almost every point $x \in \{x \in Q_0\colon \lim_{n\to\infty} \frac{\mu(Q_n(x))}{|Q_n(x)|} = 0\}$,*

$$\lim_{n\to\infty} \frac{\left|\log \frac{\mu(Q_n(x))}{|Q_n(x)|}\right|}{X_n(x)} = 1\,.$$

Also, there exists a constant C depending on the dimension and on the doubling constant of μ such that

$$C^{-1} < \frac{\left|\log \frac{\mu(Q_n(x))}{|Q_n(x)|}\right|}{A_n^2(\mu)(x)} < C\,,$$

at almost every point $x \in \{x \in Q_0\colon \lim_{n\to\infty} \frac{\mu(Q_n(x))}{|Q_n(x)|} = 0\}$, for n sufficiently large.

It should be observed that

$$\log \frac{\mu(Q_n(x))}{|Q_n(x)|} \le C(\mu) \sum_{k=1}^{n} d_k(x)$$

is the best possible pointwise estimate. However, the last result shows that this inequality can be substantially improved at almost every point on the set $\{x \in Q_0\colon \lim_{n\to\infty} \frac{\mu(Q_n(x))}{|Q_n(x)|} = 0\}$.

The second part of our work is devoted to study the relation between singularity, as well as other regularity properties of a measure, and an square function which arises from the harmonic extension of the measure. Let u be a harmonic function in the upper-half space $\mathbb{R}^{N+1}_+ = \{(x,y)\colon x \in \mathbb{R}^N,\ y > 0\}$. Given $\alpha > 0$ and $x_0 \in \mathbb{R}^N$, we denote by $\Gamma(x_0) = \Gamma_\alpha(x_0)$ the (truncated) cone with vertex at x_0 of aperture α, that is

$$\Gamma_\alpha(x_0) = \{(x,y) \in \mathbb{R}^{N+1}_+\colon |x - x_0| < \alpha y,\ 0 < y < 1\}\,.$$

Given $t \geq 0$, the (doubly) truncated cone $\Gamma_\alpha(x_0, t)$ with vertex at $x_0 \in \mathbb{R}^N$

$$\Gamma_\alpha(x_0, t) = \{(x, y) \in \mathbb{R}^{N+1}_+ \colon |x - x_0| < \alpha y\,,\ t < y < 1\}$$

and the (doubly) truncated maximal function of u is

$$M_\alpha(u)(x_0, t) = \sup\{|u(x, y)| \colon (x, y) \in \Gamma_\alpha(x_0, t)\}$$

and when $t = 0$, we simply write $M_\alpha(u)(x_0)$.

The (doubly) truncated area function of u is defined by

$$A^2_\alpha(u)(x_0, t) = \int_{\Gamma_\alpha(x_0,t)} |\nabla u(x, y)|^2 y^{1-N}\, dx\, dy$$

and when $t = 0$, we simply write $A^2_\alpha(u)(x_0)$.

The size of the area function and the non tangential maximal function intimately related. For instance, a series of results by Marcinkiewicz, Z mund, Spencer, Calderón and Stein assert that for any $\alpha > 0$, the follow two sets

$$\{x \in \mathbb{R}^N \colon u \text{ has non-tangential limit at } x\}\,, \quad \{x \in \mathbb{R}^N \colon A_\alpha(u)(x) < \infty$$

can only differ in a set of Lebesgue measure 0. Also, C. Fefferman E. Stein ([18]) proved that for any $0 < p < \infty$, there exists a const $C = C(p, N) > 0$ such that for any harmonic function u on $\mathbb{R}^{N+1}_+$ v $\lim_{t\to\infty} u(x, t) = 0$, one has

$$C^{-1}\|A_\alpha(u)\|_p \leq \|M_\alpha(u)\|_p \leq C\|A_\alpha(u)\|_p\,.$$

So that, the distribution functions of $M(u)$ and $A(u)$ have, roughly speak the same rate of growth. Many authors have proved good-λ inequalitie $M(u)$ and $A(u)$, which are direct comparisons of their distribution funct ([10, 18]). Muray and Uchiyama ([31]) proved that given $0 < \beta < \alpha$, t exists a constant $K > 1$ and positive numbers C_1, C_2 such that for $0 < \varepsilon < 1$ and any $\lambda > 0$ one has

$$\begin{aligned}|\{x \in \mathbb{R}^N \colon A_\beta(u)(x) > K\lambda\,, M_\alpha(u) < \varepsilon\lambda\}| \\ \leq C_1 e^{-C_2/\varepsilon^2} |\{x \in \mathbb{R}^N \colon A_\beta(u)(x) > \lambda\}|\end{aligned}$$

and

$$\begin{aligned}|\{x \in \mathbb{R}^N \colon M_\beta(u)(x) > K\lambda\,, A_\alpha(u)(x) < \varepsilon\lambda\}| \\ \leq C_1 e^{-C_2/\varepsilon} |\{x \in \mathbb{R}^N \colon M_\beta(u)(x) > \lambda\}|\,.\end{aligned}$$

Here C_1, C_2 are two absolute constants only depending on the dimensi and on α, β.

The proof followed the traditional method of using Green's formula on sawtooth regions. Notice, however, that the second inequality gives a worst decay than the first one. Bañuelos, Klemes and Moore ([3]) improved this estimate and proved the subgaussian estimate

$$|\{x \in \mathbb{R}^N : M_\beta(u)(x) > K\lambda, A_\alpha(u)(x) < \varepsilon\lambda\}| \leq C_3 e^{-C_4/\varepsilon^2} |\{x \in \mathbb{R}^N : M_\beta(u)(x) > \lambda\}|.$$

Their proof was based on reducing the estimate to an analogous estimate in the dyadic-martingale setting. This scheme was first used by Chang, Wilson and Wolff in ([14]).

These results give comparisons between $A_\alpha(u)$ and $M_\beta(u)$ on the set where these functions are finite. In its complement, Bañuelos, Klemes and Moore ([3]) proved a law of the iterated logarithm. This may be stated as follows: given $0 < \beta < \alpha$ and $0 < \gamma < 1$, there exists a constant $C = C(\alpha, \beta, \gamma, N) > 0$ such that for any harmonic function u on $\mathbb{R}^{N+1}_+$, one has

$$\limsup_{\substack{(y,t)\to(x,0)\\(y,t)\in\Gamma_\beta(x)}} \frac{|u(y,t)|}{\sqrt{A^2_\alpha(u)(x,\gamma t)\log\log A_\alpha(u)(x,\gamma t)}} \leq C,$$

for almost every $x \in \{x \in \mathbb{R}^N : A_\alpha(u)(x) = \infty\}$. The opposite inequality also holds if certain regularity properties of the function u are required ([4]). It is especially interesting to us that if u satisfies the Bloch condition

$$y|\nabla u(x,y)| \leq C_1, \qquad (x,y) \in \mathbb{R}^{N+1}_+,$$

for some fixed constant C_1, then

$$\limsup_{\substack{(y,t)\to(x,0)\\(y,t)\in\Gamma_\alpha(x)}} \frac{|u(y,t)|}{\sqrt{A^2_\alpha(u)(x,t)\log\log A^2_\alpha(u)(x,t)}} \geq C_2,$$

for almost every $x \in \{x \in \mathbb{R}^N : A_\alpha(u)(x) = \infty\}$.

Given a positive measure μ on Q_0, let u be its harmonic extension. As in the martingale setting, the results mentioned above are very well suited to treat additive properties of the measure μ. In our situation it is more natural to consider the function $\log u$ and its area function

$$A^2_\alpha(\log u)(x) = \int_{\Gamma_\alpha(x)} \frac{|\nabla u(w,y)|^2}{u(w,y)^2} y^{1-N}\, dwdy, \quad x \in \mathbb{R}^N. \tag{1.3}$$

This square function was first considered by J. Brossard [8], who gave a probabilistic proof of part (a) in the following result.

Theorem 1.5. *Let μ be a positive measure on $\mathbb{R}^N$ such that*

$$\int_{\mathbb{R}^N} \frac{d\mu(x)}{1+|x|^{N+1}} < \infty .$$

Let f be the density of μ with respect to Lebesgue measure. Fix $\alpha > 0$. u be the harmonic extension of μ and $A_\alpha(\log u)(x)$ its area function. Th

(a) The sets

$$\{x \in \mathbb{R}^N : f(x) > 0\} \quad \text{and} \quad \{x \in \mathbb{R}^N : A_\alpha(\log u)(x) < \infty\}$$

can only differ in a set of Lebesgue measure 0.

(b) Consider the maximal function

$$M_\alpha(\log u)(x) = \sup\{|\log u(w,y)| : (w,y) \in \Gamma_\alpha(x)\} .$$

Let Q be a cube in $\mathbb{R}^N$. Then, for $0 < p < \infty$, $M_\alpha(\log u) \in L^p(Q$ and only if $A^2_\alpha(\log u) \in L^p(Q)$.

(c) Let Q be a cube in $\mathbb{R}^N$. Then, for $1 < p < \infty$, $\log f \in L^p(Q)$ if and o if $A^2_\alpha(\log u) \in L^p(Q)$.

(d) There exists a constant $c = c(N,\alpha)$ such that $\exp(cA^2_\alpha(\log u)) \in$ implies that μ and Lebesgue measure are mutually absolutely continu

(e) There exists a constant $C = C(N,\alpha)$ such that if $A^2_\alpha(\log u)(x) < 1$ any $x \in Q$, then $\exp(CM^2_\alpha(\log u)) \in L^1(Q)$.

Corollary 1.6. *With the notations above, μ is singular if and only if*

$$A^2_\alpha(\log u)(x) = \infty \qquad \text{a.e.}(dx)x \in \mathbb{R}^N .$$

Observe that in this result the measure μ does not need to be doub Let us explain why. Given a positive measure μ in $\mathbb{R}^N$ we may cons two extensions to $\mathbb{R}^{N+1}_+$: on one hand its harmonic extension that we de by $u(x,y)$, and on the other hand the average of the measure on the $Q(x,y)$ centered at $x \in \mathbb{R}^N$ and sidelength $y > 0$ that we denote by $\widetilde{u}(x$ So $\widetilde{u}(x,y) = \mu(Q(x,y))/y^N$. The reason why we do not need to assume doubling hypothesis on μ in Theorem 1.5 is that Harnack's inequality h for u but not for $\widetilde{u}$ unless μ is doubling.

A proof of (a) and (b) can be given using the identity

$$\Delta(\log u) = -\frac{|\nabla u|^2}{u^2} ,$$

Green's Theorem and the traditional techniques of "sawtooth regions" for instance, [35, Chapter VII]). However, (d) and (e) are harder be they really depend on the martingale subgaussian estimate of [14].

As it was mentioned before, Bañuelos and Moore ([6]) proved subgaussian estimates between the distribution functions of the non-tangential maximal function of a harmonic function and its area function. Their method was based on using a truncated version of a Calderón-type reproducing formula, to approximate a harmonic function by dyadic martingales, up to an error controled by its area function. To obtain the good λ-inequalities which are needed in our setting, we apply their scheme to the logarithm of a positive harmonic function. Then, a new term, $Q(\log u)$, which is analogue to the process X_n in the martingale setting, arises but it can be estimated by the square of the area function. This is sufficient to prove (d) and (e). In our analysis it is crucial that the logarithm of a positive harmonic function shares some properties with harmonic functions, such as a submean property for its gradient. As in the martingale setting, the subgaussian estimate also leads to a Law of the Iterated Logarithm.

We need some notation. Given $y \geq 0$, the (doubly) truncated cone $\Gamma_\alpha(x,y)$ with vertex at $x \in \mathbb{R}^N$ is

$$\Gamma_\alpha(x,y) = \{(w,t) \in \mathbb{R}^{N+1}_+ : |w-x| < \alpha t\,,\ y < t < 1\}$$

and the (doubly) truncated area function of $\log u$ is defined by

$$A^2_\alpha(\log u)(x,y) = \int_{\Gamma_\alpha(x,y)} t^{1-N} \frac{|\nabla u(w,t)|^2}{u(w,t)^2}\, dw\, dt\,.$$

Given a smooth, positive, radial function K supported on $\{x \in \mathbb{R}^N : \|x\| < \rho\}$, with integral 1, consider

$$Q(x,\varepsilon) = \int_\varepsilon^1 \int_{\mathbb{R}^N} t^{1-N} K((x-w)/t) \frac{|\nabla u(w,t)|^2}{u(w,t)^2}\, dw\, dt\,, \quad x \in \mathbb{R}^N. \tag{1.4}$$

Observe that the integral is really over the cone $\Gamma_\rho(x) \subset \mathbb{R}^{N+1}_+$. So, $Q(x,\varepsilon)$ is a smooth version of the truncated area function defined above. Actually, if $0 < \beta < \rho < \alpha$, then

$$C^{-1} A^2_\beta(\log u)(x,y) < Q(x,y) < C A^2_\alpha(\log u)(x,y),$$

where C is a numerical constant depending on α, β, N and the function K. Our result shows that $\log u(x,y)$ behaves very closely to $Q(x,y)$ at the set $x \in \{x \in \mathbb{R}^N : \lim_{y\to 0} u(x,y) = 0\}$, which coincides, except for a set of Lebesgue measure 0, with $\{x \in \mathbb{R}^N : \lim_{y\to 0} A_\alpha(\log u)(x,y) = \infty\}$ and with $\{x \in \mathbb{R}^N : \lim_{y\to 0} Q(x,y) = \infty\}$.

Theorem 1.7. *Let u be a positive harmonic function on $\mathbb{R}^{N+1}_+$ and define by (1.4). Then, there exists a constant $C = C(N, K)$ such that*

$$\limsup_{y\to 0} \frac{|\log u(x,y) + Q(x,y)|}{\sqrt{Q(x,y)\log\log Q(x,y)}} \leq C\,,$$

at almost every point $x \in \{x \in \mathbb{R}^N\colon \lim_{y\to 0} u(x,y) = 0\}$.

Hence,

Corollary 1.8. *Let u be a positive harmonic function in $\mathbb{R}^{N+1}_+$. Then,*

$$\lim_{y\to 0} \frac{|\log u(x,y)|}{Q(x,y)} = 1$$

at almost every point $x \in \{x \in \mathbb{R}^N\colon \lim_{y\to 0} u(x,y) = 0\}$.

Also, given $\alpha > 0$, there exists a constant $C = C(\alpha, N)$ such that

$$C^{-1} < \limsup_{y\to 0} \frac{|\log u(x,y)|}{A^2_\alpha(\log u)(x,y)} < C\,,$$

at almost every point $x \in \{x \in \mathbb{R}^N\colon \lim_{y\to 0} u(x,y) = 0\}$.

