Mathematical Models and Methods in Applied Sciences
Vol. 14, No. 12 (2004) 1819–1848

REGULARIZATION BY MONOTONE PERTURBATIONS OF THE HYDROSTATIC APPROXIMATION OF NAVIER–STOKES EQUATIONS

FRANCISCO ORTEGÓN GALLEGO

Departamento de Matemáticas, Universidad de Cádiz,
CASEM, Campus del Río San Pedro, 11510 Puerto Real, Cádiz, Spain
francisco.ortegon@uca.es

Received 29 January 2004
Revised 18 May 2004
Communicated by C. Conca

Due to the lack of regularity of the solutions to the hydrostatic approximation of Navier–Stokes equations, an energy identity cannot be deduced. By including certain nonlinear perturbations to the hydrostatic approximation equations, the solutions to the perturbed problem are smooth enough so that they satisfy the corresponding energy identity. The perturbations considered in this paper are of the monotone class. Three kinds of problems are then studied. To do that, we introduce a functional setting and show in every case that the set of smooth functions with compact support is dense in the space where we search for solutions. When the perturbations are small enough in a certain sense, the solutions of the perturbed problem are close to those of the original one. As a result, this gives a new proof of the existence of solutions to the hydrostatic approximation of Navier–Stokes equations. Finally, this regularization technique has been applied to the analysis of a one-equation hydrostatic turbulence model.

Keywords: Nonlinear elliptic equations; monotone perturbations; hydrostatic approximation; Navier–Stokes equations; anisotropic estimates; distributions; Sobolev spaces.

AMS Subject Classification: 35J60, 35Q30, 35Q35, 47H05, 76D03

1. Introduction

The hydrostatic approximation of Navier–Stokes equations is a general model used in oceanography for the description of the circulation of water in oceans and lakes. In this model, the vertical dimension of the domain (maximum depth of a large portion of the ocean or lake) is very small compared to its horizontal dimensions. Taking into account only the essential unknowns, namely, the horizontal velocity field $u: \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^2$ and the surface pressure $p_s: \omega \subset \mathbb{R}^2 \mapsto \mathbb{R}$, the model, at climatic time scales, becomes[1,2,5,7,11,13,15,16]

$$\begin{cases} (u\cdot\nabla)u + W(u)\dfrac{\partial u}{\partial z} - \nu_1\Delta u - \nu_2\dfrac{\partial^2 u}{\partial z^2} + \gamma u^\perp + \nabla p_{\mathrm{s}} = f & \text{in } \Omega\,, \\ \nabla\cdot\left(\displaystyle\int_{-D(x,y)}^{0} u(x,y,z)\,\mathrm{d}z\right) = 0 & \text{in } \omega\,, \\ u = 0 \quad \text{on } \Gamma_{\mathrm{b}}\,, \quad \nu_2\dfrac{\partial u}{\partial z} = g_{\mathrm{s}} & \text{on } \Gamma_{\mathrm{s}}\,, \end{cases} \tag{1.1}$$

where Ω stands for the thin domain occupied with water. It is assumed that the Lipschitz-continuous set $\Omega \subset \mathbb{R}^3$ may be described through a positive function D, the depth, defined in the set $\bar{\omega} \subset \mathbb{R}^2$, in the following way

$$\Omega = \{(x,y,z) \in \mathbb{R}^3/(x,y) \in \omega,\ -D(x,y) < z < 0\}\,, \tag{1.2}$$

and, in turn, ω is a connected, bounded and open set. The depth is assumed to be strictly positive in ω (but may vanish on $\partial\omega$). All the differential operators ∇u, $\nabla\cdot v$ and Δu refer to the usual gradient, divergence and Laplacian 2D-differential operators with respect to the (x,y)-variables. The function $W(u)$ is defined as

$$W(u) = \int_z^0 \nabla\cdot u(x,y,\zeta)\,d\zeta\,. \tag{1.3}$$

The constants $\nu_1 > 0$ and $\nu_2 > 0$ are the horizontal and vertical viscosity coefficients, respectively (in practice, we have $\nu_1 \ll \nu_2$). Also, $u^\perp = (u_2, -u_1)^{\mathrm{T}}$ and so $\gamma u^\perp$ stands for the Coriolis acceleration term, γ being a function depending upon the angular velocity of the earth and the latitude. The boundary of Ω is split into two parts, namely

$$\partial\Omega = \Gamma_{\mathrm{s}} \cup \Gamma_{\mathrm{b}}\,, \quad \Gamma_{\mathrm{s}} = \omega \times \{0\}\,, \quad \Gamma_{\mathrm{b}} = \partial\Omega\backslash\Gamma_{\mathrm{s}}\,,$$

so that Γ_{s} is the sea surface, whereas Γ_{b} stands for the bottom basin together with (possible) sidewalls or taluses. The right-hand side f is a forcing term taking into account the effects of salinity, density or temperature, which are considered here decoupled from the governing equations of the flow (1.1). Finally, g_{s} is the wind stress.

An equivalent formulation of (1.1) has been studied by Besson and Laydi[2] (Azérad and Guillén[1] have analyzed the evolution case). This formulation is obtained as the singular limit of the anisotropic Navier–Stokes equations under the assumption

$$\frac{\text{vertical diameter of } \Omega}{\text{horizontal diameter of } \Omega} \to 0\,.$$

Chacón and Guillén[7] obtained an existence result for problem (1.1). In this reference, the depth function D may vanish on $\partial\omega$, and it is shown, for $f \in H^{-1}(\Omega)^2$ and $g_{\mathrm{s}} \in H^{-1/2}(\Gamma_{\mathrm{s}})$, the existence of a solution (u, p_{s}) such that $u \in H^1_{\mathrm{b}}(\Omega)$ (see (1.5) below) and $p_{\mathrm{s}} \in L^{3/2}(\Omega)$. Problem (1.1) is also considered in former works,[11,13] but these authors assume the restrictive hypothesis $\operatorname{ess\,inf}_{(x,y)\in\partial\omega} D(x,y) > 0$, that is, Ω has a sidewall along $\partial\omega$. In this case, it can be shown that the surface pressure $p_{\rm s}$ also belongs to the space $L^2(\omega)$.

There are some differences between problem (1.1) and the Navier–Stokes equations which render the theoretical analysis of (1.1) particularly difficult. The main difficulty is the lack of a differential equation for the vertical component of the velocity field. Indeed, this is due to the asymptotic analysis leading to the hydrostatic approximation,[1] and $W(u)$, as defined in (1.3), is in fact the vertical velocity. As a result, the vertical convection term $W(u)\frac{\partial u}{\partial z}$ is less regular than the horizontal convection term $(u\cdot\nabla)u$. For instance, if we search for solutions in the usual Sobolev space $u\in H^1(\Omega)^2$, then, at first sight, $W(u)\in L^2(\Omega)$ and $W(u)\frac{\partial u}{\partial z}\in L^1(\Omega)^2$. This implies that test functions in the variational formulation of (1.1) should be in $L^\infty(\Omega)^2$, and in general, the solution u itself could not be taken as a test function. Consequently, the energy identity

$$\nu_1\int_\Omega|\nabla u|^2+\nu_2\int_\Omega\left|\frac{\partial u}{\partial z}\right|^2=\langle f,u\rangle+\langle g,u\rangle \tag{1.4}$$

cannot be deduced.

