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Symmetry reductions of a particular set of equations of associativity in two-dimensional topological field theory

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Abstract
The WDVV equations of associativity arising in two-dimensional topological field theory can be represented, in the simplest nontrivial case, by a single third-order equation of the Monge–Ampère type. By investigating its Lie point symmetries, we reduce it to various nonlinear ordinary differential equations, and obtain several new explicit solutions.

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1. Introduction

As introduced by Witten, Dijkgraaf, H Verlinde and E Verlinde [10, 1], the equations of associativity involve the following unknowns: a function $\mathcal{F}(t^1,\dots,t^n) \equiv \mathcal{F}(t)$, integer numbers q_α and $r_\alpha, \alpha = 1,\dots,n$, another integer d, a constant symmetric nondegenerate matrix $(\eta^{\alpha\beta})$, other constants $A_{\alpha\beta}, B_\alpha, C$. These unknowns must obey three main sets of equations [3].

1. The equations of associativity properly said (with summation over the repeated indices)

$$\partial_\alpha\partial_\beta\partial_\lambda\mathcal{F}(t)\eta^{\lambda\mu}\partial_\mu\partial_\gamma\partial_\delta\mathcal{F}(t) = \partial_\delta\partial_\beta\partial_\lambda\mathcal{F}(t)\eta^{\lambda\mu}\partial_\mu\partial_\gamma\partial_\alpha\mathcal{F}(t). \qquad \text{(WDVV1)}$$

2. A condition singling out one variable, say, t^1,

$$\partial_\alpha\partial_\beta\partial_1\mathcal{F}(t) = \eta^{\alpha\beta}. \qquad \text{(WDVV2)}$$

in which the matrix $(\eta_{\alpha\beta})$ is the inverse of $(\eta^{\alpha\beta})$.

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3. A condition of quasi-homogeneity,

$$\sum_{\alpha=1}^{n}[(1-q_\alpha)t^\alpha + r_\alpha]\partial_\alpha \mathcal{F}(t) = (3-d)\mathcal{F}(t) + \frac{1}{2}A_{\alpha\beta}t^\alpha t^\beta + B_\alpha t^\alpha + C. \qquad \text{(WDVV3)}$$

In the simplest nontrivial case $n = 3$, there essentially exist two different choices of coordinates [2], depending on η^{11} being zero or nonzero, each choice resulting in a representation of the generating function $\mathcal{F}$ in terms of the solution of a single third-order partial differential equation (PDE) of the Monge–Ampère type, which is either [2], [4, equation (9)]

1.

$$\eta^{11} \neq 0 : \mathcal{F} = \tfrac{1}{6}(t^1)^3 + t^1 t^2 t^3 + f(t^2, t^3), \qquad (1)$$

$$f_{xxx}f_{yyy} - f_{xxy}f_{yyx} - 1 = 0, \qquad x = t^2, \qquad y = t^3, \qquad (2)$$

or [3, p 304] [4, equation (22)].

2.

$$\eta^{11} = 0 : \mathcal{F} = \tfrac{1}{2}(t^1)^2 t^3 + \tfrac{1}{2}t^1(t^2)^2 + F(t^2, t^3), \qquad (3)$$

$$(F_{tyy})^2 - F_{ttt} - F_{tty}F_{yyy} = 0, \qquad y = t^2, \qquad t = t^3. \qquad (4)$$

There exists a Legendre transformation [2] which exchanges these two solutions $\mathcal{F}$ (this transformation exchanges the coordinates t^3 of (2) and t^2 of (4), whose common value is here denoted y), and its action on the functions of two variables $f(x, y)$ and $F(y, t)$ is the hodograph transformation [6], [4, equation (23)]

$$t = f_{xx}, \qquad F_{yyy} = \frac{f_{xxy}^2}{f_{xxx}}, \qquad F_{tyy} = -\frac{f_{xxy}}{f_{xxx}}, \qquad F_{tty} = \frac{1}{f_{xxx}}, \qquad F_{ttt} = \frac{f_{xyy}}{f_{xxx}}, \quad (5)$$

whose inverse is

$$f_{xx} = t, \qquad f_{xy} = -F_{yy}, \qquad f_{yy} = F_{tt}, \qquad x = F_{ty}. \qquad (6)$$

A nice way to obtain this hodograph transformation (5), (6) is to rewrite both PDEs [6] as integrable systems of the so-called hydrodynamic type, allowing them to be mapped by a chain of standard transformations to integrable three-wave systems.

Both PDEs admit a Lax pair [2], e.g. for the PDE (2) [4, equation (10)]

$$\psi_x = \lambda \begin{pmatrix} 0 & 1 & 0 \\ 0 & f_{xxy} & f_{xxx} \\ 1 & f_{xyy} & f_{xxy} \end{pmatrix} \psi, \qquad \psi_y = \lambda \begin{pmatrix} 0 & 0 & 1 \\ 1 & f_{xyy} & f_{xxy} \\ 0 & f_{yyy} & f_{xyy} \end{pmatrix} \psi, \qquad (7)$$

in which λ is a nonzero spectral parameter.

The purpose of this paper is to obtain new explicit solutions of either the PDE (2) or the PDE (4), and therefore of the equations of associativity in the simplest nontrivial case. Any such solution f is only defined up to an arbitrary additive second degree polynomial. However, equation (4) possesses a rather complicated structure of singularities, making the search for explicit solutions not easy, while equation (2) has a simpler such structure, so we will mainly consider this latter equation. In particular, the invariance of this PDE under permutation of x and y has no simple equivalent for the PDE (4).