It is worth mentioning that there is no pointwise estimate between

$$|\log u(x,y)| \quad \text{and} \quad Q(x,y) \text{ or } A^2_\alpha(\log u)(x,y).$$

Finally, we will apply these results to two different settings, the f one is related to Schwarz's Lemma and the second one to quasiconfor mappings. The problem on Schwarz's Lemma which we now describe wa fact the first motivation for our work. Let I be a holomorphic mapping fi the unit disc into itself. Schwarz's Lemma states that I decreases hyperb distances, or infinitesimally that its derivative with respect to the hyperb metric is bounded by 1, that is,

$$D(I)(z) = \frac{(1-|z|^2)|I'(z)|}{1-|I(z)|^2} \leq 1\,,$$

for any $z \in \mathbb{D}$. We would like to describe the best decay of the hyperb derivative of such non constant holomorphic functions. It is clear th normalization of the image domain is needed. A holomorphic mapping f the unit disc onto itself is called inner if

$$\left|\lim_{r\to 1} I(re^{i\theta})\right| = 1\,, \quad \text{a.e. } e^{i\theta} \in \partial\mathbb{D}.$$

For such mappings, W. Smith ([34], see also [1]) showed that the best r decay of the hyperbolic derivative is governed by a quadratic condition

Theorem B. *(a) Let w be an increasing function on $[0,1]$, $w(0)=0$. Assume*

$$\int_0 w^2(t)\frac{dt}{t} < \infty .$$

Then, there exists no non-constant inner function I such that

$$\frac{(1-|z|^2)|I'(z)|}{1-|I(z)|^2} \le w(1-|z|)), \qquad \text{for any } z \in \mathbb{D}.$$

(b) Let w be an increasing function on $[0,1]$, $w(0)=0$, such that $w(t)/t^{1-\varepsilon}$ decreases for some $\varepsilon > 0$. Assume

$$\int_0 w^2(t)\frac{dt}{t} = \infty .$$

Then, there exists an inner function I such that

$$\frac{(1-|z|^2)|I'(z)|}{1-|I(z)|^2} \le w(1-|z|), \quad \text{for any } z \in \mathbb{D}.$$

We will apply Theorem 1.5 to find a non-radial version of this result. As it was noticed in [1], this is really a problem on positive harmonic functions. Actually, let

$$\tau(z) = \frac{1+z}{1-z}$$

be a conformal mapping from the unit disc onto the right half plane. Given a holomorphic function $I\colon \mathbb{D} \to \mathbb{D}$, let u be the real part of $\tau \circ I$. So, u is a positive harmonic function and therefore the Poisson integral of a positive measure μ on $\partial\mathbb{D}$. It is clear that I is inner if and only if μ is a singular measure. Also, a simple calculation shows

$$2\frac{|I'(z)|}{1-|I(z)|^2} = \frac{|\nabla u(z)|}{u(z)}, \quad z \in \mathbb{D}.$$

Given $\theta \in [0, 2\pi]$ and $\alpha > 1$ we denote by $\Gamma_\alpha(\theta)$ the Stölz angle with vertex at $e^{i\theta}$, that is

$$\Gamma_\alpha(\theta) = \{z \in \mathbb{D}\colon |z - e^{i\theta}| \le \alpha(1-|z|)\}.$$

We consider the following hyperbolic version of the area function

$$\mathcal{A}^2_\alpha(I)(e^{i\theta}) = \int_{\Gamma_\alpha(\theta)} \frac{|I'(z)|^2}{(1-|I(z)|^2)^2}\, dm(z),$$

which corresponds to the usual area function once one has replaced the euclidean derivative and the Lebesgue measure by the hyperbolic derivative $D(I)(z)$ and the hyperbolic density $dm(z)/(1-|z|^2)^2$.

Geometrically, $\mathcal{A}^2_\alpha(I)(e^{i\theta})$ is the hyperbolic area (in the d itir multiplicities, of $I(\Gamma_\alpha(\theta))$. With the previous notation $\mathcal{A}^2_\alpha(I)(e^{i\theta})$ coincid with $4^{-1}A^2_\alpha(\log u)(e^{i\theta})$. Part (a) of our next result can be understood as t hyperbolic analogous of the classical result by Calderón on the existence radial limits at almost every point where the usual area function is finit More concretely, since the function I is bounded, Fatou's Theorem asser that I has finite radial limits at almost every point of the unit circle. Ho ever, having limits of modulus 1 means going to infinity in the hyperbo metric of the disc. So, (a) states that the set where this occurs coincid modulo sets of Lebesgue measure 0, with the set where the hyperbolic ar function is infinite.

Theorem 1.9. *(a) Let I be a holomorphic mapping from the unit disc i itself. Fix $\alpha > 0$. Then, the sets*

$$\{e^{i\theta} \in \partial\mathbb{D} : |\lim_{r\to 1} I(re^{i\theta})| < 1\} \quad \text{and} \quad \{e^{i\theta} \in \partial\mathbb{D} : \mathcal{A}^2_\alpha(I)(e^{i\theta}) < \infty\}$$

can only differ in a set of Lebesgue measure 0.

(b) Let h be a positive function $h\colon \mathbb{D} \to (0,\infty)$ for which

$$(99/100)h(w) \le h(z) \le (101/100)h(w)\,,$$

for any pair of points z, w such that $|z-w|/|1-\overline{w}z| \le 1/2$. Given $\alpha >$ assume that

$$\int_{\Gamma_\alpha(\theta)} h^2(z)\, dm(z) = \infty\,,$$

at almost every $\theta \in [0, 2\pi)$. Assume also that, for any $0 \le \theta < 2\pi$, function $h(re^{i\theta})$ increases in $r \in [0,1)$. Then, there exists a non const inner function I such that

$$\frac{|I'(z)|}{1-|I(z)|^2} \le h(z)\,, \qquad \text{for any } z \in \mathbb{D}.$$

Observe that (a) implies part (a) in Theorem B. Actually if

$$\frac{(1-|z|^2)|I'(z)|}{1-|I(z)|^2} \le w(1-|z|)\,, \quad z \in \mathbb{D}\,,$$

one would deduce that for any $\theta \in [0, 2\pi]$,

$$\int_{\Gamma_\alpha(\theta)} \left(\frac{|I'(z)|}{1-|I(z)|^2}\right)^2 dm(z) \le \int_{\Gamma_\alpha(\theta)} \frac{w^2(1-|z|)}{(1-|z|)^2}\, dm(z) \le C(\alpha) \int_0^1 w^2(t)$$

Also, part (b) of Theorem B corresponds to the case

$$h(z) = w(1-|z|)/(1-|z|)$$

and the crucial integral assumption on h follows from the corresponding on $w(t)$.

As in the preceding situations, L^p estimates between the hyperbolic area function of I and the non-tangential maximal function of $\log(1-|I|)$ can be proved. Also, a Law of the Iterated Logarithm is obtained. Given a smooth, positive, radial function K supported on $(-\rho,\rho)$, with integral 1, consider

$$\mathcal{Q}(re^{i\theta}) = \int_{\Gamma_\rho(\theta,r)} K((\operatorname{Arg} z - \theta)/\rho)\frac{|I'(z)|^2}{(1-|I(z)|^2)^2}\,dm(z) \quad , \tag{1.5}$$

where $\Gamma_\alpha(\theta,r) = \Gamma_\alpha(\theta)\cap\{|z|<r\}$ is the truncated Stölz angle with vertex at $e^{i\theta}$. This is a smooth version of the truncated hyperbolic area function given by

$$\mathcal{A}_\rho(I)(re^{i\theta}) = \int_{\Gamma_\rho(\theta,r)} \frac{|I'(z)|^2}{(1-|I(z)|^2)^2}\,dm(z) \quad .$$

Our result is the following.

Theorem 1.10. *Let I be a holomorphic mapping from the unit disc into itself and let $\mathcal{Q}$ be defined by (1.5). Then, there exists a constant $C = C(\alpha,K)$ such that*

$$\limsup_{r\to 1} \frac{|\log(1-|I(re^{i\theta})|^2) + \mathcal{Q}(re^{i\theta})|}{\sqrt{\mathcal{Q}(re^{i\theta})\log\log\mathcal{Q}(re^{i\theta})}} \le C \,,$$

at almost every point $e^{i\theta}\in\{e^{i\theta}\colon \lim_{r\to 1}|I(re^{i\theta})|=1\}$.

The other application of Theorem 1.5 is related to the theory of quasiconformal mappings of the upper half plane $\mathbb{R}^+_2$ onto itself. Let $\rho:\mathbb{R}^+_2\to\mathbb{R}^+_2$ denote a quasiconformal mapping, $\rho(\infty)=\infty$, with complex dilatation $\sigma = \overline{\partial}\rho/\partial\rho$. Then classical results show that ρ extends quasiconformally to $\mathbb{R}^2$, in particular its restriction to the boundary $F=\rho|_{\mathbb{R}}$ defines a homeomorphism on $\mathbb{R}$ which is quasisymmetric, that is its derivative is a doubling measure. The converse is also true: any quasisymmetric map $F:\mathbb{R}\to\mathbb{R}$ can be extended quasiconformally to $\mathbb{R}^2$. There is also a one to one correspondence between such quasisymmetric mappings F and conformal weldings of quasicircles, that is

$$F(x) = \Phi[\Psi(x)]\,; \quad x\in\mathbb{R}$$

where Ψ is a normalized conformal mapping of $\mathbb{R}^-_2$ onto a quasidisc Ω which admits a quasiconformal extension to $\mathbb{R}^+_2$ with complex dilatation σ, and Φ is a conformal mapping of $\mathbb{C}\setminus\Omega$ onto $\mathbb{R}^+_2$. A very well studied problem is to investigate the smoothness properties of the boundary correspondence F in terms of the complex dilatation σ. In [13], Carleson studies this problem obtaining the conclusions of Theorem A stated at the beginning of the introduction, where $w(t)$ is, in this case, defined by

$$w(t) = \sup_{0<Imz<t} |\sigma(z)|$$

We are interested in a result [15, Th. 4] proved by Dyn'kin, which we sta below and that can be thought as a pointwise version of Carleson's result

Theorem C. *Let $\sigma(z)$ be the dilatation coefficient of a quasiconformal se mapping Ψ of $\mathbb{C} \setminus \mathbb{D}$, with $\Psi(\infty) = \infty$ and let*

$$g(\sigma)(e^{i\theta}) = \left(\int_{\mathbb{C}\setminus\mathbb{D}} \frac{|\sigma(\xi)|^2}{|\xi - e^{i\theta}|^2}\, dm(\xi) \right)^{1/2}.$$

Then there exists $a > 0$ such that if

$$\exp(ag^2(\sigma)) \in L^1(\partial\mathbb{D})$$

then Ψ is absolutely continuous on $\partial\mathbb{D}$.

Dyn'kin's proof uses the representation of $\Psi|_{\mathbb{R}}$ as the welding of a qua circle Γ and shows that under the hypothesis of the theorem, Γ is rectifiab His main tool is the pointwise estimate

$$S(e^{i\theta}) \leq \text{const}\, g(\sigma)(e^{i\theta}), \quad \forall e^{i\theta} \in \partial\mathbb{D} \tag{1.6}$$

where

$$S(e^{i\theta}) = \left(\int_{\Gamma_\alpha(\theta)} \left| \frac{f''(\xi)}{f'(\xi)} \right|^2 \right)^{1/2}$$

and where f represents the conformal mapping from $\mathbb{D}$ onto the inner dom bounded by Γ.

We will consider instead of S, the area function $A^2_\alpha(\log u)$; where u is harmonic extension of a doubling measure μ, and prove an inequality sim to (1.6), which does not involve conformal techniques, and which will a us to recover Dyn'kin's Theorem. Since we state our result on the real li we need to impose some control on the doubling constant of μ to guarar the convergence of its Poisson integral. More precisely we obtain:

Theorem 1.11. *Let μ be a doubling measure on $\mathbb{R}$, with doubling const $C < 2\sqrt{2}$, that is $\mu(2I) \leq C\mu(I)$, for all intervals $I \subset \mathbb{R}$. Denote $u = P[\mu]$ the harmonic extension of μ to $\mathbb{R}^2_+$ and let $\rho : \mathbb{R}^2 \to \mathbb{R}^2$ l quasiconformal extension of $\rho(x) = \mu[0, x]$. Then*

$$\int_{\Gamma_\alpha(x)} \frac{|\nabla u(z)|^2}{u^2(z)}\, dm(z) \leq c \int_{\mathbb{R}^2_-} \frac{|\sigma(w)|^2}{|w - x|^2}\, dm(w)$$

where σ is the complex dilatation of ρ and $c = c(\alpha, C)$ is a constant depen on α and C.

As a consequence of this theorem and Theorem 1.5, we obtain

Theorem 1.12. *With the same notation as in Theorem 1.11, the following holds:*

(a) There exists a constant $a > 0$ such that if

$$\exp\left(a\int_{\mathbb{R}^-_2}\frac{|\sigma(w)|^2}{|w-x|^2}\,dm(w)\right)\in L^1_{\mathrm{loc}}\,,$$

then μ and Lebesgue measure are mutually absolutely continuous.

(b) If μ is singular, then

$$\int_{\mathbb{R}^2_-}\frac{|\sigma(w)|^2}{|w-x|^2}\,dm(w)=\infty\,,$$

a.e. $(dx)x\in\mathbb{R}$.

(c) Let $0<p<\infty$. If

$$\int_{\mathbb{R}^-_2}\frac{|\sigma(w)|^2}{|w-x|^2}\,dm(w)\in L^p_{\mathrm{loc}}\,,$$

then $M_\alpha(\log u)\in L^p_{\mathrm{loc}}$.

The paper is organized as follows. Section 2 contains the results on dyadic doubling measures. Also an example of a positive, singular, dyadic doubling measure μ, for which the square functions X_n, $A^2_n(\mu)$ can be given explicitely, is presented.

Section 3 is devoted to the results on positive harmonic functions. A class of positive harmonic functions, for which the square function has maximal growth, is also introduced.

Section 4 contains the proof of Theorems 1.9 and 1.10 as well as an analogous result for inner functions into hyperbolic domains in the complex plane.

In Section 5, we prove the results related to quasiconformal mapppings, in particular Theorems 1.11 and 1.12.

Finally, in Section 6 a non-dyadic square function of a doubling measure is introduced. This section also contains some questions.

We would like to thank R. Bañuelos, J. L. Fernández and P. MacManus for several helpful conversations.

2. Dyadic doubling measures

The logarithmic transform of a dyadic martingale

There is a correspondence between positive dyadic martingales and positi measures. Given a positive measure μ in Q_0, we may consider the positi dyadic martingale defined by

$$S_n = \sum \frac{\mu(Q)}{|Q|}\chi_Q ,$$

where the sum is taken over all dyadic subcubes Q of Q_0 of generation Conversely, given a positive dyadic martingale $\{S_n\}$ we may define a positi measure μ on Q_0 by first defining it on dyadic subcubes Q as

$$\mu(Q) = S_n(Q)|Q| ,$$

if Q is of generation n, and extending it by standard methods. As it w mentioned at the Introduction, the martingale S_n defined above does r seem to be well suited to study multiplicative properties of the measure and roughly speaking, one should consider instead its logarithm.

Let S_n be a positive dyadic martingale on a cube Q_0 in $\mathbb{R}^N$. Then log is a supermartingale. We will show that there exists a unique (except adding constants) process X_n such that

(a) $\log S_n + X_n$ is a dyadic martingale.

(b) X_n is F_{n-1}-measurable, that is, X_n is constant on each cube of gene tion $n-1$.

Let us first show the unicity. Assume that $\{Y_n\}$ is a dyadic mart gale such that Y_n is constant on cubes of generation $n-1$, $n = 1, 2, .$ Let Q, Q' be two cubes of generation $n-1$ which are contained in the sa cube $\tilde{Q}$ of generation $n-2$. Since $Y_{n-1}(Q)$ is the arithmetic mean of on the cubes of generation n contained in Q, we have $Y_n(Q) = Y_{n-1}($ Similarly $Y_n(Q') = Y_{n-1}(Q')$. So, $Y_n(Q) = Y_n(Q')$. Hence, Y_n is constant cubes of generation $n-2$. Repeating the argument on cubes of generat $n-2, \ldots, 1$, one shows that Y_n is constant.

As the following lemma asserts, the process X_n satisfying (a) and can be given explicitely. We recall the notation given in the introduct Given a positive martingale $\{S_n\}$ in Q_0 and a dyadic sub Q generation n, we denote by a.m.(Q), (g.m.(Q)) the arithmetic (geomet mean of the values of S_n in the cubes of generation $(n+1)$ contained in

More explicitely, if Q is of generation n, then

$$\text{a.m.}(Q) = S_n(Q) = \frac{1}{2^N}\sum S_{n+1}(Q')\,,$$
$$\text{g.m.}(Q) = \prod S_{n+1}(Q')^{1/2^N}$$

where both the sum and the product are taken over the 2^N cubes Q' of generation $(n+1)$ contained in Q. Consider

$$X_n = \sum_k X_n(Q_k^{(n)})\chi_{Q_k^{(n)}}\,, \qquad n = 1, 2, \ldots, \tag{2.1}$$

where $\{Q_k^{(n)}\colon k = 1, \ldots, 2^{Nn}\}$ are the dyadic subcubes of Q_0 of generation n and

$$X_n(Q_k^{(n)}) = -\sum_j \log\left(\frac{\text{g.m.}(Q_j)}{\text{a.m.}(Q_j)}\right)\,,$$

where the sum is taken over all dyadic subcubes Q_j of Q_0 which properly contain $Q_k^{(n)}$. It is clear that X_n is positive, increasing process satisfying property (b). Also, $X_n(Q)$ measures, roughly speaking, the error the martingale does when doubling at the dyadic cubes containing Q.