Fortunately, $W(u)$ is more regular than just $L^2(\Omega)$. Indeed, if $u\in H^1(\Omega)^2$, $W(u)$ belongs to the Hilbert space $H(\partial z)$ given as

$$H(\partial z)=\left\{v\in L^2(\Omega):\frac{\partial v}{\partial z}\in L^2(\Omega)\right\};$$

it is very easy to check that the elements of $H(\partial z)$ bear a trace $v\cdot n_3|_{\partial\Omega}$ as an element of $H^{-1/2}(\partial\Omega)$ ($n=n(\mathbf{x})=(n_1,n_2,n_3)^{\rm T}$ is the outward, unitary and normal vector to $\partial\Omega$ in $x\in\partial\Omega$). On the other hand, for $1<q<+\infty$, we introduce the spaces

$$\begin{cases} W^{1,q}_{\rm b}(\Omega)=\{v\in W^{1,q}(\Omega)/v=0\text{ on }\Gamma_{\rm b}\}\,,\\ W^{-1,q'}_{\rm b}(\Omega)=W^{1,q}_{\rm b}(\Omega)'=\text{dual space of }W^{1,q}_{\rm b}(\Omega)\,,\quad 1/q+1/q'=1\,,\\ H^1_{\rm b}(\Omega)=W^{1,2}_{\rm b}(\Omega)\,,\quad H^{-1}_{\rm b}(\Omega)=H^1_{\rm b}(\Omega)'\,. \end{cases} \tag{1.5}$$

It is straightforward to show that if $u\in H^1(\Omega)^2$, then $W(u)\cdot n_3|_{\Gamma_{\rm s}}=W(u)|_{\Gamma_{\rm s}}=0$ and, if furthermore $u\in H^1_{\rm b}(\Omega)^2$, then

$$W(u)\frac{\partial u}{\partial z}\in W^{-1,3/2}_{\rm b}(\Omega)^2\,,$$

and for all $v\in W^{1,3}_{\rm b}(\Omega)^2$ one has

$$\left\langle W(u)\frac{\partial u}{\partial z},v\right\rangle_{W^{-1,3/2}_{\rm b}(\Omega)^2,W^{1,3}_{\rm b}(\Omega)^2}=\int_\Omega(\nabla\cdot u)uv-\int_\Omega W(u)\frac{\partial v}{\partial z}u\,.$$

Going a step further in the analysis of this term, the following "sharp" regularity[9] can be shown: if $u\in H^1_{\rm b}(\Omega)^2$, then

$$W(u)\frac{\partial u}{\partial z}\in W^{-1,q'}_{\rm b}(\Omega)^2\,,\quad\text{for all }q'<2\,,$$

and for all $v \in W_{\mathrm{b}}^{1,q}(\Omega)^2$, with $q > 2$ one has

$$\left\langle W(u)\frac{\partial u}{\partial z}, v\right\rangle_{W_{\mathrm{b}}^{-1,q'}(\Omega)^2, W_{\mathrm{b}}^{1,q}(\Omega)^2} = \int_\Omega (\nabla\cdot u)uv - \int_\Omega W(u)\frac{\partial v}{\partial z}u\,,$$

and there exists a constant $C = C(q,\Omega)$ such that [see (2.9)]

$$\left\| W(u)\frac{\partial u}{\partial z}\right\|_{W_{\mathrm{b}}^{-1,q'}(\Omega)^2} \le C\|u\|^2_{H^1_{\mathrm{b}}(\Omega)^2}\,. \tag{1.6}$$

These properties are deduced from the so-called anisotropic estimates[2,9] for u and $W(u)$ which give in particular $W(u)\frac{\partial v}{\partial z}u \in L^1(\Omega)$. In Sec. 2.4, we recall some of these estimates and deduce new ones adapted to our functional setting.

Since in general, the solution $u \notin W_{\mathrm{b}}^{1,q}(\Omega)^2$, for any $q > 2$, this extra regularity for the vertical convection term is not enough to use the solution u as a test function in the variational formulation of problem (1.1), and again it is not possible to deduce the energy identity (1.4) for this solution.

We insist on including the energy identity due to the interesting consequences that can be derived from its applications. For instance, it may serve to deduce certain compactness properties for bounded sequences (u_j) in $H^1(\Omega)^2$ of the velocity field. In turn, this compactness property may be used to show the existence of solutions to certain turbulence models[6] consisting of a convenient modification of (1.1) coupled with new transport-diffusion equations (turbulent kinetic energy, density, salinity, etc.). A simple example of this situation is described in Sec. 7.

In this paper a different approach is developed in order to study (1.1). We introduce a suitable monotone perturbation in the differential equations of (1.1) in order to regularize the horizontal velocity field u, yielding an energy identity. In order to set the perturbed problem, we take $\varepsilon > 0$, $q > 2$ and define the distance function $d_{\mathrm{b}} \in W^{1,\infty}(\Omega)$ as $d_{\mathrm{b}}(\mathbf{x}) = \mathrm{dist}(\mathbf{x},\Gamma_{\mathrm{b}})$, $\mathbf{x} = (x,y,z) \in \Omega$. The perturbed problem is given as

$$\begin{cases} (u^\varepsilon\cdot\nabla)u^\varepsilon + W(u^\varepsilon)\dfrac{\partial u^\varepsilon}{\partial z} - \nu_1\Delta u^\varepsilon - \nu_2\dfrac{\partial^2 u^\varepsilon}{\partial z^2} + \gamma u^{\varepsilon\perp} \\[2ex] \qquad + \nabla p_{\mathrm{s}}^\varepsilon - \varepsilon\nabla\left[|\nabla\cdot u^\varepsilon|^{q-2}\nabla\cdot u^\varepsilon\right] + \varepsilon\dfrac{|u^\varepsilon|^{q-2}u^\varepsilon}{d_{\mathrm{b}}^q} = f \quad \text{in } \Omega\,, \\[2ex] \qquad\qquad \nabla\cdot\left(\displaystyle\int_{-D(x,y)}^0 u^\varepsilon(x,y,z)\,dz\right) = 0 \quad \text{in } \omega\,, \\[2ex] \qquad\qquad u^\varepsilon = 0 \quad \text{on } \Gamma_{\mathrm{b}}\,, \quad \nu_2\dfrac{\partial u^\varepsilon}{\partial z} = g_{\mathrm{s}} \quad \text{on } \Gamma_{\mathrm{s}}\,, \end{cases} \tag{1.7}$$

and we show the existence of a solution $(u^\varepsilon, p_{\mathrm{s}}^\varepsilon)$ such that $u^\varepsilon \in H^1_{\mathrm{b}}(\Omega)^2$, $p_{\mathrm{s}}^\varepsilon \in L_D^{q'}(\omega) = \{v\colon \omega \mapsto \mathbb{R}$ measurable: $\int_\omega D(x,y)|v|^{q'} < \infty\}$, with the extra regularity $\nabla\cdot u^\varepsilon, u^\varepsilon/d_{\mathrm{b}} \in L^q(\Omega)$, and the corresponding energy identity holds

$$\nu_1\int_\Omega|\nabla u^\varepsilon|^2 + \nu_2\int_\Omega\left|\frac{\partial u^\varepsilon}{\partial z}\right|^2 + \varepsilon\int_\Omega|\nabla\cdot u^\varepsilon|^q + \varepsilon\int_\Omega\left|\frac{u^\varepsilon}{d_{\mathrm{b}}}\right|^q = \langle f, u^\varepsilon\rangle + \langle g, u^\varepsilon\rangle\,. \tag{1.8}$$

Moreover, when $\varepsilon \to 0^+$, there exist $u \in H^1_{\rm b}(\Omega)^2$ and $p_{\rm s} \in L^r_D(\omega)$, for all $r < 2$, such that, modulo a subsequence, $u^\varepsilon \to u$ in $H^1(\Omega)^2$-weakly, $p^\varepsilon_{\rm s} \to p_{\rm s}$ in $L^{q'}_D(\omega)$-weakly and $(u, p_{\rm s})$ is a solution to (1.1). In particular, this means that the perturbed problem (1.7) is a regularized approximation to the initial problem (1.1).

The paper is organized as follows. Section 2 is devoted to the introduction of some notations and hypotheses. In Sec. 3, we study a simpler version of (1.7) given in (3.1) below: we drop out the constraint about the divergence in ω so no pressure gradient will appear in this problem. We just consider homogeneous Dirichlet boundary conditions and more general monotone perturbations. The analysis of problem (3.1) relies on some known properties of monotone operators and on the density of smooth functions with compact support in the space where we search for solutions.

In Sec. 4, we take into account the divergence constraint in ω and keep the homogenous Dirichlet boundary conditions: this is problem (4.1) below. In order to solve this problem, a de Rham-like lemma is needed.[14] It is then deduced that the term $\nabla p_{\rm s}$ appears as the Lagrange multiplier related to the divergence constraint in ω.

The distinction between the boundaries $\Gamma_{\rm s}$ and $\Gamma_{\rm b}$ is taken into account from Sec. 5 onwards. Indeed, in this section we study the existence of a solution to the problem that contains (1.7) as a particular case. A different functional setting is then introduced, and new density properties are shown.