To achieve this search for solutions, we perform a systematic investigation, via the Lie point symmetries method, of the reductions of the PDE (2) to ordinary differential equations (ODEs), which *a priori* can be integrated since they inherit the integrability properties of the equations of associativity. In addition to the reductions or particular solutions of either (4) or (2) which have already been found [2, 4], we obtain several new results.

The paper is organized as follows. In section 2, we apply the classical Lie method [9, 8], derive the Lie algebra, compute the commutator table and the adjoint table [8], which then allow us to derive the optimal system of generators. In section 3, we perform all the associated classical reductions. Section 4 summarizes the solutions.

2. Classical Lie symmetries

In order to apply the classical Lie method to the Ferapontov equation (2), we consider the one-parameter Lie group of infinitesimal transformations in (x, y, f)

$$\begin{aligned} x^* &= x + \varepsilon\xi(x, y, f) + \mathcal{O}(\varepsilon^2), \\ y^* &= y + \varepsilon\eta(x, y, f) + \mathcal{O}(\varepsilon^2), \\ f^* &= f + \varepsilon\phi(x, y, f) + \mathcal{O}(\varepsilon^2), \end{aligned} \tag{8}$$

where ε is the group parameter. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$\mathbf{v} = \xi\partial_x + \eta\partial_y + \phi\partial_f. \tag{9}$$

One then requires that this transformation leaves invariant the set of solutions of equation (2). This yields an overdetermined, linear system of equations for the infinitesimals $\xi(x, y, f)$, $\eta(x, y, f)$ and $\phi(x, y, f)$. Having determined the infinitesimals, the symmetry variables are found by solving the invariant surface condition

$$\Phi \equiv \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} - \phi = 0. \tag{10}$$

Applying the classical method to equation (2) leads to a ten-parameter Lie group. Associated with this Lie group we have a Lie algebra which can be represented by the following generators:

$$\begin{aligned} &\mathbf{v}_1 = \partial_x, && \mathbf{v}_2 = \partial_y, && \mathbf{v}_3 = x\partial_x + \tfrac{3}{2}f\partial_f, && \mathbf{v}_4 = y\partial_y + \tfrac{3}{2}f\partial_f, && \mathbf{v}_5 = xy\partial_f, \\ &\mathbf{v}_6 = x^2\partial_f, && \mathbf{v}_7 = y^2\partial_f, && \mathbf{v}_8 = x\partial_f, && \mathbf{v}_9 = y\partial_f, && \mathbf{v}_{10} = \partial_f. \end{aligned}$$

2.1. Optimal system

In order to construct the optimal system, following Olver [8], we first construct the commutator table (table 1) and the adjoint table (table 2) which shows the separate adjoint actions of each element in $\mathbf{v}_i$, $i = 1, \ldots, 10$, as it acts on all other elements. This construction is done easily by summing the Lie series.

The corresponding generators of the optimal system of subalgebras are

$$\begin{aligned} &\mathbf{v}_3, \\ &\mathbf{v}_4, \\ &-a\mathbf{v}_3 + b\mathbf{v}_4, \\ &\mathbf{v}_3 - \mathbf{v}_4 + a\mathbf{v}_5 + b\mathbf{v}_{10}, \\ &3\mathbf{v}_3 + \mathbf{v}_4 + a\mathbf{v}_7, \end{aligned}$$

Table 1. Commutator table for the Lie algebra $\mathbf{v}_i$.

	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$	$\mathbf{v}_6$	$\mathbf{v}_7$	$\mathbf{v}_8$	$\mathbf{v}_9$	$\mathbf{v}_{10}$
$\mathbf{v}_1$	0	0	$\mathbf{v}_1$	0	$\mathbf{v}_9$	$2\mathbf{v}_8$	0	$\mathbf{v}_{10}$	0	0
$\mathbf{v}_2$	0	0	0	$\mathbf{v}_2$	$\mathbf{v}_8$	0	$2\mathbf{v}_9$	0	$\mathbf{v}_{10}$	0
$\mathbf{v}_3$	$-\mathbf{v}_1$	0	0	0	$-\frac{1}{2}\mathbf{v}_5$	$\frac{1}{2}\mathbf{v}_6$	$-\frac{3}{2}\mathbf{v}_7$	$-\frac{1}{2}\mathbf{v}_8$	$-\frac{3}{2}\mathbf{v}_9$	$-\frac{3}{2}\mathbf{v}_{10}$
$\mathbf{v}_4$	0	$-\mathbf{v}_2$	0	0	$-\frac{1}{2}\mathbf{v}_5$	$-\frac{3}{2}\mathbf{v}_6$	$\frac{1}{2}\mathbf{v}_7$	$-\frac{3}{2}\mathbf{v}_8$	$-\frac{1}{2}\mathbf{v}_9$	$-\frac{3}{2}\mathbf{v}_{10}$
$\mathbf{v}_5$	$-\mathbf{v}_9$	$-\mathbf{v}_8$	$\frac{1}{2}\mathbf{v}_5$	$\frac{1}{2}\mathbf{v}_5$	0	0	0	0	0	0
$\mathbf{v}_6$	$-2\mathbf{v}_8$	0	$-\frac{1}{2}\mathbf{v}_6$	$\frac{3}{2}\mathbf{v}_6$	0	0	0	0	0	0
$\mathbf{v}_7$	0	$-2\mathbf{v}_9$	$\frac{3}{2}\mathbf{v}_7$	$-\frac{1}{2}\mathbf{v}_7$	0	0	0	0	0	0
$\mathbf{v}_8$	$-\mathbf{v}_{10}$	0	$\frac{1}{2}\mathbf{v}_8$	$\frac{3}{2}\mathbf{v}_8$	0	0	0	0	0	0
$\mathbf{v}_9$	0	$-\mathbf{v}_{10}$	$\frac{3}{2}\mathbf{v}_9$	$\frac{1}{2}\mathbf{v}_9$	0	0	0	0	0	0
$\mathbf{v}_{10}$	0	0	$\frac{3}{2}\mathbf{v}_{10}$	$\frac{3}{2}\mathbf{v}_{10}$	0	0	0	0	0	0