Lemma 2.1. *Let $\{S_n\}$ be a positive dyadic martingale in a cube Q_0. Let X_n, $n = 1, 2, \ldots$, be the functions given by* (2.1). *Then*

$$T_n = \sum(\log S_n + X_n)\chi_Q\,,$$

where the sum is taken over all dyadic subcubes $\{Q\}$ of Q_0 of generation n, is a dyadic martingale.

Proof. Let Q be a dyadic subcube of Q_0 of generation n and let us denote by $\{Q_k\}$ the 2^N dyadic cubes of generation $n+1$ contained in Q. We have to show

$$T_n(Q) = \frac{1}{2^N}\sum_k T_{n+1}(Q_k)\,. \tag{2.2}$$

Since $X_{n+1}(Q_k) = X_{n+1}(Q_j)$, for any $k, j = 1, \ldots, 2^N$ and

$$X_n(Q) - X_{n+1}(Q_k) = \log\frac{\text{g.m.}(Q)}{\text{a.m.}(Q)}\,,$$

the equation (2.2) is equivalent to

$$\log S_n(Q) = \frac{1}{2^N}\sum_k \log S_{n+1}(Q_k) - \log\frac{\text{g.m.}(Q)}{\text{a.m.}(Q)}\,.$$

Since $\text{a.m.}(Q) = S_n(Q)$, this is an identity and the lemma is proved. ∎

The results mentioned at the introduction relate the behaviour of T_n $\log S_n + X_n$ with the size of the square function of the martingale. The ma observations, which will be proved below, are that when $\{S_n\}$ is a doubli martingale, X_n is pointwise comparable to $\langle T\rangle_n^2$ and that $\langle T\rangle(x)$ behaves

$$\left(\sum_n \|(\log S_n/S_{n-1})X_{Q_{n-1}(x)}\|_\infty^2\right)^{1/2}, \qquad x \in Q_0\,.$$

Comparing X_n and $\langle T\rangle_n^2$ for doubling martingales

A positive dyadic martingale $\{S_n\}$ is called doubling if there exists a consta $M \geq 1$ such that

$$M^{-1} \leq S_{n+1}/S_n \leq M\,, \qquad n = 1, 2, \ldots\,.$$

The main estimate is given in the following result.

Lemma 2.2. *Let $\{S_n\}$ be a dyadic doubling martingale in Q_0. Let X_n the process given by* (2.1) *and $T_n = \log S_n + X_n$ be the logarithm transfc of the martingale. Then, there exists a constant $C > 0$, which depends the doubling constant of the martingale, such that*

$$C^{-1} \leq \frac{\langle T\rangle_n^2(x)}{\sum_{k=1}^n \|\log(S_k/S_{k-1})\chi_{Q_{k-1}(x)}\|_\infty^2} \leq C\,,$$

$$C^{-1} \leq \frac{X_n(x)}{\sum_{k=1}^n \|\log(S_k/S_{k-1})\chi_{Q_{k-1}(x)}\|_\infty^2} \leq C\,,$$

for any $x \in Q_0$, $n = 1, 2, \ldots$.

Proof. We will show that in the corresponding sums, the terms are con rable, that is,

$$C^{-1} \leq \frac{\|(T_{j+1} - T_j)\chi_{Q_j(x)}\|_\infty^2}{\|\log(S_{j+1}/S_j)\chi_{Q_j(x)}\|_\infty^2} \leq C\,,$$

$$C^{-1} \leq \frac{-\log(\text{g.m.}(Q_j)/\text{a.m.}(Q_j))}{\|\log(S_{j+1}/S_j)\chi_{Q_j(x)}\|_\infty^2} \leq C\,,$$

To prove the first estimate note that $\|(T_n - T_{n-1})\chi_{Q_{n-1}(x)}\|_\infty$ is compai to $\max|T_n(Q_j) - T_n(Q_k)|$, where the maximum is taken over the 2^N dy cubes of generation n contained in $Q_{n-1}(x)$. Since X_n is predictible, quantity is $\max|\log(S_n(Q_j)/\ S_n(Q_k))|$. The estimate follows now fron fact that S_n is a doubling martingale.

To prove the second one we will show that given $L = 2^N$ positive numbers $a_1, \dots, a_L$, with $M^{-1} \le a_k/a_j \le M$, $k, j = 1, \dots, L$, there exists a constant $C = C(M, L) > 0$ such that

$$C^{-1} \le \frac{-\log\left(\left(\prod a_k^{1/L}\right) / \frac{1}{L}\sum a_k\right)}{\max_j \left(\log(a_j/\frac{1}{L}\sum a_k)\right)^2} \le C\,.$$

We may assume that $\sum a_k = L$, then we have to show that

$$C^{-1} \le \frac{-\log\left(\prod_k a_k^{1/L}\right)}{\max_k (\log a_k)^2} \le C\,.$$

Writing $a_k = 1 + x_k$, this follows from the following observation. ∎

Lemma 2.3. *Let $m > -1$, $M > 0$ and $x_k \in [m, M]$, $k = 1, \dots, L$, so that*

$$\sum_{k=1}^{L} x_k = 0\,.$$

Then, there exists a constant $C = C(m, M) > 0$ such that

$$C^{-1} \le \frac{-\sum_{k=1}^{L} \log(1 + x_k)}{\sum_{k=1}^{L} x_k^2} \le C\,.$$

Observe that $\sum_{k=1}^{L} \log(1 + x_k) \le 0$, because the arithmetic mean of $x_1, \dots, x_L$ vanishes. So, Lemma 2.3 shows that the trivial estimate

$$-\sum_{k=1}^{L} \log(1 + x_k) \le C \sum_{k=1}^{L} |x_k|\,,$$

can be substantially improved when $\sum_{k=1}^{L} x_k = 0$.

Proof of Lemma 2.3. One may assume that x_k, $k = 1, \dots, L$, are close to 0, because otherwise the geometric mean of $1 + x_k$ is bounded away from one and then there exists $C_1 < 0$ such that

$$\sum \log(1 + x_k) \le C_1 < 0$$

and the result follows in this case. So, we assume x_k, $k = 1, \dots, L$ are close to 0. Using that $-\log t = 1 - t + O((1 - t)^2)$ for t close to 1, one gets

$$-\log \prod_{k=1}^{N} (1 + x_k) \simeq 1 - \prod_{k=1}^{N} (1 + x_k)\,.$$

We now prove (e). It is sufficient to show that

$$N(x) = \sup \left\{ \frac{\mu(Q)}{|Q|}, \left(\frac{\mu(Q)}{|Q|} \right)^{-1} \right\} \in L^1(Q_0)$$

where the supremum is taken over all dyadic cubes which contain x. Let

$$M(x) = \sup \left| \log \frac{\mu(Q)}{|Q|} \right|.$$

Since $e^M = N$, we will show that $e^M \in L^1(Q_0)$. Observe that

$$\begin{aligned} &\int_{Q_0} (e^M - 1)\, dx = \\ &\quad = \int_0^\infty e^\lambda |\{M > \lambda\}|\, d\lambda = 3 \int_0^\infty e^{3\lambda} |\{M > 3\lambda\}|\, d\lambda \leq \\ &\quad \leq 3 \int_0^\infty e^{3\lambda} |\{M > 3\lambda,\ A < \varepsilon_1 \lambda^{1/2}\}|\, d\lambda + 3 \int_0^\infty e^{3\lambda} |\{A > \varepsilon_1 \lambda^{1/2}\}|\, d\lambda, \end{aligned}$$

where $0 < \varepsilon_1 < 1$ is a constant to be fixed later. Using the estimate (a) in Theorem 2.4 we bound the first term by

$$3C_1 \int_0^\infty \exp((3 - C_2/\varepsilon_1^2)\lambda) |\{M > \lambda\}|\, d\lambda \leq 3C_1 \int_{Q_0} M,$$

if ε_1 is chosen small enough to guarantee $3 - C_2/\varepsilon_1^2 < 0$. The second term is

$$3 \int_0^\infty e^{3\lambda} |\{A^2 > \varepsilon_1^2 \lambda\}|\, d\lambda = \int_{Q_0} \exp(3A^2/\varepsilon_1^2)\, dx.$$

So, the result follows taking $C = 3/\varepsilon^2$. The proof of (f) is similar and we omit it. ∎

It should be observed that part (e) of Theorem 2.5 implies part (a) of Theorem A mentioned in the introduction. Recall that $w(t)$ was defined as

$$w(t) = \sup \max \left| \frac{\mu(\widetilde{Q}) 2^N}{\mu(Q)} - 1 \right|,$$

where the supremum is taken over all dyadic cubes Q of sidelength smaller than t and the maximum is over the 2^N dyadic subcubes $\widetilde{Q}$ contained in Q of sidelength $l(Q)/2$. Then

$$A^2(\mu)(x) \leq \sum_{n=1}^\infty w^2(2^{-n}) \leq \int_0^1 w^2(t) \frac{dt}{t}.$$

So, if the last integral converges, the square function $A^2(\mu)$ is bounded and it follows from (e) that μ must be absolutely continuous.

The hypothesis in part (e) of Theorem 2.5 is closely related to the space BMO. Let Q_0 be a cube in $\mathbb{R}^N$. The space dyadic BMO(Q_0) consists on the integrable functions H defined on Q_0 for which

$$\|H\|_{\mathrm{BMO}} = \sup \frac{1}{|Q|} \int_Q |H - H_Q| < \infty,$$

where

$$H_Q = \frac{1}{|Q|} \int_Q H$$

and the supremum is taken over all dyadic subcubes Q of Q_0. Chebyshev inequality gives that

$$|\{x \in Q\colon |H(x) - H_Q| > \lambda\}| \leq \frac{1}{\lambda}|Q|.$$

However, the fact that the BMO condition is given at all scales, implies a stronger estimate. This is the John-Nirenberg Theorem that states that there exist constants $C_1, C_2 > 0$ such that for any dyadic cube Q and any H in dyadic BMO, one has

$$|\{x \in Q\colon |H(x) - H_Q| > \lambda\}| \leq C_1 \exp(-C_2\lambda/\|H\|_{\mathrm{BMO}})|Q|.$$

Given a measurable function H defined on Q_0, let $\varepsilon(H)$ be the infimum of $\varepsilon > 0$ for which

$$\sup \frac{1}{|Q|} \int_Q \exp(\varepsilon^{-1}|H(x) - H_Q|) < \infty,$$

where the supremum is taken over all dyadic subcubes Q of Q_0. The John-Nirenberg Theorem implies that $H \in \mathrm{BMO}$ if and only if $\varepsilon(H) < \infty$. Garnett and Jones ([20, 21]) proved that there exists a constant $C = C(N)$ such that

$$C^{-1}\varepsilon(H) < \mathrm{dist}_{\mathrm{BMO}}(H, L^\infty) = \inf\{\|H - G\|_{BMO} : G \in L^\infty\} < C\varepsilon(H). \tag{2.4}$$

Corollary 2.8. *There exists a constant $\varepsilon_0 = \varepsilon_0(N)$ such that if μ is a dyadic doubling measure on Q_0 and $\varepsilon(A^2(\mu)(x)) \leq \varepsilon_0$, then μ is absolutely continuous. Moreover $\mu = f\,dx$ with $f, 1/f \in L^1(Q_0)$.*

Proof. Applying (2.4), we have

$$A^2(\mu) = H_1 + H_2,$$

where $H_1 \in L^\infty$ and $\|H_2\|_{\mathrm{BMO}}$ is small if ε_0 is chosen sufficiently small. Thus, the John-Nirenberg Theorem tells $e^{CH_2} \in L^1(Q_0)$ if $C < C(\varepsilon_0)$ as $\varepsilon \to 0$. Then the corollary follows from part (e) of Theo

The quadratic condition given in (b) of Theorem 2.5 characterizing singular measures is also sharp in the following sense.

Proposition 2.9. *Let $h\colon \mathbb{R}^2_+ \to (0,\infty)$ be a positive function such that if $B \subset \mathbb{R}^2_+$ is a disc with $2B \subset \mathbb{R}^2_+$, then*

$$\sup_B h \leq \frac{101}{100} \inf_B h\,.$$

Fixed $\alpha > 0$, assume

$$\int_{\Gamma_\alpha(x)} h^2(z)\,\frac{dm(z)}{(Imz)^2} = \infty\,,$$

at almost every $x \in \mathbb{R}$. Then, there exists a positive, doubling, singular measure μ such that

$$\left|\frac{\mu(I)}{\mu(I')} - 1\right| \leq C\,h(x_I + i|I|)\,, \tag{2.5}$$

for any interval $I \subset \mathbb{R}$, $|I| \leq 1$. Here I' is an interval adjacent to I of the same sidelength and x_I is the center of I.

Proof. The construction of the measure follows the ideas in [13] and [24]. We will define μ by giving its mass on intervals of the form

$$J_k^{(n)} = [k4^{-n}, (k+1)4^{-n})\,, \tag{2.6}$$

where $k \in \mathbb{Z}$ and $n = 0, 1, 2, \ldots$ The intervals $\{J_k^{(n)}\colon k \in \mathbb{Z}\}$ are called of generation n. We may assume that $h \leq 1/8$. Set $\mu(J_k^{(0)}) = 1$, $k \in \mathbb{Z}$, and, by induction, assume $\mu(J_k^{(n)})$ has been defined satisfying

$$(1 + h(J_k^{(n)}))^{-1} \leq \frac{\mu(J_k^{(n)})}{\mu(J_{k+1}^{(n)})} \leq 1 + h(J_k^{(n)})\,, \quad k \in \mathbb{Z}\,,$$

where $h(J_k^{(n)}) = h(k4^{-n} + i4^{-n})$. We now proceed to define $\mu(J_k^{(n+1)})$, $k \in \mathbb{Z}$.

Let J, J' be two consecutive intervals of generation n. We divide J into four intervals $J_1, \ldots, J_4$ of generation $n+1$ and similarly J' into $J_1', \ldots, J_4'$. Let J' be to the right of J and let the numbering be from the left to the right. Let $\widetilde{h}(J) = \min\{h(J_3), h(J_4), h(J_1'), h(J_2')\}$ and $\alpha(J) = (4 + \widetilde{h}(J))^{-1}$ so that

$$\frac{1 + \alpha(J)\widetilde{h}(J)}{1 - \alpha(J)\widetilde{h}(J)} = 1 + \widetilde{h}(J)/2\,. \tag{2.7}$$

Assume $\mu(J) \le \mu(J')$. Then, set

$$\mu(J_3) = (1 - \alpha(J)\widetilde{h}(J))\frac{\mu(J)}{4},$$

$$\mu(J_4) = (1 + \alpha(J)\widetilde{h}(J))\frac{\mu(J)}{4},$$

$$\mu(J_1') = (1 - \alpha(J)\widetilde{h}(J))\frac{\mu(J')}{4},$$

$$\mu(J_2') = (1 + \alpha(J)\widetilde{h}(J))\frac{\mu(J')}{4}.$$

If $\mu(J) > \mu(J')$, we change all signes in the above formula. This formu defines the mass of μ over intervals of generation $n+1$. Next we show th

$$(1 + h(I))^{-1} \le \frac{\mu(I)}{\mu(I')} \le 1 + h(I), \tag{2.8}$$

for any pair of consecutive intervals I, I' of generation $n+1$. The ide tity (2.7) gives (2.8) when I, I' are J_3, J_4 or J_1', J_2'. If $I = J_4$ and $I' =$. we have to show

$$(1 + \widetilde{h}(J))^{-1} \le \frac{(1 + \alpha(J)\widetilde{h}(J))\mu(J)}{(1 - \alpha(J)\widetilde{h}(J))\mu(J')} \le 1 + \widetilde{h}(J).$$

Since $\mu(J) \le \mu(J')$, the right-hand inequality follows from (2.7). To pr the left-hand inequality, observe that, by induction, $\mu(J') \le (1 + h(J))\mu($. Thus, it is sufficient to prove

$$\frac{1 + h(J)}{1 + \widetilde{h}(J)} \le 1 + \frac{1}{2}\widetilde{h}(J),$$

which follows from the regularity assumption on the function h. Finally, need to show that

$$\frac{1 + \alpha(J)\widetilde{h}(J)}{1 - \alpha(J')\widetilde{h}(J')} \le 1 + \min\{\widetilde{h}(J), \widetilde{h}(J')\}.$$

Since $\alpha(J)$, $\alpha(J')$ are close to $1/4$, this follows from the regularity sumption on h.