Section 6 deals with the behavior of the solutions $(u^\varepsilon, p^\varepsilon_{\rm s})$ when $\varepsilon \to 0^+$. It is then deduced that, after extracting a convenient subsequence, $(u^\varepsilon, p^\varepsilon_{\rm s})$ converges weakly to some $(u, p_{\rm s})$, in some suitable Banach spaces, and $(u, p_{\rm s})$ is a solution to the original problem (1.1). In particular, this yields another proof, with a completely different approach, of the existence of a solution to the hydrostatic approximation of Navier–Stokes equations, not based on a mixed formulation of these equations,[7] nor on a transport truncation technique.[11]

The inclusion of monotone perturbations in this kind of problems leads to an energy identity for any solution; the key point is that, in the corresponding variational formulation, the solution itself may be taken as a test function, in spite of working in a non-Hilbert setting.

In the last section, we develop an application of the procedure described in the preceding sections; indeed, we apply this technique to a convenient monotone perturbation of a one equation hydrostatic turbulence model. An existence theorem is then shown for this modified turbulence model.

2. Notations and General Assumptions on Data

Throughout this paper, the following notations and hypotheses on data will be assumed.

2.1. *The domain* Ω

Let $\omega \subset \mathbb{R}^2$ be a bounded, connected and open set such that

$$\omega = \mathrm{int}\overline{\bigcup_{j=1}^{J} \omega_i}\,, \quad J \geq 1,\ \omega_i \cap \omega_j = \emptyset\,, \quad \text{for all } i, j,\ 1 \leq i < j \leq J\,,$$

where every ω_j, $j = 1, \dots, J$, is an open and Lipschitz-continuous set in $\mathbb{R}^2$. Then, the domain $\Omega \subset \mathbb{R}^3$ is defined as in (1.2) through a depth function $D: \omega \mapsto \mathbb{R}$, $D > 0$ in ω in such a way that Ω is Lipschitz-continuous. For instance, we may assume that D fulfills the following conditions:

(1) $D|_{\omega_j} \in W^{1,\infty}(\omega_j)$, $j = 1, \dots, J$;
(2) $D(x, y) > 0$, for all $(x, y) \in \bar{\omega}_j \backslash \partial\omega$, $j = 1, \dots, J$;
(3) For $(x, y) \in \partial\omega_i \cap \partial\omega_j \neq \emptyset$, $D(x, y)$ is defined as

$$D(x, y) = \min\{D_i(x, y), D_j(x, y)\}\,;$$

where

$$D_k(x, y) = \lim_{m\to\infty} D(x_m, y_m)\,, \quad (x_m, y_m) \in \omega_k\,, \quad (x_m, y_m) \to (x, y),\ k = i, j\,;$$

(4) Let $E_j = \{(x, y) \in \partial\omega \cap \partial\omega_j / D(x, y) = 0\}$, $1 \leq j \leq J$, then

$$\min_{1\leq j\leq J}\ \operatorname*{ess\,inf}_{(x,y)\in E_j} |\nabla D(x, y)| > 0\,;$$

Also, since the depth function D may have finite jumps along a common boundary $\partial\omega_i \cap \partial\omega_j \neq \emptyset$, the domain Ω may have a talus (inner vertical slope). On one hand, the presence of sidewalls on $\partial\omega$ is connected with a condition of the kind $D(x, y) > 0$ on points $(x, y) \in \partial\omega$. On the other hand, the depth may vanish along a portion of $\partial\omega$ (this is the case of a beach); in this case, we just assume condition (4) in order to assure the Lipschitz-continuous regularity of Ω.

2.2. *The differential operators*

As it has already been stated, the differential operators ∇, Δ, etc., refer to the usual gradient, Laplacian, etc., respectively, in the (x, y)-variables. Thus, for a function $v: \Omega \mapsto \mathbb{R}$ (or $v: \omega \mapsto \mathbb{R}$), we have

$$\nabla v = \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right)^{\mathrm{T}}\,, \quad \Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\,.$$

If $u: \Omega \mapsto \mathbb{R}^2$, then ∇u and Δu apply the former definitions to the two components of u; thus, ∇u is a 2×2 matrix, and Δu is a vector in $\mathbb{R}^2$. Also, for $u, v: \Omega \mapsto \mathbb{R}^2$, the horizontal convection term $(u \cdot \nabla)v$ is given by

$$(u \cdot \nabla)v = u_1 \frac{\partial v}{\partial x} + u_2 \frac{\partial v}{\partial y}\,, \quad u = (u_1, u_2)^{\mathrm{T}}\,.$$

Remember that the vertical transport function $W(u): \Omega \mapsto \mathbb{R}$ is defined in (1.3). It will be interesting to introduce the function $M(u): \omega \mapsto \mathbb{R}^2$ given as

$$M(u) = \int_{-D(x,y)}^{0} u(x, y, z)\, dz\,. \tag{2.1}$$

2.3. *The functional setting*

We have already introduced some functional spaces in (1.5). The method developed in the following sections make use of some other Banach spaces. Let $1 < q < \infty$ and q' the conjugate exponent of q, that is $1/q + 1/q' = 1$. We put

$$\begin{aligned}
\mathcal{D}(\Omega) &= \left\{\phi \in \mathcal{C}^\infty(\Omega) : \text{supp}\ \phi \text{ is compact in } \Omega\right\}\,,\\
\mathcal{D}'(\Omega) &= \text{space of distributions in } \Omega\,,\\
W^{1,q}(\Omega) &= \left\{\phi \in L^q(\Omega) : \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z} \in L^q(\Omega)\right\},\\
&\qquad \text{a Banach space with norm } \|\phi\|_{W^{1,q}(\Omega)}\\
&= \left(\|\phi\|^q_{L^q(\Omega)} + \left\|\frac{\partial\phi}{\partial x}\right\|^q_{L^q(\Omega)} + \left\|\frac{\partial\phi}{\partial y}\right\|^q_{L^q(\Omega)} + \left\|\frac{\partial\phi}{\partial z}\right\|^q_{L^q(\Omega)}\right)^{1/q}\,,\\
W^{1,q}_0(\Omega) &= \overline{\mathcal{D}(\Omega)}^{W^{1,q}(\Omega)},\ \text{a Banach space with norm } \|\phi\|_{W^{1,q}_0(\Omega)}\\
&= \left(\left\|\frac{\partial\phi}{\partial x}\right\|^q_{L^q(\Omega)} + \left\|\frac{\partial\phi}{\partial y}\right\|^q_{L^q(\Omega)} + \left\|\frac{\partial\phi}{\partial z}\right\|^q_{L^q(\Omega)}\right)^{1/q}\,,\\
W^{-1,q'}(\Omega) &= \text{dual of } W^{1,q}_0(\Omega)\,,\\
H^1_0(\Omega) &= W^{1,2}_0(\Omega)\,,\quad H^{-1}(\Omega) = W^{-1,2}(\Omega)\,,\\
\mathcal{D}_{\text{b}}(\Omega) &= \{\phi \in \mathcal{C}^\infty(\bar\Omega) : \text{supp}\ \phi \text{ is compact in } \bar\Omega\backslash\Gamma_{\text{b}}\}\,,\\
L^{q'}_D(\omega) &= \left\{v\colon \omega \mapsto \mathbb{R} \text{ measurable} : \int_\omega D|v|^{q'} < \infty\right\}\,,\\
L^{q'}_{D,0}(\omega) &= \left\{v \in L^{q'}_D(\omega) : \int_\omega Dv = 0\right\}.
\end{aligned}$$

Since Ω is Lipschitz-continuous, it is well known that $W^{1,q}_{\text{b}}(\Omega) = \overline{\mathcal{D}_{\text{b}}(\Omega)}^{W^{1,q}(\Omega)}$, (see (1.5) above). We also consider the function $d \in W^{1,\infty}(\Omega)$, the distance to the boundary $\partial\Omega$:

$$d(\mathbf{x}) = \text{dist}(\mathbf{x}, \partial\Omega)\,; \tag{2.2}$$

and then introduce the following spaces:

$$\left\{\begin{aligned}
X^q_0(\Omega) &= \{v \in H^1_0(\Omega)^2 : \nabla\cdot v \in L^q(\Omega),\ d^{-1}v \in L^q(\Omega)^2\}\,,\\
&\qquad \text{a Banach space with norm } \|v\|_{X^q_0(\Omega)}\\
&= (\|v\|^2_{H^1_0(\Omega)^2} + \|\nabla\cdot v\|^2_{L^q(\Omega)} + \|d^{-1}v\|^2_{L^q(\Omega)^2})^{1/2}\,,\\
X^q_{\text{b}}(\Omega) &= \{v \in H^1_{\text{b}}(\Omega)^2 : \nabla\cdot v \in L^q(\Omega),\ d_{\text{b}}^{-1}v \in L^q(\Omega)^2\}\,,\\
V^q_0(\Omega) &= \{v \in X^q_0(\Omega) : \nabla\cdot M(v) = 0 \text{ in } \omega\}\,,\\
V^q_{\text{b}}(\Omega) &= \{v \in X^q_{\text{b}}(\Omega) : \nabla\cdot M(v) = 0 \text{ in } \omega\}\,,\\
\mathcal{V}(\Omega) &= \{\varphi \in \mathcal{D}(\Omega)^2 : \nabla\cdot M(\varphi) = 0 \text{ in } \omega\}\,,\\
\mathcal{V}_{\text{b}}(\Omega) &= \{\varphi \in \mathcal{D}_{\text{b}}(\Omega)^2 : \nabla\cdot M(\varphi) = 0 \text{ in } \omega\}\,.
\end{aligned}\right. \tag{2.3}$$