Table 2. Adjoint table for the Lie algebra $\mathbf{v}_i$.

Ad	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$	$\mathbf{v}_6$	$\mathbf{v}_7$	$\mathbf{v}_8$	$\mathbf{v}_9$	$\mathbf{v}_{10}$
$\mathbf{v}_1$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3-\varepsilon\mathbf{v}_1$	$\mathbf{v}_4$	$\mathbf{v}_5-\varepsilon\mathbf{v}_9$	$\mathbf{v}_6-2\varepsilon\mathbf{v}_8+\varepsilon^2\mathbf{v}_{10}$	$\mathbf{v}_7$	$\mathbf{v}_8-\varepsilon\mathbf{v}_{10}$	$\mathbf{v}_9$	$\mathbf{v}_{10}$
$\mathbf{v}_2$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4-\varepsilon\mathbf{v}_2$	$\mathbf{v}_5-\varepsilon\mathbf{v}_8$	$\mathbf{v}_6$	$\mathbf{v}_7-2\varepsilon\mathbf{v}_9+\varepsilon^2\mathbf{v}_{10}$	$\mathbf{v}_8$	$\mathbf{v}_9-\varepsilon\mathbf{v}_{10}$	$\mathbf{v}_{10}$
$\mathbf{v}_3$	$e^{\varepsilon}\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$e^{\frac{\varepsilon}{2}}\mathbf{v}_5$	$e^{-\frac{\varepsilon}{2}}\mathbf{v}_6$	$e^{\frac{3\varepsilon}{2}}\mathbf{v}_7$	$e^{\frac{\varepsilon}{2}}\mathbf{v}_8$	$e^{\frac{3\varepsilon}{2}}\mathbf{v}_9$	$e^{\frac{3\varepsilon}{2}}\mathbf{v}_{10}$
$\mathbf{v}_4$	$\mathbf{v}_1$	$e^{\varepsilon}\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$e^{\frac{\varepsilon}{2}}\mathbf{v}_5$	$e^{\frac{3\varepsilon}{2}}\mathbf{v}_6$	$e^{-\frac{\varepsilon}{2}}\mathbf{v}_7$	$e^{\frac{3\varepsilon}{2}}\mathbf{v}_8$	$e^{\frac{\varepsilon}{2}}\mathbf{v}_9$	$e^{\frac{3\varepsilon}{2}}\mathbf{v}_{10}$
$\mathbf{v}_5$	$\mathbf{v}_1+\varepsilon\mathbf{v}_9$	$\mathbf{v}_2+\varepsilon\mathbf{v}_8$	$\mathbf{v}_3-\frac{1}{2}\varepsilon\mathbf{v}_5$	$\mathbf{v}_4-\frac{1}{2}\varepsilon\mathbf{v}_5$	$\mathbf{v}_5$	$\mathbf{v}_6$	$\mathbf{v}_7$	$\mathbf{v}_8$	$\mathbf{v}_9$	$\mathbf{v}_{10}$
$\mathbf{v}_6$	$\mathbf{v}_1+2\varepsilon\mathbf{v}_8$	$\mathbf{v}_2$	$\mathbf{v}_3+\frac{1}{2}\varepsilon\mathbf{v}_6$	$\mathbf{v}_4-\frac{3}{2}\varepsilon\mathbf{v}_6$	$\mathbf{v}_5$	$\mathbf{v}_6$	$\mathbf{v}_7$	$\mathbf{v}_8$	$\mathbf{v}_9$	$\mathbf{v}_{10}$
$\mathbf{v}_7$	$\mathbf{v}_1$	$\mathbf{v}_2+2\varepsilon\mathbf{v}_9$	$\mathbf{v}_3-\frac{3}{2}\varepsilon\mathbf{v}_7$	$\mathbf{v}_4+\frac{1}{2}\varepsilon\mathbf{v}_7$	$\mathbf{v}_5$	$\mathbf{v}_6$	$\mathbf{v}_7$	$\mathbf{v}_8$	$\mathbf{v}_9$	$\mathbf{v}_{10}$
$\mathbf{v}_8$	$\mathbf{v}_1+\varepsilon\mathbf{v}_{10}$	$\mathbf{v}_2$	$\mathbf{v}_3-\frac{1}{2}\varepsilon\mathbf{v}_8$	$\mathbf{v}_4-\frac{3}{2}\varepsilon\mathbf{v}_8$	$\mathbf{v}_5$	$\mathbf{v}_6$	$\mathbf{v}_7$	$\mathbf{v}_8$	$\mathbf{v}_9$	$\mathbf{v}_{10}$
$\mathbf{v}_9$	$\mathbf{v}_1$	$\mathbf{v}_2+\varepsilon\mathbf{v}_{10}$	$\mathbf{v}_3-\frac{3}{2}\varepsilon\mathbf{v}_9$	$\mathbf{v}_4-\frac{1}{2}\varepsilon\mathbf{v}_9$	$\mathbf{v}_5$	$\mathbf{v}_6$	$\mathbf{v}_7$	$\mathbf{v}_8$	$\mathbf{v}_9$	$\mathbf{v}_{10}$
$\mathbf{v}_{10}$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3-\frac{3}{2}\varepsilon\mathbf{v}_{10}$	$\mathbf{v}_4-\frac{3}{2}\varepsilon\mathbf{v}_{10}$	$\mathbf{v}_5$	$\mathbf{v}_6$	$\mathbf{v}_7$	$\mathbf{v}_8$	$\mathbf{v}_9$	$\mathbf{v}_{10}$