So by induction we have defined the mass of μ over all intervals of form (2.6) and it satisfies (2.8) for such intervals. Next, we will show t it also holds for any interval, that is,

$$1 + C^{-1}h(x - t + it) \le \frac{\mu(x - t, x)}{\mu(x, x + t)} \le 1 + C\,h(x - t + it). \tag{2.9}$$

Assume $x = 0$ and let $4^{-n-1} \le t < 4^{-n}$. Let $(0, t) = \cup I_k$, where $\{$ are pairwise disjoint intervals of the form (2.6) with decreasing len Hence $|I_1| = 4^{-n-1}$.

Now

$$\mu(I_k) = \mu(I_1)4^{n+1-m(k)}(1 + O(h(I_1)))^{m(k)-n-1},$$

where I_k is of generation $m(k)$. Since $t = \sum 4^{-m(k)}$, we have

$$\mu(0,t) - t\mu(I_1)4^{n+1} = \mu(I_1)4^{n+1}tO(h(I_1)).$$

The same argument applies to $\mu(-t,0)$ and (2.9) follows.

To show that μ is singular, let $x \in \mathbb{R}$ and $I_n(x)$ be the unique interval of generation n containing x. The assumption on h gives that

$$\sum_n h^2(I_n(x)) = \infty, \text{a.e. } x \in \mathbb{R}. \tag{2.10}$$

Observe that

$$\mu(I_n(x)) = 4^{-n}\prod_{k=1}^{n}(1 \pm \alpha_k(I_k(x))h(I_k(x))).$$

The signes + and − occur in equal number in the construction. Since $\alpha_k(I_k(x))$ is close to 1/4, the law of large numbers and (2.10) give that at almost every point $x \in \mathbb{R}$, one has

$$\lim_{n\to\infty} \frac{\mu(I_n(x))}{4^{-n}} = 0.$$

Hence the measure μ is singular. This finishes the proof. ■

Law of the iterated logarithm for dyadic doubling measures

The subgaussian estimate (1.1) of Chang, Wilson and Wolff implies the law of the iterated logarithm for dyadic martingales, that is, if $\{T_n\}$ is a dyadic martingale and $\langle T\rangle_n$ is its (truncated) square function, one has

$$\limsup_{n\to\infty} \frac{|T_n(x)|}{\langle T\rangle_n(x)\sqrt{2\log\log\langle T\rangle_n(x)}} \leq 1$$

a.e. $x \in \{x \in Q_0 \colon \sup_n |T_n(x)| = \infty\}$.

We will apply it to the dyadic martingale $T_n = \log S_n + X_n$, given in Lemma 2.1. When the measure is doubling, $\{T_n\}$ has bounded increments and it is known that, in this situation, we also have

$$\limsup_{n\to\infty} \frac{|T_n(x)|}{\langle T\rangle_n(x)\sqrt{2\log\log\langle T\rangle_n(x)}} \geq C > 0,$$

a.e. $x \in \{x \in Q_0 \colon \sup_n |T_n(x)| = \infty\}$. See [4]. Using Lemma 2.2 to estimate $\langle T\rangle_n(x)$, we deduce Theorem 1.3 and therefore Corollary 1.4. Also, the rate of convergence in the last two results can not be improved. To show this, let us introduce in dimension 1 a well known dyadic doubling measure.

An example

Let $I_0 = [0,1]$ be the unit interval in the real line. We will define a measure by specifying its mass on dyadic intervals. Let $0 \leq \lambda < 1$ and let $\mu(I_0) = 1$. Assume $\mu(I)$ has been defined on the dyadic intervals of generation k.

We now proceed to define it on the intervals of the next generation. Given a dyadic subinterval I of I_0 of generation k, let I_+ (I_-) be its right (left) half, that is, I_+ I_- are the two intervals of generation $k+1$ contained in I. Then set

$$\mu(I_+) = \frac{1+\lambda}{2}\mu(I)\,,$$
$$\mu(I_-) = \frac{1-\lambda}{2}\mu(I)\,.$$

It is clear that if I is a dyadic interval of generation n,

$$\frac{\mu(I)}{|I|} = (1+\lambda)^k(1-\lambda)^{n-k}\,,$$

where k is the number of dyadic intervals containing I which are at right hand position.

Actually, this is a well known example. Let us consider a biassed coin, that is, a sequence of equally distributed and independent random variables $\{X_n\}$ such that

$$p\{X_n = 0\} = \frac{(1-\lambda)}{2} \quad \text{and} \quad p\{X_n = 1\} = \frac{(1+\lambda)}{2}\,.$$

Now, consider the random variable

$$X = \sum_{n=1}^{\infty} 2^{-n} X_n\,.$$

If I is a dyadic interval of generation n, we have

$$\mu(I) = p\{X \in I\}\,.$$

Equivalently, if the left end of I is

$$a = \sum_{k=1}^{\infty} \frac{t_k}{2^k}$$

where $t_k = 0$ or 1, then $\mu(I) = p\{X_1 = t_1, \ldots, X_n = t_n\}$.

By construction,

$$\frac{\mu(I_+)}{\mu(I_-)} = \frac{1+\lambda}{1-\lambda},$$

for any pair of dyadic intervals I_+, I_- of the same generation contained in a interval of the previous one. Hence μ is a dyadic doubling measure. Also, when $0 < \lambda < 1$, μ is singular. Actually, the measure μ is concentrated in a set of Haussdorff dimension

$$\gamma = \frac{(1+\lambda)\log\frac{2}{1+\lambda} + (1-\lambda)\log\frac{2}{1-\lambda}}{2\log 2},$$

that satisfies $\gamma < 1$ if $0 < \lambda < 1$. See [22] for a more general result.

Let I be a dyadic interval of length 2^{-n}. Then, an easy calculation shows

$$X_n(I) = -\frac{n}{2}\log(1-\lambda^2),$$

$$A_n^2(\mu)(I) = n\lambda^2.$$

So, Theorem 1.3 and Corollary 1.4 state that at almost (dx) every point $x \in [0,1]$, one has

$$C^{-1}(\lambda) \le \limsup_{n\to\infty} \frac{\left|\log\left(\frac{\mu(I_n(x))}{|I_n(x)|}\right) - \frac{n}{2}\log(1-\lambda^2)\right|}{\sqrt{n\log\log n}} \le C(\lambda), \tag{2.11}$$

$$\lim_{n\to\infty} \frac{\log\frac{\mu(I_n(x))}{|I_n(x)|}}{\frac{n}{2}\log(1-\lambda^2)} = 1 \tag{2.12}$$

Next, we show that these estimates are best possible. By the law of large numbers

$$\lim_{n\to\infty} \frac{A_n(x)}{n} = 1/2, \qquad \text{a.e. } (dx)\, x \in [0,1] \tag{2.13}$$

and

$$\limsup_{n\to\infty} \frac{|A_n(x) - n/2|}{\sqrt{n\log\log n}} = 1, \qquad \text{a.e. } (dx)\, x \in [0,1]. \tag{2.14}$$

Now,

$$\log\frac{\mu(I_n(x))}{|I_n(x)|} = A_n(x)\log(1+\lambda) + (n - A_n(x))\log(1-\lambda).$$

So, (2.11) and (2.12) are equivalent to (2.13) and (2.14), except for the value of the constant $C(\lambda)$.

3. Positive harmonic functions

Logarithms of positive harmonic functions

Let u be the Poisson extension of a positive measure μ. As it was explaine in the Introduction, the area function associated to the harmonic function is not well suited to treat the doubling properties of the measure, and o should consider instead the subharmonic function $-\log u$.

The natural area function associated to a subharmonic function v in $\mathbb{R}^N$

$$A_\alpha^2(v)(x_0) = \int_{\Gamma_\alpha(x_0)} \Delta v(x,y) y^{1-N}\, dx\, dy\,.$$

Observe that if $v = u^2$, where u is harmonic, one obtains the usual ar function of u. Uchiyama ([38]), extending previous work of McConnell ([30 showed that for any $1 < p < \infty$,

$$\|A_\alpha(v)\|_p \le C(p,N,\alpha,\beta)\, \|N_\beta(v)\|_p\,.$$

Let u be a positive harmonic function in $\mathbb{R}^{N+1}_+$. In our case a sim calculation shows

$$|\nabla(\log u)|^2 = -\Delta(\log u) = \frac{|\nabla u|^2}{u^2}\,.$$

So, the natural square function for $\log u$ is

$$A_\alpha^2(\log u)(x_0) = \int_{\Gamma_\alpha(x_0)} \frac{|\nabla u(x,y)|^2}{u(x,y)^2} y^{1-N}\, dx\, dy\,.$$

We consider the non-tangential maximal function of $\log u$,

$$M_\alpha(\log u)(x_0) = \sup\{|\log u(x,y)| \colon (x,y) \in \Gamma_\alpha(x_0)\}\,.$$

We now define the local versions of the functions above. Given a c $Q \subset \mathbb{R}^N$ centered at x_Q and $x_0 \in Q$, define

$$M_\alpha(\log u, Q)(x_0) = \sup\{|\log u(x,y) - \log u(x_Q, l(Q))| \colon (x,y) \in \Gamma_\alpha(x_0)\,, \quad 0 < y < l(Q)\}\,,$$

$$A_\alpha^2(\log u, Q)(x_0) = \int_{\Gamma_\alpha(x_0)\cap Q} \frac{|\nabla u(x,y)|^2}{u(x,y)^2} y^{1-N}\, dx\, dy\,.$$

Now, the good λ-inequalities relating these two functions are descr in the following result, whose proof will be given after the next section

Theorem 3.1. *Let u be a positive harmonic function in the upper half space $\mathbb{R}^{N+1}_+$. Then there exists a constant $K_0 > 1$ and positive constants C_1, C_2 such that for any cube $Q \subset \mathbb{R}^N$, for all $K > K_0$ and $\lambda > 1$, one has*

(a) $$|\{x \in Q\colon M_\alpha(\log u, Q)(x) > K\lambda\,, A^2_\alpha(\log u, Q)(x) < \lambda\}|$$
$$\leq C_1 \exp(-C_2\lambda K^2)|\{x \in Q\colon M_\alpha(\log u, Q)(x) > \lambda\}|$$

(b) $$|\{x \in Q\colon A^2_\alpha(\log u, Q)(x) > K\lambda\,, M_\alpha(\log u, Q)(x) < \lambda\}|$$
$$\leq C_1 \exp(-C_2 K\lambda)|\{x \in Q\colon A^2_\alpha(\log u, Q)(x) > \lambda\}|$$

The proof follows the Bañuelos-Moore scheme of approximating a harmonic function by a dyadic martingale (see [6, Chapter 2]). In our case the function $\log u$ is not harmonic. However, the approximation scheme is flexible enough to deal with it. As in the martingale case, Theorem 1.5 stated in the Introduction will follow easily from the result above. However Theorem 1.5 (a) and therefore Corollary 1.6 can be proved in a more elementary way, as presented in the next paragraph.

Characterization of singular measures

Let u be a positive harmonic function in the upper half space $\mathbb{R}^{N+1}_+$. Harnack's inequality states that there exists a constant $C = C(N)$ such that

$$C^{-1}u(w,t) \leq u(x,y) \leq Cu(w,t)\,,$$

for any pair of points $(x,y), (w,t) \in \mathbb{R}^{N+1}_+$ such that

$$\|(w,t) - (x,y)\| \leq y/2\,.$$

The infinitessimal version of Harnack's inequality is the following estimate

$$\frac{y|\nabla u(x,y)|}{u(x,y)} \leq N\,,$$

which holds for any $(x,y) \in \mathbb{R}^{N+1}_+$. So, the function $\log u$ satisfies the Lipschitz condition

$$|\log u(z) - \log u(w)| \leq C\rho(z,w)\,,$$

where $\rho(z,w)$ is the hyperbolic distance in $\mathbb{R}^{N+1}_+$. By Harnack's inequality and the submean property of gradient of u, we obtain

Lemma 3.2. *Let u be a positive harmonic function in $\mathbb{R}^{N+1}_+$. Let B be a ball centered at $z \in \mathbb{R}^{N+1}_+$ such that $2B \subset \mathbb{R}^{N+1}_+$. Then, there exists a constant $C = C(N)$ such that*

$$\left|\frac{\nabla u(z)}{u(z)}\right| \leq \frac{C}{|B|}\int_B \frac{|\nabla u(w)|}{u(w)}\,dm(w)\,.$$

Recall that a positive harmonic function u in $\mathbb{R}^{N+1}_+$ has a finite non-ta gential limit at almost every point $x \in \mathbb{R}^N$. However, the limit may be 0 a large set. The result we prove next is the following:

Theorem 3.3. *Let u be a positive harmonic function in the upper h space $\mathbb{R}^{N+1}_+$. Fix $\alpha > 0$. Then, the sets*

$$\{x \in \mathbb{R}^N : \lim_{y\to 0} u(x,y) > 0\} \quad \text{and} \quad \{x \in \mathbb{R}^N : \int_{\Gamma_\alpha(x)} \frac{|\nabla u(w,y)|^2}{u(w,y)^2} y^{1-N}\, dw\, dy <$$

can only differ in a set of Lebesgue measure 0.

As a consequence we get

Corollary 3.4. *Let μ be a positive measure and let u be its harmonic tension. Fix $\alpha > 0$. Then μ is singular if and only if*

$$\int_{\Gamma_\alpha(x)} \frac{|\nabla u(w,y)|^2}{u(w,y)^2} y^{1-N}\, dwdy = \infty\,, \quad \text{a.e. } x \in \mathbb{R}^N$$

Proof of Theorem 3.3. The proof is based on well known arguments us Green's formula on saw-tooth regions (see, for instance [35, Chap. VII]).

$$A = \{x \in \mathbb{R}^N : \lim_{y\to 0} u(x,y) > 0\} \quad \text{and} \quad B = \{x \in \mathbb{R}^N : A_\alpha(u)(x) < \infty\}$$

We first show that almost every point of A is in B. So, let E be a comp set and $\varepsilon > 0$ such that

$$\varepsilon^{-1} > \lim_{y\to 0} u(x,y) > \varepsilon\,.$$

Fix $\beta > \alpha$, Harnack's inequality provides constants $C = C(\beta,\varepsilon)$, $K = K(\beta$ such that

$$\inf_{(w,y)\in\Gamma_\beta(x)} u(w,y) > C\,, \qquad \sup_{(w,y)\in\Gamma_\beta(x)} u(w,y) < K\,,$$

for any $x \in E$. Renormalizing, we may assume

$$\inf_{(w,y)\in\Gamma_\beta(x)} u(w,y) > \varepsilon\,, \qquad \sup_{(w,y)\in\Gamma_\beta(x)} u(w,y) < \varepsilon^{-1}\,,$$

for any $x \in E$. Now, let

$$R = \cup_{x\in E}\Gamma_\alpha(x)$$

and note that $\varepsilon^{-1} > u > \varepsilon$ on R. Fubini's theorem gives

$$\int_E A^2_\alpha(\log u)(x)dx \le C \int_R y \frac{|\nabla u(x,y)|^2}{u(x,y)^2}\, dx\, dy\,.$$

Since $\Delta(\log u) = -|\nabla u|^2/u^2$, Green's formula applied to the functions $\log u$ and y on the domain R yields

$$\int_R y\frac{|\nabla u(x,y)|^2}{u(x,y)^2}\,dx\,dy = \int_{\partial R}(\log u)\partial_{\tilde{n}}y - \int_{\partial R} y\partial_{\tilde{n}}(\log u)\,.$$

The first integral on the right side is bounded by $|\log\varepsilon|\,|\partial R|$ and the second one by $N|\partial R|$ because by Harnack's inequality $y|\nabla(\log u)(x,y)| \le N$. Hence,

$$\int_E A_\alpha^2(\log u)(x)\,dx < \infty$$

and the proof is finished.