It is very easy to check that $X_0^q(\Omega)' = H^{-1}(\Omega)^2 + Y + Z$, where $Y = \{\nabla g\colon g \in L^{q'}(\Omega)\}$ and $Z = \{d^{-1}v\colon v \in L^{q'}(\Omega)^2\}$; thus, $X_0^q(\Omega)$ is a separable and reflexive Banach space.

Owing to Poincaré's inequality, $X_{\mathrm{b}}^q(\Omega)$, $V_0^q(\Omega)$ and $V_{\mathrm{b}}^q(\Omega)$ are also Banach spaces with the same norm $\|\cdot\|_{X_0^q(\Omega)}$. Also, from Hardy's inequality (if $\phi \in H_0^1(\Omega)$ then $d^{-1}\phi \in L^2(\Omega)$ and there exists a constant $C > 0$ such that $\|d^{-1}\phi\|_{L^2(\Omega)} \leq C\|\phi\|_{H_0^1(\Omega)}$, for all $\phi \in H_0^1(\Omega)$) we have $X_0^q(\Omega) = H_0^1(\Omega)^2$ whenever $q \leq 2$. In the same way, if $q \leq 2$, then $X_{\mathrm{b}}^q(\Omega) = H_{\mathrm{b}}^1(\Omega)^2$ since there is a Hardy inequality related to d_{b}, namely, $\|d_{\mathrm{b}}^{-1}\phi\|_{L^2(\Omega)} \leq C\|\phi\|_{H_{\mathrm{b}}^1(\Omega)}$ for all $\phi \in H_{\mathrm{b}}^1(\Omega)$. This means that we are defining something new only for $q > 2$. From this point on we will apply the following assumption

$$q > 2\,. \tag{2.4}$$

The value of q also appears in (2.7) below. Note that the linear operator W given in (1.3) is continuous from $X_0^q(\Omega)$ [or $X_{\mathrm{b}}^q(\Omega)$] to $L^q(\Omega)$. Indeed, if $u \in X_0^q(\Omega)$, then

$$\begin{aligned} |W(u)| &\leq \int_{-D(x,y)}^0 |\nabla \cdot u(x,y,\zeta)|\,d\zeta \\ &\leq \|D\|_{L^\infty(\omega)}^{1/q'} \left(\int_{-D(x,y)}^0 |\nabla \cdot u(x,y,\zeta)|^q\,d\zeta\right)^{1/q}, \end{aligned}$$

whence

$$\int_{-D(x,y)}^0 |W(u)|^q \leq \|D\|_{L^\infty(\omega)}^{1+q/q'} \int_{-D(x,y)}^0 |\nabla \cdot u(x,y,\zeta)|^q\,d\zeta\,,$$

and integrating over ω, it yields

$$\|W(u)\|_{L^q(\Omega)} \leq \|D\|_{L^\infty(\omega)}\|\nabla \cdot u\|_{L^q(\Omega)}\,. \tag{2.5}$$

As far as the coefficient function γ of the Coriolis term is concerned, it will be assumed that

$$\gamma \in L^{3/2}(\Omega)\,; \tag{2.6}$$

note that in this case, if $u \in H_{\mathrm{b}}^1(\Omega)$ then $\gamma u \in L^{6/5}(\Omega) \subset H_{\mathrm{b}}^{-1}(\Omega)$.

Finally, we consider nonlinear functions $\Phi\colon \mathbb{R} \mapsto \mathbb{R}$ (respectively $\Phi\colon \mathbb{R}^2 \mapsto \mathbb{R}^2$) from which the monotone operators are built. We assume the following assumptions for these functions:

$$\begin{cases} \text{(a) } \Phi \in \mathcal{C}^0(\mathbb{R})\,, \quad [\text{respectively } \Phi \in \mathcal{C}^0(\mathbb{R}^2)]\,; \\ \text{(b) there exists a constant } C_0 > 0 \text{ such that} \\ \qquad |\Phi(s)| \leq C_0(|s|^{q-1} + 1)\,, \quad \text{for all } s \in \mathbb{R} \text{ (respectively } s \in \mathbb{R}^2)\,; \\ \text{(c) there exists a constant } C_1 > 0 \text{ such that} \\ \qquad \Phi(s)s \geq C_1(|s|^q - 1)\,, \quad \text{for all } s \in \mathbb{R} \text{ (respectively } s \in \mathbb{R}^2)\,; \\ \text{(d) } (\Phi(s_1) - \Phi(s_2))(s_1 - s_2) \geq 0\,, \quad \text{for all } s_1, s_2 \in \mathbb{R} \text{ (respectively } s_1, s_2 \in \mathbb{R}^2)\,, \end{cases} \tag{2.7}$$

where $q > 2$. An example of a function Φ fulfilling these conditions is $\Phi(s) = |s|^{q-2}s$. Note that if $u \in X_0^q(\Omega)$ [respectively $u \in X_{\mathrm{b}}^q(\Omega)$], then $\Phi(\nabla \cdot u) \in L^{q'}(\Omega)$ and $\Phi(d^{-1}u) \in L^{q'}(\Omega)^2$ [respectively $\Phi(d_{\mathrm{b}}^{-1}u) \in L^{q'}(\Omega)^2$].

2.4. *The anisotropic estimates*

The particular geometry of the domain Ω, as has been introduced in (1.2), allows us to distinguish between the regularity of functions defined in Ω with respect to, on one hand, the (x, y)-variables and, on the other hand, the z-variable, in a separate manner. This regularity distinction has led to the derivation of the so-called anisotropic estimates. In turn, these estimates has been used in order to give a sense to the vertical convection term $W(u)\frac{\partial u}{\partial z}$. For the sake of completeness, we recall some well-known anisotropic estimates, and then derive new ones that are valid in our functional setting.

For every $z \in (-\|D\|_{L^\infty(\omega)}, 0)$, we define the set $S_z = \{(x, y) \in \omega/(x, y, z) \in \Omega\}$. Also, for $\alpha, \beta \in [1, +\infty]$, we introduce the following Banach spaces (here and henceforth, a.e. will stand for "almost everywhere")

$$L_z^\alpha L_{xy}^\beta(\Omega) = \{v \in L^1(\Omega) : v(\cdot\,, z) \in L^\beta(S_z) \text{ for a.e. } z \in (-\|D\|_{L^\infty(\omega)}, 0)$$
$$\text{and } \|v(\cdot\,, z)\|_{L^\beta(S_z)} \in L^\alpha(-\|D\|_{L^\infty(\omega)}, 0)\},$$

endowed with the norm

$$\|v\|_{L_z^\alpha L_{xy}^\beta(\Omega)} = \|\|v(\cdot\,, z)\|_{L^\beta(S_z)}\|_{L^\alpha(-\|D\|_{L^\infty(\omega)}, 0)}\,.$$

The next result holds.[9]

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^3$ like in* (1.2). *If $u \in H^1(\Omega)$ is such that $un_1 = un_2 = 0$ on Γ_{b}, then $u \in L_z^2 L_{xy}^\beta(\Omega)$ for all $\beta < \infty$ and there exists a constant C_β such that $C_\beta \to +\infty$ as $\beta \to +\infty$ and*

$$\|u\|_{L_z^2 L_{xy}^\beta(\Omega)} \le C_\beta \|u\|_{L^2(\Omega)}^{1-\theta(\beta)} \|u\|_{H^1(\Omega)}^{\theta(\beta)} \le C_\beta \|u\|_{H^1(\Omega)}\,, \tag{2.8}$$

where θ is an increasing function and $0 < \theta(\beta) < 1$.