$$\begin{aligned}
&\mathbf{v}_3+3\mathbf{v}_4+a\mathbf{v}_6,\\
&-3\mathbf{v}_3+\mathbf{v}_4+a\mathbf{v}_8,\\
&\mathbf{v}_3-3\mathbf{v}_4+a\mathbf{v}_9,\\
&a\mathbf{v}_2+b\mathbf{v}_3,\\
&a\mathbf{v}_1+b\mathbf{v}_4,\\
&a\mathbf{v}_1+b\mathbf{v}_2+c\mathbf{v}_5+d\mathbf{v}_6+e\mathbf{v}_7,
\end{aligned}\tag{11}$$

where a, b, c, d, e are arbitrary real nonzero constants.

3. Classical reductions

Each generator of the optimal system defines a reduction of equation (2) to an ODE. Because of the invariance of (2) under permutation of x and y, these ten generators only define seven different reductions to an ODE, which we now consider.

Although these reductions are probably integrable in some sense, performing their explicit integration is a difficult task. Moreover, since the Lax pair (7) is not isospectral, its reductions,

which are also Lax pairs for the reduced ODEs, cannot generate any first integral, so the Lax pair is unfortunately of no use for integrating the reduced ODEs.

From the scaling invariance of the two considered PDEs, an obvious solution is

$$f = 2\mathrm{i}\frac{\sqrt{2}}{3}(xy)^{3/2}, \qquad F = \frac{y^4}{8t}, \qquad 2t^2x + y^3 = 0, \qquad \mathrm{i}^2 = -1. \tag{12}$$

3.1. Reduction with the generator $\mathbf{v}_3$ *or* $\mathbf{v}_4$

The generators $\mathbf{v}_3$ and $\mathbf{v}_4$ define a reduction to the same autonomous linear ODE [4, equation (30) p 46],

$$\begin{cases} z = y, \qquad f = [x^3\Phi(z)]^{1/2}, \quad \text{or} \quad z = x, \qquad f = [y^3\Phi(z)]^{1/2} \\ \Phi''' + 16/3 = 0. \end{cases} \tag{13}$$

This contains the scaling solution (12).

3.2. Reduction with the generator $-a\mathbf{v}_3 + b\mathbf{v}_4$

With the notation $s = a + b$, $p = ab$, a symmetric definition of this reduction is

$$\begin{cases} z = x^b y^a, \qquad f = (xy)^{3/2}\varphi(z), \\ [-16p^2sz^5\varphi'' - 8p(4p + 2ps + s^2)z^4\varphi' - 3s^3z^3\varphi]\varphi''' \\ \quad + 8p(2p - 6ps - 3s^2)\varphi''^2 - (64p^2 + 72ps + 64p^2s + 72ps^2 + 9s^3)z^3\varphi'\varphi'' \\ \quad - 9(2 + s)s^2z^2\varphi\varphi'' - (40p + 16p^2 + 72ps + 16p^2s + 18s^2 + 32ps^2 + 9s^3)z^2\varphi'^2 \\ \quad - (33 + 18s + 3s^2)sz\varphi\varphi' - 9\varphi^2 - 8 = 0. \end{cases} \tag{14}$$

An equivalent, shorter expression is obtained by suppressing the term φ^2 [4, p 46],

$$\begin{cases} z = xy^{-\mu}, \qquad f = \left(\dfrac{xy}{z}\right)^{3/2}\varphi(z), \quad \text{or} \quad z = yx^{-\mu}, \qquad f = \left(\dfrac{xy}{z}\right)^{3/2}\varphi(z), \\ [16\mu^2(\mu - 1)z^2\varphi'' - 8\mu(3\mu + 1)(\mu + 1)z\varphi' + 3(3\mu + 1)(3\mu - 1)(\mu + 1)\varphi]\varphi''' \\ \quad - 8\mu(\mu - 3)z\varphi''^2 + (\mu - 3)(\mu + 3)(\mu + 1)\varphi'\varphi'' - 8 = 0. \end{cases} \tag{15}$$

As results from the scaling solution (12), the ODE (15) admits the particular zero-parameter solution

$$\forall\mu : \varphi = 2\mathrm{i}\frac{\sqrt{2}}{3}z^{3/2}, \qquad f = 2\mathrm{i}\frac{\sqrt{2}}{3}(xy)^{3/2}. \tag{16}$$

For generic values of (a, b), this ODE is unfortunately outside the class

$$\varphi''' = \sum_{j=0}^{3} A_j(z, \varphi, \varphi')\varphi''^j, \tag{17}$$

an equation which for some A_j can be linearized by a contact transformation. However, there exist particular values of μ for which the integration can be performed at least partially. The invariance of (14) under $(a, b) \to (b, a)$ induces an invariance of (15) under $\mu \to \mu^{-1}$.