Now, we want to show that almost every point of B is in A. We argue by contradiction. So, we may assume

$$|\{x \in \mathbb{R}^N\colon A_\alpha^2(\log u)(x) \le 1\,,\ \lim_{y\to 0} u(x,y) = 0\}| > 0\,.$$

Then one may find a compact set E of positive measure such that u is non-tangentially bounded on E and

$$A_\alpha^2(\log u)(x) \le 1\,, \qquad \lim_{y\to 0} u(x,y) = 0\,,$$

for any $x \in E$. Observe that Harnack's inequality implies that

$$u(w,t) \to 0\,,$$

whenever (w,t) tends non-tangentially to a point $x \in E$. Considering a subset of E if necessary, one can assume that $\log u$ is bounded from above on the set

$$R = \cup_{x\in E}\Gamma_\alpha(x)\,.$$

Now applying Green's formula as above we get

$$\int_R y\frac{|\nabla u(x,y)|^2}{u(x,y)^2}\,dx\,dy = \int_{\partial R}(\log u)\partial_{\tilde{n}}y - \int_{\partial R} y\partial_{\tilde{n}}(\log u)\,.$$

By Fubini, the volume integral is bounded by a multiple of

$$\int_E A_\alpha^2(\log u)(x)\,dx \le |E|$$

and by Harnack's inequality

$$\left|\int_{\partial R} y\partial_{\tilde{n}}(\log u)\right| \le N|\partial R|\,.$$

However, since $\partial_{\tilde{n}}y \ge 1/2$ at $\partial R \cap \{y = 1\}$ and $\log u$ is bounded from above, we deduce

$$\int_{\partial R}(\log u)\partial_{\tilde{n}}y = -\infty\,,$$

which gives the contradiction. ∎

Last proof combined with stopping time arguments give versions of the estimates in Theorem 3.1 with constants having worst decay. Next we present a proof of the good-λ-inequalities stated in that Theorem. As explained before, the idea is to adapt the Bañuelos-Moore method to reduce the result to the martingale setting.

Approximating $\log u$ by a dyadic martingale.

The idea of approximating a function by a dyadic martingale using Calderón reproducing formula goes back to Chang, Wilson and Wolff [14]. Later Bañuelos and Moore [6] related a harmonic function in the upper half space to a truncated version of a Calderón-type integral formula. Let us explain briefly their idea.

Let u be a harmonic function in $\mathbb{R}^{N+1}_+$. Given $\rho > 0$, fix a positive, smooth radial function K supported on $B(0,\rho) \subset \mathbb{R}^N$, with integral 1. Then write

$$P(x) = -\sum_{i=1}^{N} x_i \frac{\partial K}{\partial x_i}(x) - (N-1)K(x)\,, \tag{3.1}$$

$$q(x) = (-x_1 P(x) + \frac{\partial K}{\partial x_1}(x), \ldots, -x_N P(x) + \frac{\partial K}{\partial x_N}(x))\,. \tag{3.2}$$

Finally write $q_t(x) = t^{-N} q(x/t)$ and a similar definition for $K_t(x)$. Define

$$v(x,t) = \int_{\mathbb{R}^N} P_t(x-y)u(y,t)\,dy\,.$$

The Bañuelos-Moore approach may be divided into the following steps. Let $\varepsilon_m = 2^{-m-2}/4\rho$ and recall that $A_\alpha(u)(x,t)$ denotes the truncated area function of u.

(I) Given $\alpha > 0$, one can choose ρ sufficiently small such that for $m = 1, 2, \ldots$ if $\varepsilon_{m+1} < t_0 \le \varepsilon_m$,

$$|u(x,t_0) - v(x,\varepsilon_m)| \le C A_\alpha(u)(x,\gamma t_0)\,,$$

where $0 < \gamma < 1$ is a constant depending on ρ. See [6, p. 72].

(II) For $0 < \varepsilon < \widetilde{\varepsilon} \le 1/4$,

$$v(x,\widetilde{\varepsilon}) - v(x,\varepsilon) = \int_{\varepsilon}^{\widetilde{\varepsilon}} \int_{\mathbb{R}^N} q_t(x-w)\nabla_w u(w,t)dw\,dt + A(x)$$

where

$$A(x) = \int_{\mathbb{R}^N} \widetilde{\varepsilon} K_{\widetilde{\varepsilon}}(x-w)\frac{\partial u}{\partial t}(w,\widetilde{\varepsilon})dw - \int_{\mathbb{R}^N} \varepsilon K_\varepsilon(x-w)\frac{\partial u}{\partial t}(w,\varepsilon)\,dw$$

Moreover, if ρ is chosen sufficiently small with respect to α, $A(x)$ is bounded by a fixed multiple of $A_\alpha(u)(x,\gamma\varepsilon)$. Here $0 < \gamma < 1$ is a constant depending on ρ. See [6, p. 56, p. 70].

(III) Consider the function

$$V(x,\varepsilon_m)=\int_{\varepsilon_m}^{1/4}\int_{\mathbb{R}^N} q_t(x-w)\nabla_w u(w,t)\,dw\,dt\,.$$

If ρ is taken sufficiently small with respect to α, there exist a positive number $L=L(N)$ and martingales (with respect to different translates of the usual dyadic filtration), $\{S_n^{(1)}\},\dots,\{S_n^{(L)}\}$, such that

$$\left|V(x,\varepsilon_m)-\sum_{k=1}^{L}S_m^{(k)}(x)\right|\le C+CA_\alpha(u)(x,\gamma\varepsilon_m)\,,$$

$$\sum_{k=1}^{L}(S^{(k)})_m(x)\le CA_\alpha(u)(x,\gamma\varepsilon_m)\,.$$

See [6, p. 50].

So, if ρ is taken sufficiently small, combining (I), (II) and (III) we get that for $m=1,2,\dots$ and $\varepsilon_{m+1}<t<\varepsilon_m$,

$$\left|u(x,t)-\sum_{k=1}^{L}S_m^{k}(x)\right|\le C+CA_\alpha(u)(x,\gamma t)\,,$$

$$\sum_{k=1}^{L}(S^{k})_m(x)\le CA_\alpha(u)(x,\gamma t)\,,$$

where γ is a fixed constant. These estimates allow us to transfer problems on the growth of the harmonic function u on regions where its (truncated) area function is bounded to the martingale setting, that is, to problems on growth of martingales on regions where its quadratic variation remains bounded, since in this context the subgaussian estimate is available. In this scheme, the harmonicity of u is used in different ways. The steps (I) and (III) only depend on the submean property of $|\nabla u|$, that is,

$$|\nabla u(x_0,y_0)|\le\frac{1}{|B|}\int_B|\nabla u(x,y)|\,dx\,dy\,,$$

where B is a ball centered at (x_0,y_0) contained in $\mathbb{R}^{N+1}_+$. However the identity in (II) uses the whole information $\Delta u=0$.

Now, given a positive harmonic function u, our strategy consists on replacing u by $\log u$ in the previous scheme. So we consider

$$v_*(x,t)=\int_{\mathbb{R}^N}P_t(x-w)\log u(w,t)\,dw\,.$$

Recall that by Lemma 3.2 the submean value property for $|\nabla(\log u)$ holds. So the proof of (I) and (III) by Bañuelos and Moore applies in ou situation, that is,

(I*) Given $\alpha > 0$, one can choose ρ sufficiently small such that for $m = 1, 2, \ldots$ if $\varepsilon_{m+1} < t_0 \leq \varepsilon_m$,

$$|\log u(x, t_0) - v_*(x, \varepsilon_m)| \leq C A_\alpha(\log u)(x, \gamma t_0) .$$

(III*) Consider the function

$$V_*(x, \varepsilon_m) = \int_{\varepsilon_m}^{1/4} \int_{\mathbb{R}^N} q_t(x - w) \nabla_w \log u(w, t)\, dw\, dt .$$

If ρ is chosen sufficiently small with respect to α, there exists a positi number $L = L(N)$ and martingales (with respect to different translat of the usual dyadic filtration), $\{S_n^{(1)}\}, \ldots, \{S_n^{(L)}\}$ such that

$$\left| V_*(x, \varepsilon_m) - \sum_{k=1}^{L} S_m^{(k)}(x) \right| \leq C + C A_\alpha(\log u)(x, \gamma \varepsilon_m) ,$$

$$\sum_{k=1}^{L} \langle S^k \rangle_m(x) \leq C A_\alpha(\log u)(x, \gamma \varepsilon_m) .$$

The analogue of (II) for $\log u$ is harder to establish and it is the ma content of the two following results.

Lemma 3.5. *Let u be a positive harmonic function in $\mathbb{R}^{N+1}_+$. Given positive, smooth, radial function K supported on $\{x \in \mathbb{R}^N : \|x\| < \rho\}$, w integral 1, consider the functions P, q given in* (3.1) *and* (3.2). *Define*

$$v_*(x, t) = \int_{\mathbb{R}^N} P_t(x - w) \log u(w, t)\, dw .$$

Then, for $0 < \varepsilon < 1/4$,

$$v_*(x, 1/4) - v_*(x, \varepsilon) = V_*(x, \varepsilon) + Q(x, \varepsilon) + R(x, \varepsilon) ,$$

where

$$V_*(x, \varepsilon) = \int_{\varepsilon}^{1/4} \int_{\mathbb{R}^N} q_t(x - w) \nabla_w (\log u)(w, t)\, dw\, dt ,$$

$$Q(x, \varepsilon) = \int_{\varepsilon}^{1/4} \int_{\mathbb{R}^N} t K_t(x - w) \frac{|\nabla u(w, t)|^2}{u(w, t)^2}\, dw\, dt ,$$

$$R(x, \varepsilon) = \int_{\mathbb{R}^N} \frac{1}{4} K_{1/4}(x - w) \frac{\partial \log u}{\partial t}(w, 1/4)\, dw - \int_{\mathbb{R}^N} \varepsilon K_\varepsilon(x - w) \frac{\partial \log u}{\partial t}(w, \varepsilon)\, dw .$$

Proof. We follow the proof of (II) by Bañuelos and Moore (see [6, p. 56]). Let $0 < \varepsilon < \widetilde{\varepsilon}$. Applying the fundamental Theorem of Calculus, we get

$$v_*(x,\widetilde{\varepsilon}) - v_*(x,\varepsilon) = \int_\varepsilon^{\widetilde{\varepsilon}} \int_{\mathbb{R}^N} \frac{\partial}{\partial t}\left(t^{-N}P((x-w)/t)\right)\right) \log u(w,t)\, dw\, dt + \int_\varepsilon^{\widetilde{\varepsilon}} \int_{\mathbb{R}^N} t^{-N}P((x-w)/t)\partial_t(\log u)(w,t)\, dw\, dt\,.$$

Using that

$$\frac{\partial}{\partial t}(t^{-N}P((x-w)/t)) = \sum_{i=1}^N \frac{\partial}{\partial w_i}\left(\frac{x_i - w_i}{t^{N+1}}P((x-w)/t)\right)$$

and integrating by parts, the first integral becomes

$$-\int_\varepsilon^{\widetilde{\varepsilon}} \int_{\mathbb{R}^N} \sum_{i=1}^N \frac{x_i - w_i}{t^{N+1}}P((x-w)/t)\frac{\partial \log u}{\partial w_i}(w,t)\, dw\, dt\,.$$

Using that

$$t^{-N}P((x-w)/t) = \frac{\partial}{\partial t}(t^{-N+1}K((x-w)/t))\,,$$

and an integration by parts, the second integral becomes

$$\begin{aligned} &-\int_\varepsilon^{\widetilde{\varepsilon}} \int_{\mathbb{R}^N} t^{-N+1}K((x-w)/t)\frac{\partial^2}{\partial t^2}(\log u)(w,t)\, dw\, dt \\ &+\int_{\mathbb{R}^N} \widetilde{\varepsilon}K_{\widetilde{\varepsilon}}(x-w)\frac{\partial \log u}{\partial t}(w,\widetilde{\varepsilon})\, dw - \int_{\mathbb{R}^n} \varepsilon K_\varepsilon(x-w)\frac{\partial \log u}{\partial t}(w,\varepsilon)\, dw \\ &= A + B + C\,. \end{aligned}$$

Since

$$\frac{\partial^2 \log u}{\partial t^2} = -\sum_{i=1}^N \frac{\partial^2 \log u}{\partial w_i^2} - \frac{|\nabla u|^2}{u^2}\,,$$

we get

$$A = \int_\varepsilon^{\widetilde{\varepsilon}} \int_{\mathbb{R}^N} t^{-N+1}K((x-w)/t)\sum_{i=1}^N \frac{\partial^2 \log u}{\partial w_i^2}(w,t)\, dw\, dt + \int_\varepsilon^{\widetilde{\varepsilon}} \int_{\mathbb{R}^N} t^{-N+1}K((x-w)/t)\frac{|\nabla u(w,t)|^2}{u(w,t)^2}\, dw\, dt\,.$$

Another integration by parts shows that the first term can be written as

$$-\int_\varepsilon^{\widetilde{\varepsilon}} \int_{\mathbb{R}^N} t^{-N+1}\sum_{i=1}^N \left(\frac{\partial K}{\partial w_i}((x-w)/t)\right)\left(\frac{-1}{t}\frac{\partial \log u}{\partial w_i}(w,t)\right) dw\, dt\,.$$

So, collecting all the identities, one gets

$$
\begin{aligned}
v(x,\widetilde{\varepsilon}) - v(x,\varepsilon) = &\int_{\varepsilon}^{\widetilde{\varepsilon}} \int_{\mathbb{R}^N} q_t(x-w)\nabla_w(\log u)(w,t)\,dw \\
&+ \int_{\varepsilon}^{\widetilde{\varepsilon}} \int_{\mathbb{R}^N} t^{-N+1}K((x-w)/t)\frac{|\nabla u(w,t)|^2}{u(w,t)^2}\,dw\,dt \\
&+ \int_{\mathbb{R}^N} \widetilde{\varepsilon}K_{\widetilde{\varepsilon}}(x-w)\frac{\partial \log u}{\partial t}(w,\widetilde{\varepsilon})\,dw \\
&- \int_{\mathbb{R}^N} \varepsilon K_{\varepsilon}(x-w)\frac{\partial \log u}{\partial t}(w,\varepsilon)\,dw\,.
\end{aligned} \tag{3.3}
$$

This finishes the proof. ∎

Theorem 3.6. *Let u be a positive harmonic function on $\mathbb{R}^{N+1}_+$. Fix $\alpha > 0$. Let K be a smooth, positive, radial function supported on $\{x \in \mathbb{R}^N : \|x\| < \rho\}$ with integral 1. Consider the function*

$$
Q(x,\varepsilon) = \int_{\varepsilon}^{1/4} \int_{\mathbb{R}^N} tK_t(x-w)\frac{|\nabla u(w,t)|^2}{u(w,t)^2}\,dw\,dt\,.
$$

If ρ/α is small enough, there exists a positive number $L = L(N)$ and martingales (with respect to different translates of the dyadic filtration)

$$
\{S_n^{(1)}\}, \ldots, \{S_n^{(L)}\}
$$

such that for $m = 1, 2, \ldots,$ and $\varepsilon_{m+1} < y \le \varepsilon_m$,

$$
\begin{aligned}
|\log u(x,y) + Q(x,\varepsilon_m) - \sum_{k=1}^{L} S_m^{(k)}(x)| &\le C + CA_\alpha(\log u)(x,\gamma y)\,, \\
\sum_{k=1}^{L} \langle S^{(k)}\rangle_m(x) &\le A_\alpha(\log u)(x,\gamma y)\,,
\end{aligned}
$$

where $\varepsilon_m = 2^{-m-2}/4\rho$ and $0 < \gamma < 1$ is a constant depending on ρ. Here C is a constant depending on α and N.