If $h \in L^2(\Omega)$ and $W(x, y, z) = \int_z^0 h(x, y, \zeta)\, d\zeta$. Then, $W \in L_z^\infty L_{xy}^2(\Omega)$ and

$$\|W\|_{L_z^\infty L_{xy}^2(\Omega)} \le \|D\|_{L^\infty(\omega)}^{1/2} \|h\|_{L^2(\Omega)}\,.$$

Let $u \in H_{\mathrm{b}}^1(\Omega)^2$ and take $h = \nabla \cdot u$, then Lemma 2.1 claims that $W(u) \in L_z^\infty L_{xy}^2(\Omega)$. This regularity, together with the estimate (2.8) applied to both components of u, yield the regularity of the vertical convection term. Indeed, if $v \in W_{\mathrm{b}}^{1,q}(\Omega)^2$, $q > 2$, it is not difficult to see that

$$\left|\int_\Omega W(u)\frac{\partial v}{\partial z}u\right| \le \|W(u)\|_{L_z^\infty L_{xy}^2(\Omega)} \left\|\frac{\partial v}{\partial z}\right\|_{L^q(\Omega)^2} \|u\|_{L_z^2 L_{xy}^{r(q)}(\Omega)^2}\,, \quad r(q) = \frac{2q}{q-2}$$

and thus

$$\left|\int_\Omega W(u)\frac{\partial v}{\partial z}u\right| \le C_{r(q)}\|D\|^{1/2}_{L^\infty(\omega)}\|\nabla\cdot u\|_{L^2(\Omega)}\|u\|_{H^1(\Omega)^2}\left\|\frac{\partial v}{\partial z}\right\|_{L^q(\Omega)^2}, \tag{2.9}$$

which implies that $W(u)\frac{\partial u}{\partial z} \in W_{\rm b}^{-1,q'}(\Omega)^2$.

The problems studied below involve spaces like $X_{\rm b}^q(\Omega)$ (or a subspace); since $W_{\rm b}^{1,q}(\Omega)^2 \subset X_{\rm b}^q(\Omega)$, the estimate (2.9) does not guarantee that $W(u)\frac{\partial u}{\partial z} \in X_{\rm b}^q(\Omega)'$ for $u \in X_{\rm b}^q(\Omega)$. The next result will be useful in order to determine that assertion. It is obtained through the derivation of new anisotropic estimates.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^3$ like in (1.2) and $q > 2$. Let $h \in L^q(\Omega)$ and $W(x,y,z) = \int_z^0 h(x,y,\zeta)\,d\zeta$. Then $W \in L_z^\infty L_{xy}^q(\Omega)$ and*

$$\|W\|_{L_z^\infty L_{xy}^q(\Omega)} \le \|D\|^{1/q'}_{L^\infty(\omega)}\|h\|_{L^q(\Omega)}\,.$$

Moreover, if $u, v \in X_{\rm b}^q(\Omega)$, then

$$\left|\int_\Omega W(u)\frac{\partial v}{\partial z}u\right| \le C_{r(q)}\|D\|^{1/q'}_{L^\infty(\omega)}\|\nabla\cdot u\|_{L^q(\Omega)}\|u\|_{H^1(\Omega)^2}\left\|\frac{\partial v}{\partial z}\right\|_{L^2(\Omega)^2}. \tag{2.10}$$

Proof. We have $|W(x,y,z)| \le \int_{-D(x,y)}^0 |h(x,y,\zeta)|\,d\zeta$, thus, for almost everywhere $z \in (-\|D\|_{L^\infty(\omega)}, 0)$,

$$|W(x,y,z)|^q \le \|D\|^{q/q'}_{L^\infty(\omega)}\int_{-D(x,y)}^0 |h(x,y,\zeta)|^q\,d\zeta\,,$$

integrating over S_z,

$$\begin{aligned}\int_{S_z}|W(x,y,z)|^q\,dxdy &\le \|D\|^{q/q'}_{L^\infty(\omega)}\int_{S_z}\int_{-D(x,y)}^0|h(x,y,\zeta)|^q\,d\zeta dxdy\\ &\le \|D\|^{q/q'}_{L^\infty(\omega)}\int_{\omega}\int_{-D(x,y)}^0|h(x,y,\zeta)|^q\,d\zeta dxdy\\ &= \|D\|^{q/q'}_{L^\infty(\omega)}\int_\Omega|h(x,y,\zeta)|^q\,d\zeta dxdy\,,\end{aligned}$$

thus, for almost everywhere $z \in (-\|D\|_{L^\infty(\omega)}, 0)$,

$$\|W(\cdot\,,\,z)\|_{L^q(S_z)} \le \|D\|^{1/q'}_{L^\infty(\omega)}\|h\|_{L^q(\Omega)}$$

and taking the essential supremum in z, it yields the desired result. □

In order to show (2.10), let $u, v \in X_{\rm b}^q(\Omega)$, then $W(u) \in L_z^\infty L_{xy}^q(\Omega)$, and putting again $r(q) = 2q/(q-2)$, we have

$$\begin{aligned}\left|\int_\Omega W(u)\frac{\partial v}{\partial z}u\right| &\le \int_{-D(x,y)}^0\int_{S_z}\left|W(u)\frac{\partial v}{\partial z}u\right|\\ &\le \int_{-D(x,y)}^0\|W(u)\|_{L^q(S_z)}\left\|\frac{\partial v}{\partial z}\right\|_{L^2(S_z)^2}\|u\|_{L_{xy}^{r(q)}(S_z)^2}\end{aligned}$$

$$\leq \|W(u)\|_{L_z^\infty L_{xy}^q(\Omega)} \left\| \frac{\partial v}{\partial z} \right\|_{L^2(\Omega)^2} \|u\|_{L^2 L_{xy}^{r(q)}(\Omega)^2}$$

$$\leq C_{r(q)} \|D\|_{L^\infty(\omega)}^{1/q'} \|\nabla \cdot u\|_{L^q(\Omega)} \|u\|_{H^1(\Omega)^2} \left\| \frac{\partial v}{\partial z} \right\|_{L^2(\Omega)^2} .$$

3. Analysis of a Simpler Version of the Original Problem

In order to analyze the perturbed problem (1.7), we first study a simpler version in which the divergence constraint $\nabla \cdot M(u) = 0$ in ω is not taken into account. Also, we just consider homogeneous Dirichlet boundary conditions. The problem is the following: To find $u \in X_0^q(\Omega)$ such that

$$\begin{cases} (u \cdot \nabla)u + W(u)\dfrac{\partial u}{\partial z} - \nu_1 \Delta u - \nu_2 \dfrac{\partial^2 u}{\partial z^2} + \gamma u^\perp \\ \qquad - \nabla \Phi_1(\nabla \cdot u) + d^{-1}\Phi_2(d^{-1}u) = f \quad \text{in } \Omega \,, \\ \qquad\qquad\qquad\qquad\qquad\qquad u = 0 \quad \text{on } \partial\Omega \,, \end{cases} \tag{3.1}$$

where $W(u)$ is given in (1.3), $\Phi_1\colon \mathbb{R} \mapsto \mathbb{R}$ and $\Phi_2\colon \mathbb{R}^2 \mapsto \mathbb{R}^2$ verify (2.7), and $f \in X_0^q(\Omega)'$, the dual space of $X_0^q(\Omega)$.

In order to solve (3.1), we introduce the operators A, B: $X_0^q(\Omega) \mapsto X_0^q(\Omega)'$, defined as

$$\begin{cases} Au = -\nu_1 \Delta u - \nu_2 \dfrac{\partial^2 u}{\partial z^2} + \gamma u^\perp - \nabla \Phi_1(\nabla \cdot u) + d^{-1}\Phi_2(d^{-1}u) \,, \\ Bu = (u \cdot \nabla)u + W(u)\dfrac{\partial u}{\partial z} \,, \end{cases} \tag{3.2}$$

that is, A keeps the monotone part of the differential equation of problem (3.1) while B retains the convection terms. Notice that (3.1) now becomes

$$\text{To find } u \in X_0^q(\Omega) \text{ such that } Au + Bu = f \,. \tag{3.3}$$

To begin with, we first establish some interesting properties on A and B.