1. For $\mu = 0, 1, -1, -2, -1/2$, a first integral K is known,

$$\begin{cases} \mu = 0, & K = -8z - 3\varphi\varphi'' - 3\varphi'^2 = \left[-\frac{4}{3}z^3 - \frac{3}{2}\varphi^2\right]', \\ \mu = 1, & K = \text{any rational function of } a, b, c, \text{ see (22)}, \\ \mu = -1, & K = z + 2z^2\varphi''^2, \\ \mu = -2, & K = -8z - \varphi'^2 - 105\varphi\varphi'' + 112z\varphi'\varphi'' - 96z^2\varphi''^2, \\ \mu = -\frac{1}{2}, & K = -8z^2 + \frac{15}{4}(z\varphi\varphi'' + z\varphi'^2 - \varphi\varphi') - 10z^2\varphi'\varphi'' - 6z^3\varphi''^2. \end{cases} \tag{18}$$

2. For $\mu = 1$, the third-order equation [4, equation (31) p 46],

$$\begin{cases} z = x/y, \qquad f = y^3\varphi(z), \quad \text{or} \quad z = y/x, \qquad f = x^3\varphi(z), \\ 2(3\varphi - 2z\varphi')\varphi''' + 2z\varphi''^2 - 2\varphi'\varphi'' - 1 = 0, \end{cases} \tag{19}$$

is linearizable since its derivative factorizes into

$$2(3\varphi - 2z\varphi')\varphi'''' = 0, \tag{20}$$

so its general solution is

$$\varphi = \alpha z^3 + 3\beta z^2 + 3\gamma z + \delta, \qquad 36(\alpha\delta - \beta\gamma) - 1 = 0, \ (\alpha, \beta, \gamma) \text{ arbitrary}. \tag{21}$$

It is interesting to note that, knowing the three first integrals a, b, c,

$$\begin{cases} 12a = \dfrac{1 + 2\varphi'\varphi'' - 2z\varphi''^2}{3\varphi - 2z\varphi'} = 2\varphi''', \\ 4b = \dfrac{-z + 6\varphi\varphi'' - 6z\varphi'\varphi'' + 2z^2\varphi''^2}{3\varphi - 2z\varphi'} = 2\varphi'' - 2z\varphi''', \\ 4c = \dfrac{z^2 + 12\varphi\varphi' - 8z\varphi'^2 - 12z\varphi\varphi'' + 10z^2\varphi'\varphi'' - 2z^3\varphi''^2}{3\varphi - 2z\varphi'} = 4\varphi' - 4z\varphi'' + 2z^2\varphi''', \end{cases} \tag{22}$$

there exists no first integral which would be polynomial in $(\varphi, \varphi', \varphi'')$.

3. For $\mu = -1$, the ODE reduces to a linear equation for φ''^2, identical to the particular case $r_1 = r_2 = s_1 = s_2 = 0$ of reduction (27) given below.
4. For $\mu = 3, 1/3$ and $\mu = -3, -1/3$ respectively, the ODE is just the subcase $a = 0$ of reductions (30) and (32) given below.
5. For $\mu = 2, 1/2$, two rational solutions for φ^2 can be obtained,

$$\begin{cases} \mu = 2, \qquad \varphi = \dfrac{2}{15c}(z - c)^{5/2}, \qquad f = \dfrac{2y^2}{15c}\left(\dfrac{x}{y} - cy\right)^{5/2}, \\ \mu = \dfrac{1}{2}, \qquad \varphi = \dfrac{2}{15c}z^{-1/2}(1 - cz^2)^{5/2}, \qquad f = \dfrac{2x^2}{15c}\left(\dfrac{y}{x} - cx\right)^{5/2}, \end{cases} \tag{23}$$

and

$$\begin{cases} \mu = 2, \qquad \varphi = 2\mathrm{i}\dfrac{\sqrt{2}}{3}z^{3/2}(1 - cz), \qquad f = \dfrac{2\mathrm{i}\sqrt{2}}{3}(xy)^{3/2}\left(1 - \dfrac{cx}{y^2}\right) \\ \mu = \dfrac{1}{2}, \qquad \varphi = 2\mathrm{i}\dfrac{\sqrt{2}}{3}z^{-1/2}(z^2 - c), \qquad f = \dfrac{2\mathrm{i}\sqrt{2}}{3}(xy)^{3/2}\left(1 - \dfrac{cy}{x^2}\right) \end{cases} \tag{24}$$

in which c is arbitrary.
The first solution (23) represents the octahedron solution B_3 of Dubrovin, see [4, p 41].
The second solution f extrapolates the scaling solution (12).

6. For $\mu = 5/3, 3/5$, one rational solution exists, which depends on one arbitrary parameter c,

$$\begin{cases} \mu = \dfrac{5}{3}, \qquad \varphi = \dfrac{c}{6}z^3 + \dfrac{1}{24c}, \\ \mu = \dfrac{3}{5}, \qquad \varphi = \dfrac{c}{6z} + \dfrac{z^4}{24c}. \end{cases} \tag{25}$$

This represents the tetrahedron solution A_3 of Dubrovin, see [4, p 41].