Proof. Observe that Harnack's inequality and (I)* assert that

$$
|\log u(x,y) - v_*(x,\varepsilon_m)| \le CA_\alpha(\log u)(x,\gamma\varepsilon_m) + C\,.
$$

By Lemma 3.5,

$$
v_*(x,1/4) - v_*(x,\varepsilon_m) = V_*(x,\varepsilon_m) + Q(x,\varepsilon_m) + R(x,\varepsilon_m)\,.
$$

Now, (III)* provides the martingales which approximate $V_*(x, \varepsilon_m)$. Hence, it only remains to show that

$$R(x, \varepsilon_m) \leq C + CA_\alpha(\log u)(x, \gamma\varepsilon_m)\,.$$

It is enough to prove

$$\left| \int_{\mathbb{R}^N} \varepsilon K_\varepsilon(x-w) \frac{\partial \log u}{\partial t}(w, \varepsilon) dw \right| \leq CA_\alpha(\log u)(x, \gamma\varepsilon)\,.$$

Observe that the integral is over all the w contained in the ball centered at x of radius $\rho\varepsilon$. By the submean property of the gradient

$$\left| \frac{\partial \log u}{\partial t}(w, \varepsilon) \right| \leq \left(\frac{1}{|B(w)|} \int_{B(w)} \left| \frac{\nabla u}{u} \right|^2 \right)^{1/2},$$

where $B(w)$ is a ball centered at (w, ε) of radius $\eta\varepsilon$. Now if η and ρ are sufficiently small, there exists $\delta < 1$ such that

$$B_w \subset B_\varepsilon = \{z \in \mathbb{R}^{N+1}_+ \colon \|z - (x, \varepsilon)\| < \delta\varepsilon\} \subset \Gamma_\alpha(x)$$

for any w such that $\|w - x\| < \rho\varepsilon$. Then

$$\left| \frac{\partial \log u}{\partial t}(w, \varepsilon) \right| \leq C \left(\frac{1}{\varepsilon^{N+1}} \int_{B_\varepsilon} \frac{|\nabla u|^2}{u^2} \right)^{1/2}.$$

So,

$$\begin{aligned} \left| \int_{\mathbb{R}^N} \varepsilon K_\varepsilon(x-w) \frac{\partial \log u}{\partial t}(w, \varepsilon)\, dw \right| &\leq C \left(\int_{B_\varepsilon} \varepsilon^{-N+1} \frac{|\nabla u|^2}{u^2} \right)^{1/2} \\ &\leq A_\alpha(\log u)(x, \gamma\varepsilon)\,. \end{aligned}$$

This finishes the proof. ∎

Next we will show that given $0 < \beta < \alpha$, one can choose ρ in such a way that the resulting quantity $Q(x, \varepsilon)$ is, roughly speaking, intermediate between $A^2_\alpha(\log u)(x, \varepsilon)$ and $A^2_\beta(\log u)(x, \varepsilon)$.

Lemma 3.7. *Let u, K, Q be as in Theorem 3.6. Let $\{x \in \mathbb{R}^N \colon \|x\| \leq \rho\}$ be the closed support of K. Let $0 < \beta < \alpha$.*

(a) If $\rho < \alpha$,

$$Q(x, \varepsilon) \leq CA^2_\alpha(\log u)(x, \varepsilon)\,.$$

(b) If $\rho > \beta$,

$$Q(x, \varepsilon) \geq C^{-1} \int_{\Gamma_\beta(x,\varepsilon) \cap \{t < 1/4\}} t^{-N+1} \frac{|\nabla u(w, t)|^2}{u(w, t)^2}\, dw\, dt\,.$$

Here C is a constant depending on the function K, α and β.

Proof. Since K is supported in $\{x \in \mathbb{R}^N : \|x\| < \rho\}$, the integral over w in the definition of $Q(x,\varepsilon)$, is really over w in the ball of center x and radius ρt. So, letting $C = \sup|K|$, if $\rho < \alpha$, $Q(x,\epsilon)$ is bounded by

$$C\int_\varepsilon^{1/4} t^{-N+1}\int_{\|w-x\|<\rho t}\frac{|\nabla u(w,t)|^2}{u(w,t)^2}\,dw\,dt \le C\int_{\Gamma_\alpha(x)\cap\{t\ge\varepsilon\}} t^{-N+1}\frac{|\nabla u(w,t_{j_1}}{u(w,t)^2}\,dw\,dt$$

This gives the first estimate.

Given $\beta < \rho$, we choose $0 < \delta < 1$ such that

$$\Gamma_\beta(x) \subset \{(w,t) : \|w-x\| < (1-\delta)\rho t\}\,.$$

Now let $m = \min\{K(x) : \|x\| < \rho(1-\delta)\}$. Thus

$$\begin{aligned} Q(x,\varepsilon) &\ge m\int_\varepsilon^{1/4} t^{1-N}\int_{\{w:\,\|w-x\|<(1-\delta)\rho t\}}\frac{|\nabla u(w,t)|^2}{u(w,t)^2}\,dw\,dt \\ &\ge m\int_{\Gamma_\beta(x)\cap\{t\le 1/4\}} t^{1-N}\frac{|\nabla u(w,t)|^2}{u(w,t)^2}\,dw\,dt\,. \end{aligned}$$

■

The sets $\{x \in \mathbb{R}^N : A_\alpha(\log u)(x) < \infty\}$ do not depend on α, except for sets of Lebesgue measure 0. In their complement, next result shows that the order of growth of $A_\alpha(\log u)(x,y)$ does not depend on α.

Lemma 3.8. *Let u be a positive harmonic function in $\mathbb{R}^{N+1}_+$. Let $0 < \beta < \alpha$. Then, there exists a constant $C = C(\alpha,\beta,N) > 0$ such that*

$$\limsup_{t\to 0}\frac{A_\alpha(\log u)(x.t)}{A_\beta(\log u)(x,t)} \le C\,,$$

at almost every point $x \in \{x \in \mathbb{R}^N : A_\alpha(\log u)(x) = \infty\}$.

Proof. By contradiction, assume that for any positive constant C there exists a compact set $F = F(C)$ of positive Lebesgue measure such that

$$\begin{aligned} &\limsup_{t\to 0}\frac{A_\alpha(\log u)(x,t)}{A_\beta(\log u)(x,t)} > C, \\ &\lim_{t\to 0} A_\alpha(\log u)(x,t) = \infty\,, \end{aligned}$$

for any $x \in F$. Considering a point of density of F, one can assume that F is contained in a cube Q and $|F \cap Q| \ge (1-\varepsilon)|Q|$. Thus, for arbitrarily large $\lambda > 0$, one can find a set $F_\lambda \subset Q$, $|F_\lambda| \ge (1-2\varepsilon)|Q|$, such that for any $x \in F_\lambda$ there exists $t = t(x,\lambda) > 0$ for which $A_\alpha(\log u)(x,t) = \lambda$ and $A_\beta(\log u)(x,t) < C^{-1}\lambda$.

Next we run a stopping time argument. Let $\{Q_j\}$ be the maximal dyadic subcubes of Q for which there exists $x \in Q_j$ satisfying

$$A_\alpha(\log u)(x, l(Q_j)) \geq \lambda .$$

Since $F_\lambda \subset \cup Q_j$, we have

$$\sum |Q_j| \geq (1 - 2\varepsilon)|Q| .$$

Observe that Harnack's inequality yields that whenever $|x - w| < y$,

$$|A_\alpha(\log u)(x, y) - A_\alpha(\log u)(w, y)| \leq C(N, \alpha) .$$

Thus, for any $x \in Q_j$ one has

$$A_\alpha(\log u)(x, l(Q_j)) \geq \lambda - C(N, \alpha) .$$

Given a cube Q in $\mathbb{R}^N$ we denote by $\widehat{Q}$ the set $\{(x, y) : x \in Q, 0 < y < l(Q)\}$. Let $\Omega = \widehat{Q} \setminus \cup \widehat{Q}_j$. Then Fubini gives

$$\int_\Omega y \frac{|\nabla u(w, y)|^2}{u(w, y)^2} \, dw \, dy \geq C(\alpha) \int_Q \left(\int_{\Gamma_\alpha(x) \cap \Omega} y^{1-N} \frac{|\nabla u(w, y)|^2}{u(w, y)^2} \, dw \, dy \right) dx$$

$$\geq C_1(\alpha) \lambda^2 (1 - 2\varepsilon)|Q| .$$

On the other hand, by Fubini again

$$\int_\Omega y \frac{|\nabla u(w, y)|^2}{u(w, y)^2} \, dw \, dy \leq C(\beta) \int_Q \left(\int_{\Gamma_\beta(x) \cap \Omega} y^{1-N} \frac{|\nabla u(w, y)|^2}{u(w, y)^2} \, dw \, dy \right) dx$$

and applying Txebicheff's inequality one can find a set $G \subset Q$, $|G| \geq C(\alpha, \beta)|Q|$ such that

$$\int_{\Gamma_\beta(x) \cap \Omega} y^{1-N} \frac{|\nabla u(w, y)|^2}{u(w, y)^2} \, dw \, dy \geq C(\alpha, \beta) \lambda^2 .$$

Let $\mathcal{B}$ be the family of the cubes $\{Q_j\}$ for which

$$A_\beta(\log u)(x, l(Q_j)) > 2C^{-1}\lambda$$

for some $x \in Q_j$. Then

$$\sum_{\mathcal{B}} |Q_j| < 2\varepsilon |Q| .$$

If C is large enough this gives the contradiction and finishes the proof. ■

We are now ready to prove Theorem 3.1

Proof of Theorem 3.1. Given a cube Q of center x_Q and sidelength $l(Q)$, let us denote $z_Q = (x_Q, l(Q))$ and

$$T(Q) = \{(x,y) \in \mathbb{R}^{N+1}_+ : x \in Q, l(Q)/2 < y < l(Q)\}.$$

To prove (a), we consider $\{Q_j\}$ the maximal dyadic subcubes of Q such that

$$\sup_{z \in T(Q_j)} |\log u(z) - \log u(z_Q)| > \lambda + C(\alpha)\,,$$

where $C(\alpha)$ is a constant depending on α (and independent on λ) to be chosen later. It follows from Harnack's inequality that if $C(\alpha)$ is chosen conveniently, then

$$\{x \in Q\colon M_\alpha(\log u, Q)(x) > \lambda\} \supset \bigcup_j Q_j\,.$$

Also, by maximality and Harnack's inequality, we have

$$|\log u(z_{Q_j}) - \log u(z_Q)| \le \lambda + C_1(\alpha)\,,$$

where $C_1(\alpha)$ is another positive constant depending on α. Now, in each Q_j we will use the martingales $S_m^{(i)}$ given in Theorem 3.6. Let

$$S_m = \sum_{i=1}^{L} (S_m^{(i)} - S_m^{(i)}(Q_j))\chi_{Q_j}.$$

Then, the estimates in Theorem 3.6 and Lemma 3.7 give

$$\begin{aligned}\{x \in Q_j\colon & M_\alpha(\log u, Q_j)(x) > (K-1)\lambda\,, A_\alpha^2(\log u, Q_j)(x) < \lambda\} \\ & \subseteq \{x \in Q_j\colon \sup_m |S_m(x)| > (K-2C)\lambda - C, \langle S\rangle^2(x) < C + \lambda C\}\,.\end{aligned}$$

The subgaussian estimate (1.1) applied to $S_m^{(i)}$, $i = 1, \dots, L$, implies that the measure of this set is bounded by

$$C_1 \exp(-C_2 \lambda K^2)|Q_j|\,.$$

Adding up, we deduce that

$$\begin{aligned}|\{x \in Q\colon & M_\alpha(\log u, Q)(x) > K\lambda\,, A_\alpha^2(\log u, Q)(x) < \varepsilon\lambda\}| \\ & \le C_1 \exp(-C_2 K^2 \lambda) \sum |Q_j| \\ & \le C_1 \exp(-C_2 K^2 \lambda) |\{x \in Q\colon M_\alpha(\log u, Q)(x) > \lambda\}|\,,\end{aligned}$$

which is (a).

The proof of (b) is similar. Let $\{Q_j\}$ be the maximal dyadic subcubes of Q such that

$$A_\alpha^2(\log u, Q)(x, l(Q_j)) \geq \lambda + C_1,$$

for some $x \in Q_j$, where $C_1 = C_1(\alpha)$ is a constant depending on α (and independent on λ) to be chosen later. It follows from Harnack's inequality and the maximality that if C is chosen conveniently, then

$$\lambda \leq A_\alpha^2(\log u, Q)(x, l(Q_j)) \leq \lambda + 2C_1\,,$$

for any $x \in Q_j$. So,

$$\{x \in Q : A_\alpha^2(\log u, Q)(x) > \lambda\} \supset \cup_j Q_j.$$

Now, in each Q_j we will run a stopping time argument to find cubes where $A^2(\log u, Q)$ has increased $k\lambda$ units. More concretely, for $j = 1, 2, \ldots$, let $\{Q_{j,l} : l = 1, 2, \ldots\}$ be the maximal dyadic subcubes of Q_j such that

$$A_\alpha^2(\log u, Q_j)(x, l(Q_{j,l})) \geq (k-1)\lambda - C_2$$

for some $x \in Q_{j,l}$. By Harnack's inequality

$$(k-1)\lambda - C_3 \leq A_\alpha^2(\log u, Q_j)(x, l(Q_{j,l})) \leq (k-1)\lambda + C_3,$$

for any $x \in Q_{j,l}$, where C_3 is a constant depending on α, C_2 and the dimension. If the constants are chosen conveniently the set $\{x : A_\alpha^2(\log u, Q)(x) > k\lambda\}$ is contained in $\cup_{j,l} Q_{j,l}$. Let A be the subfamily of $\{Q_{j,l}, j = 1, 2 \ldots, l = 1, 2 \ldots\}$ for which $M_\alpha(\log u, Q)(x) \leq \lambda$, for any $x \in Q_{j,l}$. Now in each Q_j we will use the martingale $S_m^{(i)}$ given in Theorem 3.6. Let

$$S_m = \sum_{i=1}^{L} (S_m^{(i)} - S_m^{(i)}(Q_j))\chi_{Q_j}.$$

The estimates in Theorem 3.6 and Lemma 3.7 give that for any $Q_{j,l}$ in A, if $2^{-m-2}/4\rho = l(Q_{j,l})$,

$$S_m(x) \geq C_4 k\lambda,$$
$$\langle S \rangle_m(x) \leq C_5 \sqrt{k\lambda},$$

for any $x \in Q_{j,l}$. Hence, the subgaussian estimate (1.1) implies that

$$\sum_l |Q_{j,l}| \leq C_6 \exp(-C_7 k\lambda)|Q_j|\,.$$

Adding up, we obtain (b).

It is worth noticing that if one tries to adapt the proof of the corresponding subgaussian estimate for harmonic functions to our setting, a worst decay on the constants is obtained, as we show next. Let $\beta > \alpha > 0$ and set

$$E = \{x \in Q\colon M_\alpha(\log u, Q)(x) < \lambda\}$$

and

$$R = \bigcup_{x \in E} \Gamma_\beta(x)\,.$$

Thus, R is a Lipschitz domain say $R = \{(x,y)\colon y > \phi(x)\}$, where ϕ is a Lipschitz function. Observe that

$$|\log u(x,y)| < \lambda + C(\alpha,\beta)\,,\ (x,y) \in R\,,$$

where $C(\alpha,\beta)$ is a constant depending on α, β.