Lemma 3.1. *The operator A: $X_0^q(\Omega) \mapsto X_0^q(\Omega)'$ is well defined and satisfy the following properties*:

(1) *A is continuous.*
(2) *A is monotone, that is, for all $u, v \in X_0^q(\Omega)$ we have*

$$\langle Au - Av, u - v \rangle \geq 0 \,, \tag{3.4}$$

where $\langle \cdot\,, \cdot \rangle$ stands for the duality product between $X_0^q(\Omega)'$ and $X_0^q(\Omega)$.

(3) *A is coercive, that is*

$$\lim_{\|u\| \to \infty} \frac{\langle Au, u \rangle}{\|u\|} = +\infty \,, \tag{3.5}$$

where $\|u\| = \|u\|_{X_0^q(\Omega)}$.

Proof. Let $u, v \in X_0^q(\Omega)$. According to the definition of A, we have

$$\langle Au, v\rangle = \nu_1 \int_\Omega \nabla u \nabla v + \nu_2 \int_\Omega \frac{\partial u}{\partial z}\frac{\partial v}{\partial z} + \int_\Omega \gamma u^\perp v$$
$$+ \int_\Omega \Phi_1(\nabla\cdot u)\nabla\cdot v + \int_\Omega \Phi_2(d^{-1}u)d^{-1}v\,,$$

whence, using (b) of (2.7), for some constant $C_0' > 0$ we have

$$|\langle Au, v\rangle| \le \nu_1\|\nabla u\|_{L^2}\|\nabla v\|_{L^2} + \nu_2\left\|\frac{\partial u}{\partial z}\right\|_{L^2}\left\|\frac{\partial v}{\partial z}\right\|_{L^2} + \|\gamma\|_{L^{3/2}}\|u\|_{L^6}\|v\|_{L^6}$$
$$+ C_0'(\|\nabla\cdot u\|_{L^q} + 1)\|\nabla\cdot v\|_{L^q} + C_0'(\|d^{-1}u\|_{L^q} + 1)\|d^{-1}v\|_{L^q}$$
$$\le K\|v\|_{X_0^q(\Omega)}$$

where $K = K(\|u\|_{X_0^q(\Omega)}, \nu_1, \nu_2, \|\gamma\|_{L^{3/2}(\Omega)}, C_0', \Omega)$, and we have used $\Omega \subset \mathbb{R}^3$ so that $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$ with continuous embedding. Consequently, $Au \in X_0^q(\Omega)'$. □

In order to show the continuity of A, let $u \in X_0^q(\Omega)$ and $(u_m) \subset X_0^q(\Omega)$ such that $u_m \to u$ in $X_0^q(\Omega)$-strongly. Then,

$$-\nu_1\Delta u_m - \nu_2\frac{\partial^2 u_m}{\partial z^2} + \gamma u_m^\perp \to -\nu_1\Delta u - \nu_2\frac{\partial^2 u}{\partial z^2} + \gamma u^\perp\,,$$
$$\text{in } H^{-1}(\Omega)^2\text{-strongly}.$$

Also, since $\nabla\cdot u_m \to \nabla\cdot u$ in $L^q(\Omega)$-strongly, and $d^{-1}u_m \to d^{-1}u$ in $L^q(\Omega)^2$-strongly, then from conditions (a) and (b) of (2.7), we deduce[10]

$$\Phi_1(\nabla\cdot u_m) \to \Phi_1(\nabla\cdot u) \quad \text{in } L^{q'}(\Omega)\text{-strongly}\,,$$
$$\Phi_2(d^{-1}u_m) \to \Phi_2(d^{-1}u) \quad \text{in } L^{q'}(\Omega)^2\text{-strongly}.$$

These three convergences lead directly to $Au_m \to Au$ in $X_0^q(\Omega)'$-strongly.

To see the monotone character of A, consider $u, v \in X_0^q(\Omega)$, then putting $\nu = \min(\nu_1, \nu_2)$ and using (d) of (2.7), it yields

$$\langle Au - Av, u - v\rangle = \nu_1\int_\Omega |\nabla(u-v)|^2 + \nu_2\int_\Omega \left|\frac{\partial(u-v)}{\partial z}\right|^2$$
$$+ \int_\Omega (\Phi_1(\nabla\cdot u) - \Phi_1(\nabla\cdot v))(\nabla\cdot u - \nabla\cdot v)$$
$$+ \int_\Omega (\Phi_2(d^{-1}\cdot u) - \Phi_2(d^{-1}v))(d^{-1}u - d^{-1}v)$$
$$\ge \nu\|u - v\|^2_{H_0^1(\Omega)} \ge 0\,.$$

Finally, we show the coerciveness of A: let $u \in X_0^q(\Omega)$, then using (2.4) and (c) of (2.7) we have

$$\begin{aligned}\langle Au, u\rangle &= \nu_1 \int_\Omega |\nabla u|^2 + \nu_2 \int_\Omega \left|\frac{\partial u}{\partial z}\right|^2 + \int_\Omega \Phi_1(\nabla\cdot u)\nabla\cdot u + \int_\Omega \Phi_2(d^{-1}\cdot u)d^{-1}u \\ &\geq \nu\|u\|^2_{H_0^1(\Omega)} + C_1\left[\int_\Omega (|\nabla\cdot u|^q - 1) + \int_\Omega (|d^{-1}u|^q - 1)\right] \\ &\geq k(\|u\|^2_{H_0^1(\Omega)} + \|\nabla\cdot u\|^q_{L^q(\Omega)} + \|d^{-1}u\|^q_{L^q(\Omega)}) - C \\ &\geq k(\|u\|^2_{H_0^1(\Omega)} + \|\nabla\cdot u\|^2_{L^q(\Omega)} + \|d^{-1}u\|^2_{L^q(\Omega)} - 2) - C \\ &= k\|u\|^2_{X_0^q(\Omega)} - C', \end{aligned} \tag{3.6}$$

which implies the coerciveness of A. This ends the proof of Lemma 3.1.

As far as the operator B is concerned, the next density result will be needed.

Lemma 3.2. *The space* $\mathcal{D}(\Omega)^2$ *is dense in* $X_0^q(\Omega)$.

Proof. The proof is divided into two steps.

Step 1. Let $u \in X_0^q(\Omega)$ such that supp$u$ is compact in Ω. We consider a sequence of mollifiers $(\rho_\varepsilon) \subset \mathcal{D}(\mathbb{R}^3)$ in the usual way. Thus, for $\varepsilon > 0$ small enough, we have

$$\rho_\varepsilon * u \in \mathcal{D}(\Omega)^2\,, \quad \nabla\cdot(\rho_\varepsilon * u) = \rho_\varepsilon * (\nabla\cdot u)\,,$$

and so, as $\varepsilon \to 0^+$, $\rho_\varepsilon * u \to u$ in $H_0^1(\Omega)^2$-strongly, and $\nabla\cdot(\rho_\varepsilon * u) \to \nabla\cdot u$ in $L^q(\Omega)$-strongly. On the other hand, since supp$u$ is compact, it is straightforward that $d^{-1}\rho_\varepsilon * u \to d^{-1}u \in L^q(\Omega)$. In conclusion, we have $\rho_\varepsilon * u \to u$ in $X_0^q(\Omega)$-strongly.