3.3. Reduction with the generator $\mathbf{v}_3 - \mathbf{v}_4 + a\mathbf{v}_5 + b\mathbf{v}_{10}$

This reduction to a nonautonomous ODE,

$$z = xy, \qquad f = \varphi(z) + (az + b)\log x, \tag{26}$$

can be defined more symmetrically as [4, p 45, example 2]

$$\begin{cases} z = xy, \qquad f = \varphi(z) + (r_1 z + r_2)\log x + (s_1 z + s_2)\log y, \\ z^2\varphi''^2 + (r_1 + s_1)z\varphi'' - (r_2 + s_2)\varphi'' - \dfrac{r_1 s_2 + r_2 s_1}{z} + \dfrac{r_2 s_2}{z^2} + \dfrac{z}{2} + \dfrac{(r_1 + s_1)^2}{4} + k_0 = 0, \end{cases} \tag{27}$$

in which k_0 is a constant of integration. Its general solution is obtained by quadratures,

$$\begin{cases} \varphi = k_1 z + k_2 - \dfrac{r_1 + s_1}{2}(z\log z - z) - \dfrac{r_2 + s_2}{2}\log z \\ \quad \pm \displaystyle\int \mathrm{d}z \int \mathrm{d}z\, \frac{\sqrt{-2z^3 - 4k_0 z^2 - 2(r_1 - s_1)(r_2 - s_2)z + (r_2 - s_2)^2}}{2z^2}, \\ f = -s_2 \log x - r_2 \log y + \dfrac{r_1 - s_1}{2} xy \log\dfrac{x}{y} \pm \displaystyle\int \mathrm{d}z \int \mathrm{d}z\, \frac{\sqrt{\cdots}}{2z^2}, \end{cases} \tag{28}$$

and it generically involves elliptic integrals. A particular solution is

$$\begin{cases} f = 2\mathrm{i}\dfrac{\sqrt{2}}{3}(xy)^{3/2} + cxy\log\dfrac{x}{y}, \\ F = \mathrm{i}\dfrac{\sqrt{2}}{8}x^{-1/2}y^{3/2}\left(4c^2\log\dfrac{x}{y} - xy\right) + \dfrac{cy^2}{4} + \dfrac{c^3 y}{x} + \left(\dfrac{c^3 y}{x} - \dfrac{cy^2}{2}\right)\log\dfrac{x}{y}, \end{cases} \tag{29}$$

which is another extrapolation of the scaling solution (12).

3.4. Reduction with the generator $3\mathbf{v}_3 + \mathbf{v}_4 + a\mathbf{v}_7$ *or* $\mathbf{v}_3 + 3\mathbf{v}_4 + a\mathbf{v}_6$

These two generators define a reduction to the same nonautonomous ODE,

$$\begin{cases} z = xy^{-3}, \qquad f = y^6\varphi(z) - \dfrac{a}{4}y^2, \quad \text{or} \quad z = yx^{-3}, \qquad f = x^6\varphi(z) - \dfrac{a}{4}x^2, \\ 12(3z^2\varphi'' - 8z\varphi' + 10\varphi)\varphi''' - 1 = 0, \end{cases} \tag{30}$$

which a linear transformation can make second order in φ',

$$\begin{cases} z = xy^{-3}, \qquad f = x^2\varphi(z) - \dfrac{a}{4}y^2, \quad \text{or} \quad z = yx^{-3}, \qquad f = y^2\varphi(z) - \dfrac{a}{4}x^2, \\ [36z^6\varphi'' + 48z^5\varphi']\varphi''' + 216z^5\varphi''^2 + 504z^4\varphi'\varphi'' + 288z^3\varphi'^2 - 1 = 0. \end{cases} \tag{31}$$

Since f is defined up to an arbitrary additive second degree polynomial, the reduced ODE does not depend on a, and this case is identical to the case $\mu = 3, 1/3$ of (15), in which no solution is known other than (16).

3.5. Reduction with the generator $-3\mathbf{v}_3 + \mathbf{v}_4 + a\mathbf{v}_8$ *or* $\mathbf{v}_3 - 3\mathbf{v}_4 + a\mathbf{v}_9$

These two generators define a reduction to the same second order, nonautonomous ODE for φ',

$$\begin{cases} z = xy^3, \quad f = x\varphi(z) - \dfrac{a}{3}x\log x, \quad \text{or} \quad z = yx^3, \quad f = y\varphi(z) - \dfrac{a}{3}y\log y, \quad a \neq 0, \\ [-72z^4\varphi'' - 84z^3\varphi' + 9az^2]\varphi''' - 234z^3\varphi''^2 \\ \quad -324z^2\varphi'\varphi'' + 18az\varphi'' - 72z\varphi'^2 + 2a\varphi' - 1 = 0, \end{cases} \tag{32}$$

but, with $a \neq 0$, we could not find any solution to this ODE.

Table 3. Summary of solutions $F(y,t)$, $f(x,y)$ of equations (4), (2). A3, B3, H3, label solutions linked to regular polyhedra [2]; Dubn, solutions found by Dubrovin [2]; Fn, additional solutions listed in [4, p 41]; and Nn, solutions apparently new. A prime ($'$) labels the solution deduced by permuting x and y in f. A blank field in the column 'Equation' indicates a solution not arising from a known reduction. The irrelevant constant k reflects the scaling invariance and can be set to 1. P_n denotes a polynomial of degree n.