Claim. Let

$$A_\alpha^2(\log u)(x,\phi(x)) = \int_{\Gamma_R((x,\phi(x)),\alpha)} \frac{|\nabla u(w,y)|^2}{u(w,y)^2}(y-\phi(x))^{1-N}\,dw\,dy\,,$$

where $\Gamma_R((x,\phi(x)),\alpha)$ is a vertical cone with vertex at $(x,\phi(x))$ of aperture α. Then

$$\|A_\alpha^2(\log u)(x,\phi(x))\|_{\mathrm{BMO}(Q)} \le C_1\lambda\,.$$

This is the main step in Uchiyama's proof [38]. Then, the John-Niremberg Theorem implies

$$\begin{aligned}|\{x \in Q\colon\ & A_\alpha^2(\log u, Q)(x) > K\lambda\,, M_\alpha(\log u, Q)(x) < \lambda\}| \\ &= |\{x \in Q\colon A_\alpha^2(\log u)(x,\phi(x)) > K\lambda\}| \\ &\le C_1 \exp(-C_2K)|\{x \in Q\colon A_\alpha^2(\log u, Q)(x) > \lambda\}|\,.\end{aligned}$$

■

As in the martingale setting, the good-λ-inequalities have a number of consequences.

Theorem 3.9. *Let μ be a positive measure on $\mathbb{R}^N$ such that*

$$\int_{\mathbb{R}^N} \frac{d\mu(x)}{1+|x|^{N+1}} < \infty\,.$$

Let $\mu = f\,dx + \mu_s$, where $f\,dx$ and μ_s are its absolutely continuous and singular part with respect to Lebesgue measure. Fix $\alpha > 0$. Let $u = P[\mu]$ be the harmonic extension of μ and

$$A_\alpha^2(\log u)(x) = \int_{\Gamma_\alpha(x)} \frac{|\nabla u(w,y)|^2}{u(w,y)^2} y^{1-N}\,dw\,dy$$

its square function. Then,

(a) The sets

$$\{x \in \mathbb{R}^N : f(x) > 0\} \quad \text{and} \quad \{x \in \mathbb{R}^N : A_\alpha^2(\log u)(x) < \infty\}$$

can only differ in a set of Lebesgue measure 0.

(b) μ *is a singular measure if and only if*

$$A_\alpha^2(\log u)(x) = \infty, \quad \text{a.e. } x \in \mathbb{R}^N.$$

(c) Consider the non-tangential maximal function of $\log u$,

$$M_\alpha(\log u)(x) = \sup_{\Gamma_{\alpha(x)}} |\log u|.$$

Let Q *be a cube in* $\mathbb{R}^N$. *Then, for* $0 < p < \infty$ *and any* $\alpha > 0$, $M_\alpha(\log u) \in L^p(Q)$ *if and only if* $A_\alpha^2(\log u) \in L^p(Q)$. *Also, for* $1 < p < \infty$, $\log f \in L^p(Q)$ *if and only if* $A_\alpha^2(\log u) \in L^p(Q)$.

(d) There exists a constant $C = C(N, \alpha)$ *such that* $\exp(CA_\alpha^2(\log u)) \in L^1_{\text{loc}}$ *implies that* μ *and Lebesgue measure are mutually absolutely continuous.*

(e) There exists a constant $c = c(N, \alpha)$ *such that if* $A_\alpha^2(\log u)(x) < 1$ *for any* $x \in Q$, *then*

$$\exp(cM_\alpha^2(\log u)) \in L^1(Q).$$

The proof is identical to the one in the dyadic martingale setting and we omit it.

Law of the iterated logarithm

Let u be a positive harmonic function on $\mathbb{R}^{N+1}_+$. Given a positive, smooth, radial function K supported on $B(0, \rho) \subset \mathbb{R}^N$, recall that $Q(x, \varepsilon)$ was defined as

$$Q(x, \varepsilon) = \int_\varepsilon^{1/4} \int_{\mathbb{R}^N} tK_t(x - w) \left| \frac{\nabla u(w, t)}{u(w, t)} \right|^2 dw\, dt.$$

Observe that the integral is really over the (w, t) in the cone $\Gamma_\rho(x, \epsilon)$ in $\mathbb{R}^{N+1}_+$. So, $Q(x, \epsilon)$ is a smooth version of the truncated area function $A_\rho^2(\log u)(x, \varepsilon)$. Hence, the sets

$$\{x \in \mathbb{R}^N : \lim_{y \to 0} u(x, y) = 0\} \quad \text{and} \quad \{x \in \mathbb{R}^N : \lim_{y \to 0} Q(x, y) = \infty\}$$

can only differ in a set of Lebesgue measure 0.

Next result is an analogous of the LIL for dyadic measures given in Theorem 1.3.

Theorem 3.10. *With the notations above*

$$\limsup_{y\to 0} \frac{|\log u(x,y) + Q(x,y)|}{\sqrt{Q(x,y)\log\log Q(x,y)}} \le C$$

at almost every point

$$x \in \{x \in \mathbb{R}^N : \lim_{y\to 0} u(x,y) = 0\}.$$

Here C is a constant depending on N and K.

Proof. We use Theorem 3.6 to get a collection of L dyadic martingales $\{S_m^{(j)}\}_{j=1}^L$ such that

$$\begin{aligned} &\left|\log u(x,y) + Q(x,y) - \sum_{j=1}^{L} S_m^{(j)}(x)\right| \le C + CA_\alpha(\log u)(x,\gamma y)\,, \\ &\sum_{j=1}^{L} \langle S^j\rangle_m(x) \le CA_\alpha(\log u)(x,\gamma y)\,. \end{aligned} \tag{3.4}$$

Here m is chosen such that $\varepsilon_{m+1} < y \le \varepsilon_m$, where $\varepsilon_m = 2^{-m-2}/4\rho$, $0 < \gamma < 1$ is a constant depending on ρ, and ρ/α is sufficiently small. Now, the LIL for martingales states

$$\limsup_{m\to\infty} \frac{S_m^{(j)}(x)}{\sqrt{\langle S^{(j)}\rangle_m^2(x)\log\log\langle S^{(j)}\rangle_m(x)}} \le C_1\,,$$

at a.e. $x \in \{x : \langle S^{(j)}\rangle(x) = \infty\}$. Hence, for any $j = 1, \dots, L$

$$\limsup_{m\to\infty} \frac{S_m^{(j)}(x)}{\sqrt{A_\alpha^2(\log u, \varepsilon_m)(x)\log\log A_\alpha(\log u, \varepsilon_m)(x)}} \le C_1\,,$$

for a.e. $x \in \{x \in \mathbb{R}^N : A_\alpha(\log u)(x) = \infty\}$. Then the result follows from (3.4) and Lemmas 3.7 and 3.8. ∎

Applying the Theorem above and Lemmas 3.7 and 3.8, we deduce Corollary 1.8.

An example

There is a class of positive harmonic functions for which these LIL's give sharp results. Let u be a positive harmonic function on $\mathbb{R}^{N+1}_+$. Harnack's inequality asserts

$$y|\nabla u(x,y)| \leq Nu(x,y)\,, \qquad (x,y) \in \mathbb{R}^{N+1}_+\,.$$

Now given $\varepsilon > 0$ and $R < 1$, let $M(\varepsilon, R)$ be the class of positive harmonic functions u for which

$$\sup\left\{ y\frac{|\nabla u(x,y)|}{u(x,y)} \colon \|(x,y) - (x_0,y_0)\| \leq Ry_0 \right\} > \varepsilon$$

for any $(x_0, y_0) \in \mathbb{R}^{N+1}_+$. This is analogous to a class of holomorphic functions considered by P. Jones ([23]). For $\varepsilon > 0$ small and R close to 1, the class $M(\varepsilon, R)$ is non trivial. For instance, if one considers the measure μ given in Proposition 2.9 and u its Poisson extension, then $u \in M(\varepsilon, R)$ for some ε and R. Now, if $u \in M(\varepsilon, R)$, the submean property given in Lemma 3.2, gives that

$$Q(x,y) \geq C(\varepsilon, R)\log 1/y\,,$$

while Harnack's inequality gives

$$Q(x,y) \leq C\log 1/y\,.$$

Similarly

$$c(\alpha)\log 1/y \leq A^2_\alpha(\log u)(x,y) \leq C(\alpha)\log 1/y\,.$$

So if $u \in M(\varepsilon, R)$, then $A_\alpha(\log u)(x) = \infty$ at almost every $x \in \mathbb{R}^N$, that is, $\lim_{y\to 0} u(x,y) = 0$ a.e. $x \in \mathbb{R}^N$, or equivalently u is the Poisson integral of a positive singular measure. Moreover, since

$$\limsup_{y\to 0} \frac{|\log u(x,y) + Q(x,y)|}{\sqrt{\log 1/y \log\log\log 1/y}} \leq C\,, \quad \text{a.e. } x \in \mathbb{R}^N\,,$$

and

$$C(\varepsilon, R)\log 1/y \leq Q(x,y) \leq C\log 1/y\,,$$

we deduce that for a.e. $x \in \mathbb{R}^N$, one has

$$\limsup_{y\to 0} \frac{|\log u(x,y)|}{\log 1/y} \leq C$$

and

$$\liminf_{y\to 0} \frac{|\log u(x,y)|}{\log 1/y} > C(\varepsilon, R)\,.$$

4. Hyperbolic derivatives of holomorphic mappings

Decay in Schwarz's lemma

This subsection is devoted to the proof of Theorem 1.9. As explained in the introduction, (a) of Theorem 1.9 follows from Theorem 1.5 (a). To prove (b) it is more convenient to consider the upper half plane. An analytic function I from the upper-half plane $\mathbb{R}^2_+$ to the unit disc is called inner if

$$\lim_{y\to 0} I(x+iy)$$

has modulus 1 at almost every $x \in \mathbb{R}$.

We apply Proposition 2.9 to obtain a positive, singular, doubling measure μ such that

$$\left|\frac{\mu(I_z)}{\mu(I_z')} - 1\right| \le Ch(z)\operatorname{Im} z\,, \quad 1 \ge \operatorname{Im} z > 0\,, \tag{4.1}$$

where I_z is the interval of the real line centered at $\operatorname{Re}(z)$ of length $\operatorname{Im} z$. Let u be the harmonic extension of μ to the upper half plane and $H = u + i\widetilde{u}$ where $\widetilde{u}$ is the unique harmonic function in $\mathbb{R}^2_+$ with $\widetilde{u}(i) = 0$ such that $H = u + i\widetilde{u}$ is analytic in $\mathbb{R}^2_+$. Finally $I = \tau^{-1} \circ H$, where τ is a Möbius map from the unit disc to the right- half plane. Since μ is singular, the function I is inner. Also,

$$2\frac{|I'(z)|}{1-|I(z)|^2} = \frac{|\nabla u(z)|}{u(z)}\,, \quad \operatorname{Im} z > 0.$$

So, we have to show that

$$\frac{|\nabla u(z)|}{u(z)} \le Ch(z)\,,$$

for any $z = x + iy \in \mathbb{R}^2_+$. An integration by parts shows

$$\frac{\partial u}{\partial y}(x,y) = -\frac{1}{2}\int_{\mathbb{R}} \frac{\partial^2 P_y(t)}{\partial y \partial t}\operatorname{sign}(t)[\mu(x,x+t) - \mu(x-t,x)]\,dt$$

and

$$\frac{\partial u}{\partial x}(x,y) = \frac{1}{2}\int_{\mathbb{R}} \frac{\partial^2 P_y(t)}{\partial t^2}\operatorname{sign}(t)[\mu(x,x+t) - \mu(x-t,x)]\,dt$$

(see Lemma 2.2 in [16]). Here $P_y(t) = \pi^{-1}y(t^2+y^2)^{-1}$ is the Poisson kern in the upper-half plane. We split these integrals dyadically, that is, ov the intervals $I_k \setminus I_{k-1}$, $k = 1, 2, \ldots$, where I_k is the interval associated $x + i2^k y$. Using the estimate

$$\left|\frac{\partial^2 P_y(t)}{\partial t^2}\right| + \left|\frac{\partial^2 P_y(t)}{\partial y \partial t}\right| \le \frac{C_1}{2^{4k}y^3}\,, \quad \text{if } |t| < 2^k y\,,$$

and (4.1), one deduces

$$|\nabla u(x,y)| \le C_2 \sum_{k=0}^{\infty} \frac{1}{2^{4k}y^3}\mu(x, x+2^k y)h(x+i2^k y)(2^k y)^2$$
$$= C_2 \sum_{k=0}^{\infty} \frac{1}{2^{2k}y}\mu(x, x+2^k y)h(x+i2^k y)\,.$$

Since we are now working in the upper-half plane, the function $h(x+iy)$ is decreasing for $y \in (0, 1/2]$. Therefore

$$|\nabla u(x,y)| \le C_2 h(x+iy) \sum_{k=0}^{\infty} \frac{1}{2^{2k}y}\mu(x, x+2^k y) \le C_3 h(x+iy)u(x,y)\,,$$

because $P_y(x-t)$ is comparable to $1/2^{2k}y$ for $t \in I_k \setminus I_{k-1}$. This finishes the proof.

Law of the iterated logarithm

For $0 \le \gamma < 2\pi$, let $\tau_\gamma(z) = (e^{i\gamma}+z)(e^{i\gamma}-z)^{-1}$ be a Möbius transformation from the unit disc onto the right-half plane. Given a holomorphic mapping I from the unit disc into itself, let u_γ be the real part of $\tau_\gamma \circ I$. Thus

$$u_\gamma(z) = \frac{1-|I(z)|^2}{|e^{i\gamma}-I(z)|^2}\,, \quad z \in \mathbb{D}\,,$$

and a simple calculation shows that

$$2\frac{|I'(z)|}{1-|I(z)|^2} = \frac{|\nabla u_\gamma(z)|}{u_\gamma(z)}\,, \quad z \in \mathbb{D}\,,$$

which is independent of γ. So, moving the notation from the upper-half plane to the disc, the truncated area function $\mathcal{A}^2_\alpha(I)(re^{i\theta})$ coincides with $4^{-1}A^2_\alpha(\log u_\gamma)(re^{i\theta})$ and $\mathcal{Q}(re^{i\theta})$ is $4^{-1}Q_\alpha(\log u_\gamma)(re^{i\theta})$. Thus by Theorem 1.7

$$\limsup_{r\to 1} \frac{|\log(u_\gamma(re^{i\theta})) + \mathcal{Q}(re^{i\theta})|}{\sqrt{\mathcal{Q}(re^{i\theta})\log\log \mathcal{Q}(re^{i\theta})}} \le C\,,.$$

Now, since for any $z \in \mathbb{D}$, one has

$$\int_o^{2\pi} \log|e^{i\gamma}-z|^2 d\gamma = 0\,,$$

Theorem 1.10 follows integrating on γ.

Inner functions in complex domains

Let Ω be a hyperbolic domain in the complex plane, that is, $\Omega \subset \mathbb{C}$ suc that $\mathbb{C} \setminus \Omega$ contains at least two points. Let $\lambda_{\mathbb{D}}(z)|dz| = (1-|z|^2)^{-1}|dz|$ be the Poincaré (or hyperbolic) metric in $\mathbb{D}$ and $\Pi\colon \mathbb{D} \to \Omega$ a holomorphi covering map onto Ω. Then $\lambda_{\mathbb{D}}(z)$ projects to the Poincaré . . th domain Ω, $\lambda_\Omega(z)|dz|$, that is,

$$\lambda_\Omega(\Pi(z))|\Pi'(z)| = \lambda_{\mathbb{D}}(z)\,, \quad z \in \mathbb{D}\,.$$

Let f be a holomorphic mapping from $\mathbb{D}$ into Ω. Then, Schwarz's lemm asserts that

$$\lambda_\Omega(f(z))|f'(z)| \le \lambda_{\mathbb{D}}(z)\,, \quad z \in \mathbb{D}\,,$$

that is,

$$(1-|z|^2)|f'(z)|\lambda_\Omega(f(z)) \le 1\,, \quad z \in \mathbb{D}\,.$$

A holomorphic mapping I from the unit disc into Ω is called inner into if the set

$$\{e^{i\theta} \in \partial\mathbb{D}\colon \lim_{r\to 1} I(re^{i\theta}) \quad \text{exists and belongs to } \Omega\}$$

has length 0. R. Nevanlinna proved that a holomorphic mapping from t unit disc into a domain whose complement has positive logarithmic capaci has radial limit along almost every radius. (see [32, p. 209]). So, given su a domain $\Omega \subset \mathbb{C}$, a holomorphic mapping I from $\mathbb{D}$ into Ω is inner into Ω and only if

$$\lim_{r\to 1} I(re^{i\theta}) \in \partial\Omega$$

for almost every $e^{i\theta} \in \partial\mathbb{D}$.