Step 2. Let u be any function in $X_0^q(\Omega)$. In this step, we show that there exists $(u_m) \subset X_0^q(\Omega)$, supp u_m compact in Ω, such that u_m is arbitrarily close to u in $X_0^q(\Omega)$, for $m \geq 1$ large enough. To do so, we first introduce the function $H \in \mathcal{C}^\infty(\mathbb{R})$ defined as follows

$$H(s) = \begin{cases} 0\,, & \text{if } s \leq 1\,, \\ h(s)\,, & \text{if } 1 < s < 2\,, \\ 1\,, & \text{if } s \geq 2\,, \end{cases} \qquad h(s) = \frac{1 + e^{1/(s-2)}}{1 + e^{1/(s-1)+1/(s-2)}}\,,$$

and put $u_m(\mathbf{x}) = u(\mathbf{x})H(md(\mathbf{x}))$. Since $d \in W^{1,\infty}(\Omega)$ and H is smooth, it is easy to check that $(u_m) \subset X_0^q(\Omega)$ and supp u_m is compact in Ω. On the other hand,

$$\begin{aligned}\nabla u_m(\mathbf{x}) &= \nabla u(\mathbf{x})H(md(\mathbf{x})) + u(\mathbf{x})mH'(md(\mathbf{x}))\nabla d(\mathbf{x}) \\ &= \nabla u(\mathbf{x})H(md(\mathbf{x})) + d^{-1}(\mathbf{x})u(\mathbf{x})md(\mathbf{x})H'(md(\mathbf{x}))\nabla d(\mathbf{x})\,,\end{aligned}$$

with a similar expression for $\frac{\partial u_m}{\partial z}$. Now, since $u \in H_0^1(\Omega)^2$, then (from Hardy's inequality) $d^{-1}u \in L^2(\Omega)^2$. Also $(md(\mathbf{x})H'(md(\mathbf{x})))$ is bounded in $L^\infty(\Omega)$; thus, $u_m \to u$ in $H_0^1(\Omega)^2$-strongly. □

In the same way,

$$\nabla \cdot u_m(\mathbf{x}) = \nabla \cdot u(\mathbf{x}) H(md(\mathbf{x})) + d^{-1}(\mathbf{x}) u(\mathbf{x}) md(\mathbf{x}) H'(md(\mathbf{x})) \nabla \cdot d(\mathbf{x}) \,,$$

and since $\nabla \cdot u \in L^q(\Omega)$ and $d^{-1}u \in L^q(\Omega)^2$ [because $u \in X_0^q(\Omega)$], it yields $\nabla \cdot u_m \to \nabla \cdot u$ in $L^q(\Omega)$-strongly.

Finally, $d^{-1}(\mathbf{x})u_m(\mathbf{x}) = d^{-1}(\mathbf{x})u(\mathbf{x})H(md(\mathbf{x}))$, and again, thanks to $d^{-1}u \in L^q(\Omega)^2$, we obtain $d^{-1}u_m \to d^{-1}u$ in $L^q(\Omega)^2$-strongly.

Thus, we have deduced that $u_m \to u$ in $X_0^q(\Omega)$-strongly.

Remark 3.1. The introduction of the distance d through the function Φ_2 in problem (3.1) is now clear: it has been used in the definition of the space $X_0^q(\Omega)$ and the regularity condition $d^{-1}u \in L^q(\Omega)^2$ has led to the density of smooth functions with compact support in $X_0^q(\Omega)$, as it has been stated in Step 2 of Lemma 3.2.

Corollary 3.1. *The operator* B: $X_0^q(\Omega) \mapsto X_0^q(\Omega)'$ *is well defined and is continuous* (1) *from* $X_0^q(\Omega)$*-strong to* $X_0^q(\Omega)'$*-strong, and* (2) *from* $X_0^q(\Omega)$*-weak to* $X_0^q(\Omega)'$*-weak. Moreover, we have the following representation*

$$\langle Bu, v\rangle = -\int_\Omega (u\cdot\nabla)vu - \int_\Omega W(u)\frac{\partial v}{\partial z}u\,, \quad \textit{for all } u,\ v \in X_0^q(\Omega)\,, \tag{3.7}$$

in particular

$$\langle Bu, u\rangle = 0\,, \quad \textit{for all } u \in X_0^q(\Omega)\,. \tag{3.8}$$

Proof. Let $u, v, w \in X_0^q(\Omega)$, and define the trilinear operator b: $X_0^q(\Omega) \times X_0^q(\Omega) \times X_0^q(\Omega) \mapsto \mathbb{R}$ given as

$$b(u, v, w) = \int_\Omega (u\cdot\nabla)vw + \int_\Omega W(u)\frac{\partial v}{\partial z}w\,;$$

repeating the same steps as in the proof of Lemma 2.2, it is easy to check that there exists a constant $M = M(q,\Omega)$ such that

$$|b(u,v,w)| \le M\|u\|_{X_0^q(\Omega)}\|v\|_{X_0^q(\Omega)}\|w\|_{X_0^q(\Omega)}\,,$$

and consequently b is a continuous operator. The relation $\langle Bu, v\rangle = b(u,u,v)$ yields that B: $X_0^q(\Omega) \mapsto X_0^q(\Omega)'$ is continuous for the strong topologies in both spaces and also

$$\|Bu\|_{X_0^q(\Omega)'} \le M\|u\|^2_{X_0^q(\Omega)}\,. \tag{3.9}$$

If $u, v \in \mathcal{D}(\Omega)^2$, then (3.7) is easily obtained by means of an integration by parts. Owing to the density of $\mathcal{D}(\Omega)^2$ in $X_0^q(\Omega)$ (Lemma 3.2), and the continuity of B, we have that, in fact, (3.7) holds true for all $u, v \in X_0^q(\Omega)$. □

Finally, to see that B is continuous for the weak topologies in both spaces, let $(u_j) \subset X_0^q(\Omega)$ and $u \in X_0^q(\Omega)$ such that $u_j \to u$ in $X_0^q(\Omega)$-weakly. According to (3.9), $(Bu_j) \subset X_0^q(\Omega)'$ is bounded, so that there exists $\chi \in X_0^q(\Omega)'$ and a subsequence, still denoted in the same way, such that $Bu_j \to \chi$ in $X_0^q(\Omega)'$-weakly. Let $v \in \mathcal{D}(\Omega)^2$; on one hand we have $\langle Bu_j, v\rangle \to \langle \chi, v\rangle$; on the other, thanks to (3.7), we obtain

$$\begin{aligned}\langle Bu_j, v\rangle &= -\int_\Omega (u_j\cdot\nabla)vu_j - \int_\Omega W(u_j)\frac{\partial v}{\partial z}u_j\\ &\to -\int_\Omega (u\cdot\nabla)vu - \int_\Omega W(u)\frac{\partial v}{\partial z}u = \langle Bu, v\rangle\,,\end{aligned}$$

thus $\langle Bu, v\rangle = \langle \chi, v\rangle$ for all $v \in \mathcal{D}(\Omega)^2$, and using again the density of this space in $X_0^q(\Omega)$, we deduce that $Bu = \chi$, and consequently, it is the whole sequence Bu_j that converges to Bu.

3.1. *An existence result*

Now, we are ready to state an existence result for problem 3.1.

Theorem 3.1. *Assume* (2.4) *and* (2.7) *for both* Φ_1 *and* Φ_2; *then for every* $f \in X_0^q(\Omega)'$ *there exists a solution* $u \in X_0^q(\Omega)$ *to problem* 3.1. *Moreover, the following energy identity holds*

$$\nu_1\int_\Omega |\nabla u|^2 + \nu_2\int_\Omega \left|\frac{\partial u}{\partial z}\right|^2 + \int_\Omega \Phi_1(\nabla\cdot u)\nabla\cdot u + \int_\Omega \Phi_2(d^{-1}u)d^{-1}u = \langle f, u\rangle\,. \tag{3.10}$$

Proof. We implement a Faedo–Galerkin procedure, then apply a well-known result derived from Brower's fixed point theorem, and finally pass to limit.

Let $\{w_j\}_{j\geq 1} \subset \mathcal{D}(\Omega)^2$ be a complete system in $X_0^q(\Omega)$, that is

(1) $E_j = \langle w_1, \ldots, w_j\rangle$, $\dim E_j = j$ for all $j \geq 1$;
(2) $X_0^q(\Omega) = \overline{\bigcup_{j\geq 1} E_j}^{X_0^q(\Omega)}$. □

For every $j \geq 1$, we consider the corresponding projected problem in E_j, namely

$$\begin{cases}\text{To find } u_j \in E_j \text{ such that}\\ \langle Au_j, v\rangle + \langle Bu_j, v\rangle = \langle f, v\rangle\,, \quad \text{for all } v \in E_j\,.\end{cases} \tag{3.11}$$

In order to solve (3.11), we apply the following variant of Brower's fixed point theorem[12]:

Lemma 3.3. *Let* $P\colon \mathbb{R}^N \mapsto \mathbb{R}^N$ *be a continuous function and assume that there exists* $\rho > 0$ *such that*

$$P(\xi)\xi \geq 0\,, \quad \text{for all } \xi \in \mathbb{R}^N \text{ with } |\xi| = \rho\,,$$

then, there exists ξ *such that* $|\xi| \leq \rho$ *and* $P(\xi) = 0$.