Label	$F(y,t)$	$f(x,y)$	Equation	Link (t,x,y)
	$\dfrac{y^4}{8t}$	$2\mathrm{i}\dfrac{\sqrt{2}}{3}(xy)^{3/2}$	(12)	$x=-\dfrac{y^3}{2t^2}$
F1	$\mathrm{i}\sqrt{2}y^{5/2}x^{-15/2}\lambda^{-5}P_8(x)$	$2\mathrm{i}\dfrac{\sqrt{2}}{3}(xy)^{3/2}\lambda$, $\lambda^2=1+\dfrac{\alpha}{x}+\dfrac{\beta}{x^2}+\dfrac{\gamma}{x^3}$	(13)	$t=\mathrm{i}\sqrt{2}y^{3/2}x^{-9/2}\lambda^{-3}P_4(x)$
F1′	$\dfrac{4\alpha\gamma-\beta^2+12\gamma y+6\beta y^2+4\alpha y^3+3y^4}{24t}$	$2\mathrm{i}\dfrac{\sqrt{2}}{3}(xy)^{3/2}\lambda$, $\lambda^2=1+\dfrac{\alpha}{y}+\dfrac{\beta}{y^2}+\dfrac{\gamma}{y^3}$	(13)	$x=-\dfrac{\lambda^2y^3}{t^2}$
F2		$\dfrac{r_1-s_1}{2}xy\log\dfrac{x}{y}$ $-s_2\log x-r_2\log y$ $\pm\int\mathrm{d}z\int\mathrm{d}z\dfrac{\sqrt{\cdots}}{2z^2}$, $z=xy$	(28)	$t=f_{xx}$
F3	(29)	$2\mathrm{i}\dfrac{\sqrt{2}}{3}(xy)^{3/2}+cxy\log\dfrac{x}{y}$	(29)	$t=\mathrm{i}\dfrac{\sqrt{2}}{2}\left(\dfrac{y^3}{x}\right)^{1/2}+c\dfrac{y}{x}$
F4	$\dfrac{12(\beta^2-\alpha\gamma)y^3-6\beta y^2t+yt^2+2\gamma t^3}{12\alpha}$	$\alpha x^3+3\beta x^2y+3\gamma xy^2+\delta y^3$ $36(\alpha\delta-\beta\gamma)-1=0$	(21) (35)	$t=6(\alpha x+\beta y)$
octa B3	$\dfrac{ky^3t}{3}+\dfrac{2k^2y^2t^3}{3}+\dfrac{8k^4t^7}{105}$	$\dfrac{2y^2}{15k}\left(\dfrac{x}{y}-ky\right)^{5/2}$	(23)	$x=4k^2yt^2+ky^2$
octa′	$\dfrac{\lambda^{11}}{528k^3}+\dfrac{\lambda^7x^2}{6k}+\dfrac{\lambda^5x^3}{3}+\dfrac{7k\lambda^3x^4}{3}$ $+\dfrac{4k^2\lambda x^5}{3}$	$\dfrac{2x^2}{15k}\left(\dfrac{y}{x}-kx\right)^{5/2}$, $y=kx^2+x\lambda^2$	(23)	$t=2k\lambda x^2+\dfrac{\lambda^5}{10k}$
N1	$\dfrac{\mathrm{i}\sqrt{2}}{24}x^{1/2}y^{-7/2}$ $\times\left(\dfrac{25k^3x^3}{7}-5k^2x^2y^2-7kxy^4-3y^6\right)$	$2\mathrm{i}\dfrac{\sqrt{2}}{3}(xy)^{3/2}\left(1-\dfrac{kx}{y^2}\right)$	(24)	$t=\dfrac{\mathrm{i}\sqrt{2}}{2(xy)^{1/2}}(y^2-5kx)$
N1′	$\dfrac{\mathrm{i}\sqrt{2}}{24}x^{-11/2}y^{5/2}\left(\dfrac{125k^3y^3}{11}-25k^2x^2y^2\right.$ $\left.+5kx^4y-3x^6\right)$	$2\mathrm{i}\dfrac{\sqrt{2}}{3}(xy)^{3/2}\left(1-\dfrac{ky}{x^2}\right)$	(24)	$t=\dfrac{\mathrm{i}\sqrt{2}}{2}x^{-5/2}y^{3/2}(x^2-ky)$
tetra A3	$\dfrac{y^2t^2}{4k}+\dfrac{t^5}{60k^2}$	$\dfrac{kx^3}{6y}+\dfrac{y^4}{24k}$	(25)	$x=ty/k$
tetra′	$\dfrac{x^3y}{3k}+\dfrac{3ky^4}{8x^2}+\dfrac{k^3y^7}{28x^7}$	$\dfrac{ky^3}{6x}+\dfrac{x^4}{24k}$	(25)	$t=\dfrac{ky^3}{3x^3}+\dfrac{x^2}{2k}$
Dub1	$\dfrac{\mathrm{e}^{2kt}}{8k^3}+\dfrac{y^2\,\mathrm{e}^{kt}}{2k}-\dfrac{ky^4}{48}$	$\dfrac{kxy^3}{12}-\dfrac{x^2}{2k}\log\dfrac{x}{y}-\dfrac{3x^2}{4k}$		$x=y\,\mathrm{e}^{kt}$
Dub1′	$\dfrac{y^4}{32k^3x^4}\left(4\log\dfrac{y}{x}-3\right)$ $+\left(\dfrac{y^3}{8kx}+\dfrac{kx^2y^2}{16}\right)\left(2\log\dfrac{y}{x}+3\right)$	$\dfrac{kyx^3}{12}-\dfrac{y^2}{2k}\log\dfrac{y}{x}-\dfrac{3y^2}{4k}$		$t=\dfrac{kxy}{2}+\dfrac{y^2}{2kx^2}$
Dub2	$-\dfrac{ky^4}{24}+\dfrac{y}{k}\,\mathrm{e}^{kt}$	$\dfrac{kxy^3}{6}+\dfrac{x^2}{2k}\log x-\dfrac{3x^2}{4k}$		$x=\mathrm{e}^{kt}$