The functions which are inner into $\mathbb{D}$ are the usual inner functions. A c ering map Π from $\mathbb{D}$ onto a domain Ω is inner into Ω and, as a matter of fa if f is a holomorphic mapping from $\mathbb{D}$ into Ω which factorizes as $f = \Pi \circ$ where $b\colon \mathbb{D} \to \mathbb{D}$, then f is inner into Ω if and only if b is an inner functi (see [19]). Also,

$$\lambda_\Omega(f(z))|f'(z)| = \lambda_\Omega(\Pi(b(z)))|\Pi'(b(z))||b'(z)| = \lambda_{\mathbb{D}}(b(z))|b'(z)|\,.$$

Thus, Theorem 1.9 gives the following result.

Corollary 4.1. *Let Ω be a hyperbolic domain in the complex plane. , $\alpha > 0$. Let f be a holomorphic mapping from the unit disc into Ω. Ther is inner into Ω if and only if*

$$\int_{\Gamma_\alpha(\theta)} \lambda_\Omega^2(f(z))|f'(z)|^2\, dm(z) = \infty$$

for almost every $\theta \in [0, 2\pi]$.

5. Quasiconformal Mappings

This section is devoted to the proof of Theorem 1.11. We start by proving some auxiliary results.

Lemma 5.1. *With the notation of Theorem 1.11, the following identity holds:*

$$|\partial u(z)| = \left| \int_{\mathbb{R}^2_-} \frac{\overline{\partial}\rho(w)}{(w-z)^3}\, dm(w) \right|, \quad \text{for any } z \in \mathbb{R}^2_+.$$

Proof. Since $u(z) = Re\left(i \int_{\mathbb{R}} \frac{d\mu(t)}{z-t}\right)$, one has

$$|\partial u(z)| = \left| \partial_z \int_{\mathbb{R}} \frac{d\mu(t)}{z-t} \right| = \left| \int_{\mathbb{R}} \frac{d\mu(t)}{(z-t)^2} \right|.$$

Approximating ρ by smooth functions with compact support, we can assume that $d\mu(t) = \rho'(t)dt$. Therefore the above integral is equal to

$$\left| \int_{\mathbb{R}} \frac{\rho(t)}{(t-z)^3}\, dt \right|.$$

Now, Green's Theorem is applied to get

$$|\partial u(z)| = \left| \int_{\mathbb{R}^2_-} \overline{\partial}_w \left(\frac{\rho(w)}{(w-z)^3} \right) dm(w) \right| = \left| \int_{\mathbb{R}^2_-} \frac{\overline{\partial}\rho(w)}{(w-z)^3}\, dm(w) \right|. \quad \blacksquare$$

Note that the lemma above holds under weaker assumptions: ρ does not need to be quasiconformal.

Lemma 5.2. *With the same notation as in Theorem 1.11,*

$$\frac{|\partial u(z)|}{u(z)} \leq \frac{C}{y^{1/2}} \left(\int_y^\infty \omega^2(z,t)\, \frac{dt}{t^2} \right)^{1/2}, \quad \text{for all } z = x+iy \in \mathbb{R}^2_+,$$

where

$$\omega^2(z,t) = \frac{1}{t^2} \int_{B(z,t)\cap\mathbb{R}^-_2} |\sigma(w)|^2\, dm(w),$$

and $B(z,t)$ denotes the ball centered at z and radius t.

Proof. Since $\overline{\partial}\rho = \sigma\partial\rho$, by Lemma 5.1 we get

$$|\partial u(z)| \leq \int_{\mathbb{R}^-_2} \frac{|\sigma(w)|\,|\partial\rho(w)|}{|w-z|^3}\, dm(w).$$

Consider now the balls B_k centered at z of radius $2^k y$, and define

$$D_k = B_k \cap \mathbb{R}^2_- .$$

Denote by $J_\rho(w)$ the Jacobian of the quasiconformal mapping ρ, then

$$\begin{aligned}
|\partial u(z)| &\lesssim \sum_{k=1}^{\infty} \frac{1}{2^{3k} y^3} \int_{D_k} |\sigma(w)|\, |J_\rho(w)|^{1/2}\, dm(w) \\
&\cong \sum_{k=1}^{\infty} \frac{1}{2^k y} \fint_{D_k} |\sigma(w)|\, |J_\rho(w)|^{1/2}\, dm(w) \\
&\leq \sum_{k=1}^{\infty} \frac{1}{2^k y} \left(\fint_{D_k} |\sigma(w)|^2 \right)^{1/2} \left(\fint_{D_k} |J_\rho(w)| \right)^{1/2},
\end{aligned}$$

where the symbol $\fint_D g$ represents the mean of the function g over D. Now, by the circular distortion theorem the quantity Area $(\rho(D_k)) = \int_{D_k} |J_\rho(w)|$ is comparable to the square of the diameter of the interval $\rho(I_k)$, where I_k is the interval centered at $x = Re z$ and length $2^k y$. Therefore

$$|\partial u(z)| \lesssim \sum_{k=1}^{\infty} \frac{1}{2^k y} \left(\fint_{D_k} |\sigma(w)|^2 \right)^{1/2} \frac{\mu(I_k)}{|I_k|},$$

with all comparison constants depending only on the quasiconformal constant of the mapping ρ. Since μ is doubling, $\mu(I_k) \leq C^k \mu(I_z)$ where I_z denotes the interval centered at $x = Re z$ and length y, and C is the doubling constant. Writing $C = 2^{1+\varepsilon}$, we get

$$\frac{\mu(I_k)}{|I_k|} \leq 2^{k\varepsilon} \frac{\mu(I_z)}{|I_z|}.$$

Hence if $\varepsilon < 1/2$

$$\begin{aligned}
|\partial u(z)| &\leq \sum_k \frac{1}{2^k y} \left(\fint_{D_k} |\sigma(w)|^2 \right)^{1/2} 2^{k\varepsilon} \frac{\mu(I_z)}{|I_z|} \\
&\leq u(z) \sum_k \frac{1}{2^{k(1-\varepsilon)} y} \left(\fint_{D_k} |\sigma(w)|^2 \right)^{1/2} \\
&= \frac{u(z)}{y^{1/2}} \sum_k \frac{1}{2^{k(1/2-\varepsilon)}} \left(\frac{1}{2^k y} \fint_{D_k} |\sigma(w)|^2 \right)^{1/2} \\
&\leq \frac{u(z)}{y^{1/2}} \left(\sum_k \frac{1}{2^{k(1-2\varepsilon)}} \right)^{1/2} \left(\sum_k \frac{1}{2^k y} \fint_{D_k} |\sigma|^2 \right)^{1/2} \\
&\lesssim K(C) \frac{u(z)}{y^{1/2}} \left(\int_y^{\infty} \omega^2(z,t) \frac{dt}{t^2} \right)^{1/2}.
\end{aligned}$$

∎

We finally proceed to prove Theorem 1.11.

Proof of Theorem 1.11. We have to show that

$$\int_{\Gamma_\alpha(x)} \left(\frac{1}{y} \int_y^\infty \omega^2(z,t)\, \frac{dt}{t^2} \right) dm(z) \leq C \int_{\mathbb{R}^-_2} \frac{|\sigma(w)|^2}{|w-x|^2}\, dm(w)\,.$$

By Fubini, the left term can be expressed as:

$$\int_{\mathbb{R}^-_2} |\sigma(w)|^2 \left(\int_{\Gamma_\alpha(x)} \frac{1}{y} \left(\int_y^\infty \frac{1}{t^4} \chi_{B(z,t)}(w)\, dt \right) dm(z) \right) dm(w)\,.$$

On the other hand

$$\int_y^\infty \frac{1}{t^4} \chi_{B(z,t)}(w)\, dt \simeq \int_{|z-w|}^\infty \frac{dt}{t^4} \simeq \frac{1}{|z-w|^3}$$

and

$$\int_{\Gamma_\alpha(x)} \frac{1}{y} \frac{dm(z)}{|z-w|^3} \lesssim \frac{1}{|x-w|^2}$$

which ends the proof of the theorem. ∎

6. Concluding remarks

Centered square function

Let μ be a doubling measure in Q_0. It is natural to consider analogues of the square function in Section 1 which do not use the dyadic filtration. Given a point $(w,y) \in \mathbb{R}^{N+1}_+$, let us consider

$$d(w,y) = \max_k \left| \frac{2^N \mu(Q_k)}{\mu(Q)} - 1 \right|,$$

where Q is the cube in $\mathbb{R}^N$ centered at w of sidelength y and $\{Q_k\colon k = 1,\dots,2^N\}$ are the 2^N disjoint subcubes of Q of sidelength $y/2$. So, $d(w,y)$ measures the error done by μ when doubling at a cube centered at w of sidelength y. Define

$$\mathcal{A}^2_\alpha(\mu)(x) = \int_{\Gamma_\alpha(x)} d^2(w,y) \frac{dw\, dy}{y^{N+1}}\,, \quad x \in \mathbb{R}^N\,.$$

Again, doubling measures which are singular can be characterized in terms of this square function.

Theorem 6.1. *Let μ be a doubling measure in Q_0. Fix $\alpha > 0$. Then μ is singular if and only if*

$$\mathcal{A}^2_\alpha(\mu)(x) = \infty\,,$$

at almost every $x \in Q_0$.

Proof of Theorem 6.1. Given $0 < \varepsilon < \eta$, let us consider the (doubly) truncated square function

$$\mathcal{A}^2_{\alpha,\varepsilon,\eta}(\mu)(x) = \int_{\Gamma_\alpha(x)\cap\{\varepsilon<y<\eta\}} d^2(w,y)\frac{dw\,dy}{y^{N+1}}\,.$$

The main step is to prove the following comparison estimate between integrals of the logarithm of the density of μ and the truncated square function $\mathcal{A}^2_{\alpha,\varepsilon,\eta}$. More concretely, there exists a constant $C = C(\mu,\alpha) > 0$ such that for any cube $Q \subset Q_0$ and $0 < y < l(Q)/2$, one has

$$\begin{aligned} C^{-1}(1 + \frac{1}{|Q|}\int_Q \mathcal{A}^2_{\alpha,y,l(Q)}(\mu)(x)\,dx) \\ \leq \frac{1}{|Q|}\int_Q \log\frac{\mu(Q(x,y))}{y^N}\,dx - \log\frac{\mu(Q)}{|Q|} \\ \leq C(1 + \frac{1}{|Q|}\int_Q \mathcal{A}^2_{\alpha,y,l(Q)}(\mu)(x)\,dx)\,. \end{aligned} \tag{6.1}$$

To show this, first note that, as in Lemma 2.2,

$$d^2(w,y) \simeq -\log\frac{\text{g.m.}(Q(w,y))}{\text{a.m.}(Q(w,y))}\,,$$

where g.m. and a.m. are defined as in the martingale setting. By Fubini's Theorem

$$\int_Q \mathcal{A}^2_{\alpha,y,l(Q)}(\mu)(x)\,dx \simeq -\int_{w\in Q}\int_y^{l(Q)} \log\frac{\text{g.m.}(Q(w,t))}{\text{a.m.}(Q(w,t))}\,\frac{dt}{t}\,dw = \int_y^{l(Q)} \Phi(t)\frac{dt}{t}\,,$$

where

$$\Phi(t) = -\int_Q \log\frac{\text{g.m.}(Q(w,t))}{\text{a.m.}(Q(w,t))}\,dw\,.$$

Then, splitting the integral of $\Phi(t)$ dyadically and using the following cancellation property,

$$\left|\int_Q \log\Big(\frac{\text{g.m. } Q(w,t)}{\mu(Q(w,t/2))}\,|Q(w,t/2)|\Big)\,dw\right| \leq C(\mu)t^N\,.$$

we obtain (6.1). Now Theorem 6.1 follows from standard stopping time arguments. We omit the details. ∎

Given a doubling measure μ, let us consider the maximal function

$$M_\alpha(\log \mu)(x) = \sup \left| \log \left(\frac{\mu(Q(w,y))}{y^N} \right) \right|,$$

where the supremum is taken over the points (w,y) in the cone $\Gamma_\alpha(x)$. Again, using (6.1), standard methods also give the analogue of part (b) in Theorem 1.1, that is, for $0 < p < \infty$,

$$M_\alpha(\log \mu) \in L^p(Q_0) \quad \text{if and only if} \quad \mathcal{A}_\alpha^2(\mu) \in L^p(Q_0).$$

However, we have not explored the analogues of (d) and (e) in Theorem 1.1 and the LIL of Theorem 1.3 for the square function $\mathcal{A}$. For this, it seems necessary to prove the analogue of the good λ-inequality, given in Theorem 2.4 (a).

Sharp control of A by M

Theorem 2.4 and Theorem 3.1 establish comparisons between the distribution function of a non-tangential maximal function and a square function in two different settings: dyadic doubling measures and positive harmonic functions. Part (a) in both results provide the same subgaussian decay. However the estimate in part (b) in both results gives a worst decay and it probably could be improved. This is particularly convenient if one is interested on LIL's relating the growth of the (truncated) square function with the non-tangential maximal function. For harmonic functions, such LIL's were considered by Bañuelos and Moore (see [6, Chapter 4]).

Non-harmonic extensions

It is natural to consider the non-harmonic version of the results in Section 2. Let $\varphi\colon \mathbb{R}^N \to \mathbb{R}$ be a smooth function with integral 1. Let μ be a positive measure on $\mathbb{R}^N$ such that

$$\int_{\mathbb{R}^N} |\varphi(x)|\, d\mu(x) < \infty\,.$$

Consider

$$u(x,t) = (\varphi_t * d\mu)(x),$$

where

$$\varphi_t(x) = t^{-N}\varphi\Big(\frac{x}{t}\Big), \quad x \in \mathbb{R}^N, t > 0.$$

Under some additional restrictions on the function φ, it is known that, for $0 < p < \infty$, the non-tangential maximal function of u is in L^p if and only if the square function

$$A_\alpha^2(\mu, \varphi)(x) = \int_{\Gamma_\alpha(x)} |\nabla u(w,t)|^2 t^{1-N}\, dw\, dt$$

is in L^p (see [18]). In our setting, the natural square function to be considered is

$$A_\alpha^2(\log \mu, \varphi)(x) = \int_{\Gamma_\alpha(x)} \left|\frac{\nabla u(w,t)}{u(w,t)}\right|^2 t^{1-N}\, dw\, dt\,.$$

We have not explored the square function in this generality. It is worth mentioning that in this generality, even the analogue of the subgaussian estimates of Chang, Wilson and Wolff is open (see [6, p. 113] and [36]). So, proving sharp estimates for $A^2(\log \mu, \varphi)$ seems to be a difficult problem.

One could also ask for results analogous to the ones in Section 2, with the area function replaced by the g-function

$$g^2(\log u)(x) = \int_0^1 y \frac{|\nabla u(x,y)|^2}{u(x,y)^2}\, dx\,.$$

Again the main difficulty is the subgaussian estimate of Theorem 3.1, which is open in the context of harmonic functions (see [6, p. 114])

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Recibido: 23 de mayo de 2002

María J. González
Departamento de Matemáticas (CASEM)
Universidad de Cádiz
11510 Puerto Real, Cádiz (Spain)
majose.gonzalez@uca.es

Artur Nicolau
Departament de Matemàtiques
Facultat de Ciencies
Universitat Autònoma de Barcelona
08193 Bellaterra, Barcelona (Spain)
artur@mat.uab.es

The authors are partially supported by the grants BFM2002-00571 and 2001SGR00431.