Table 3. (Continued.)

Label	$F(y,t)$	$f(x,y)$	Equation	Link (t,x,y)
Dub2′	$\frac{t^2\log y}{2k}$	$\frac{kxy^3}{6}+\frac{x^2}{2k}\log x-\frac{3x^2}{4k}$		$x=\frac{t}{ky}$
icosa H3	$\frac{ky^3t^2}{6}+\frac{k^2y^2t^5}{20}+\frac{k^4t^{11}}{3960}$	$\frac{k^2y^4t^3}{6}+\frac{7k^3y^3t^6}{30}$ $+\frac{4k^4y^2t^9}{45}+\frac{ky^5}{60}$		$x=ky^2t+\frac{k^2yt^4}{2}$
icosa′	(36)	$\frac{4k^4x^2T^9}{45}+\frac{7k^3x^3T^6}{30}$ $+\frac{k^2x^4T^3}{6}+\frac{kx^5}{60}$		$y=kx^2T+\frac{k^2xT^4}{2}$ $t=\frac{kx^3}{3}+k^2x^2T^3+\frac{k^4T^9}{36}$

3.6. Reduction with the generator $a\mathbf{v}_2+b\mathbf{v}_3$ or $a\mathbf{v}_1+b\mathbf{v}_4$

They lead to the same autonomous ODE,

$$\begin{cases} z=bx-a\log y, \quad f=y^{3/2}\varphi(z), \quad \text{or} \quad z=ay-b\log x, \quad f=x^{3/2}\varphi(z), \quad ab\neq 0, \\ [16a^2\varphi''-8a\varphi'+3\varphi]\varphi'''-24a\varphi''^2+9\varphi'\varphi''+8b^{-3}=0. \end{cases} \tag{33}$$

We could not find a particular solution for this ODE.

3.7. Reduction with the generator $-a\mathbf{v}_1+b\mathbf{v}_2+c\mathbf{v}_5+d\mathbf{v}_6+e\mathbf{v}_7$

The reduced ODE is autonomous and linear [4, p 44, example 1],

$$\begin{cases} z=bx+ay, \qquad f=\varphi(z)+c_3x^3+c_2x^2y+c_1xy^2+c_0y^3, \qquad ab\neq 0, \\ c=-2ac_2+2bc_1, \qquad d=-3ac_3+bc_2, \qquad e=-ac_1+3bc_0, \\ 2(3a^3c_3-a^2bc_2-ab^2c_1+3b^3c_0)\varphi'''+36c_0c_3-4c_1c_2-1=0 \end{cases} \tag{34}$$

and the solution $f(x,y)$ (always defined up to an arbitrary polynomial of degree two in (x,y)) is identical to that defined by equation (21), i.e. the third degree polynomial depending on three arbitrary independent constants,

$$f(x,y)=\alpha x^3+3\beta x^2y+3\gamma xy^2+\delta y^3, \qquad 36(\alpha\delta-\beta\gamma)-1=0. \tag{35}$$

4. Summary of solutions

The explicit solutions to (2) are summarized in table 3. This table does not include the reductions for which no solution could be found. The too long expression for the 'icosa′' solution is

$$\begin{cases} F(y,t)=\dfrac{k^2x^6T}{4}+\dfrac{29k^3x^5T^4}{24}+\dfrac{29k^4x^4T^7}{30}+\dfrac{k^5x^3T^{10}}{10}+\dfrac{3k^6x^2T^{13}}{80}+\dfrac{k^8T^{19}}{3040}, \\ f(x,y)=\dfrac{4k^4x^2T^9}{45}+\dfrac{7k^3x^3T^6}{30}+\dfrac{k^2x^4T^3}{6}+\dfrac{kx^5}{60}, \\ y=kx^2T+\dfrac{k^2xT^4}{2}, \qquad t=\dfrac{kx^3}{3}+k^2x^2T^3+\dfrac{k^4T^9}{36}. \end{cases} \tag{36}$$

5. Conclusion

Finding additional solutions to the obtained reductions could generate algebraic solutions of the sixth Painlevé equation P6 [2], in which the four monodromy exponents of P6 could depend on one arbitrary constant, as in some particular cases (tetrahedron and octahedron solutions) found by Kitaev [7]. In particular, the two solutions labelled N1 and N1′ in table 3 obey the quasi-homogeneity condition (WDVV3) recalled in the introduction. This question is currently under investigation.

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