Putting $u_j = \sum_{k=1}^{j} \xi_k w_k \in E_j$, $\xi = (\xi_1, \dots, \xi_j)^{\mathrm{T}} \in \mathbb{R}^j$ and $P = (P_1, \dots, P_j)^{\mathrm{T}}$, where $P_k : \mathbb{R}^j \mapsto \mathbb{R}$ is given as

$$P_k(\xi) = \langle Au_j + Bu_j - f, w_k \rangle \,, \quad 1 \le k \le j,\ j \ge 1 \,,$$

then P is continuous, and owing to (3.6) and (3.8)

$$\begin{aligned} P(\xi)\xi &= \langle Au_j + Bu_j - f, u_j \rangle = \langle Au_j - f, u_j \rangle \\ &\ge k \|u_j\|^2_{X_0^q(\Omega)} - C' - \|f\|_{X_0^q(\Omega)'} \|u_j\|_{X_0^q(\Omega)} \,, \end{aligned}$$

for some constants $k, C' > 0$. Consequently, there exists $R > 0$ large enough (independent of j) such that

$$P(\xi)\xi \ge 0 \,, \quad \text{for } \|u_j\|_{X_0^q(\Omega)} \ge R \,,$$

or, equivalently,

$$P(\xi)\xi \ge 0 \,, \quad \text{for } |\xi| \text{ large enough} \,.$$

We can then apply Lemma 3.3 and deduce the existence of a solution $u_j \in E_j$ to problem (3.11) such that

$$\|u_j\|_{X_0^q(\Omega)} \le R \,, \quad \text{for all } j \ge 1 \,.$$

Thus (u_j) is bounded in the reflexive space $X_0^q(\Omega)$, and using (2.7), it is straightforward to see that (Au_j) is bounded in $X_0^q(\Omega)'$. As a result, we can extract a subsequence, still denoted in the same way, such that

$$u_j \to u \quad \text{in } X_0^q(\Omega)\text{-weakly} \,, \quad \text{for some } u \in X_0^q(\Omega) \,,$$

and also

$$\begin{cases} u_j \to u \begin{cases} \text{in } H_0^1(\Omega)\text{-weakly} \,, \\ \text{in } L^r(\Omega)\text{-strongly for all } r < 6 \,, \quad \text{and} \\ \text{almost everywhere in } \Omega \,; \end{cases} \\ \nabla \cdot u_j \to \nabla \cdot u \quad \text{in } L^q(\Omega)\text{-weakly} \,, \\ d^{-1} u_j \to d^{-1} u \quad \text{in } L^q(\Omega)\text{-weakly} \,, \\ W(u_j) \to W(u) \quad \text{in } L^q(\Omega)\text{-weakly} \,, \\ Au_j \to \chi \quad \text{in } X_0^q(\Omega)'\text{-weakly} \,, \quad \text{for some } \chi \in X_0^q(\Omega)' \,. \end{cases} \tag{3.12}$$

On the other hand,

$$\langle Au_j, u_j \rangle = \langle f, u_j \rangle \to \langle f, u \rangle = \langle f - Bu, u \rangle \,.$$

Next we show that $\chi = f - Bu$. To do so, we take $v = w_k \in E_j$, $j \ge k$ in (3.11). It yields $\langle Au_j, w_k \rangle = \langle f, w_k \rangle - \langle Bu_j, w_k \rangle$, and, according to Lemma 3.7, $\langle Bu_j, w_k \rangle \to \langle Bu, w_k \rangle$. Consequently, $\langle Au_j, w_k \rangle \to \langle \chi, w_k \rangle = \langle f - Bu, w_k \rangle$, for all $k \ge 1$, and this implies that $\chi = f - Bu$.

Summing up, we have shown (a) $u_j \to u$ in $X_0^q(\Omega)$-weakly, (b) $Au_j \to f - Bu$ in $X_0^q(\Omega)'$-weakly, and (c) $\langle Au_j, u_j\rangle \to \langle f - Bu, u\rangle$; owing to the monotonicity trick (see Lemma 7.1) conditions (a), (b) and (c) implies that[12] $Au = f - Bu$.

Finally, the energy identity is easily derived using (3.8).

Remark 3.2. The continuity of the monotone operator A is not mandatory in order to apply the monotonicity trick; indeed, it is enough the hemicontinuity A.

3.2. *A comment about the uniqueness of solutions*

No uniqueness result is known up to date for the 3D hydrostatic approximations of Navier–Stokes equations (1.1). Not much can be said about the uniqueness of solutions to problem (3.1).

In fact, assume without loss of generality that $\Phi_1(0) = 0 \in \mathbb{R}$ and $\Phi_2(0) = 0 \in \mathbb{R}^2$. We also consider the following hypothesis: there exists a constant $C_2 > 0$ such that

$$\begin{aligned}(\Phi(s_1) - \Phi(s_2), s_1 - s_2)& \\ \geq C_2|s_1 - s_2|^q \quad \text{for all } s_1, s_2 \in \mathbb{R} \ (\text{respectively} \in \mathbb{R}^2)\,.&\end{aligned} \tag{3.13}$$

The following result gives some estimates for the difference of two solutions.

Lemma 3.4. *Assume the same hypothese of Theorem* 3.1 *and also* (3.13). *Let* $u \in X_0^q(\Omega)$ *be any solution of problem* (3.1). *Then,*

$$\|u\|^2_{H^1_0(\Omega)} + \|\nabla\cdot u\|^q_{L^q(\Omega)} + \|d^{-1}u\|^q_{L^q(\Omega)} \leq \beta\|f\|_{X_0^q(\Omega)'}\,,$$

where $\beta = \beta(\|f\|_{X_0^q(\Omega)'})$; *in particular, for* $f = 0$, *the trivial solution* $u = 0$ *is unique.*

Let $u, v \in X_0^q(\Omega)$ *be two solutions of problem* (3.1). *Then, there exists a constant* $C_3 > 0$ *such that*

$$\|u - v\|^2_{H^1_0(\Omega)} + \|\nabla\cdot(u - v)\|^q_{L^q(\Omega)} \leq C_3\beta\|f\|_{X_0^q(\Omega)'}\|\nabla\cdot(u - v)\|^2_{L^q(\Omega)}\,,$$

$$\|u - v\|^2_{H^1_0(\Omega)} + \|\nabla\cdot(u - v)\|^q_{L^q(\Omega)} \leq C_3\beta\|f\|_{X_0^q(\Omega)'}\|u - v\|^{q'}_{H^1_0(\Omega)}\,.$$

The first estimate is derived directly from the energy identity. The last two inequalities can be obtained in the same way as in the case of the steady-state Navier–Stokes equations (in the case $2 < q < 3$, the anisotropic estimates are needed). Unfortunately, we cannot go any further. Indeed, since $q > 2$, these two last inequalities are not enough to achieve the conclusion $u = v$ even for $\|f\|_{X_0^q(\Omega)'}$ small enough. Thus, the uniqueness of solutions still remains open for problem (3.1).

4. Monotone Perturbations of the Hydrostatic Approximation with Homogeneous Dirichlet Boundary Conditions

In this section we consider the hydrostatic approximation of Navier–Stokes equations with homogeneous Dirichlet boundary conditions and with monotone perturbations like those introduced in the former section. The description of the problem now follows: let $f \in W^{-1,q'}(\Omega)^2$ be given. We search for $u: \Omega \mapsto \mathbb{R}^2$ and $p_{\mathrm{s}}: \omega \mapsto \mathbb{R}$ such that

$$\begin{cases} (u \cdot \nabla) u + W(u) \dfrac{\partial u}{\partial z} - \nu_1 \Delta u - \nu_2 \dfrac{\partial^2 u}{\partial z^2} + \gamma u^{\perp} + \nabla p_{\mathrm{s}} \\ \qquad - \nabla \Phi_1(\nabla \cdot u) + d^{-1} \Phi_2(d^{-1} u) = f \quad \text{in } \Omega \,, \\ \qquad \nabla \cdot M(u) = 0 \quad \text{in } \omega \,, \\ \qquad u = 0 \quad \text{on } \partial\Omega \,, \end{cases} \tag{4.1}$$

the operators W and M being defined in (1.3) and (2.1), respectively. In order to deal with the gradient of the surface pressure, ∇p_{s}, a de Rham-like lemma is needed. This enables us to give an equivalent variational formulation to problem (4.1) where no pressure term appears; the space where we search for solutions is $V_0^q(\Omega)$ [see (2.3)].

In this section, we also show that (4.1) is equivalent, in some sense, to the next problem

$$\begin{cases} \text{To find } u \in V_0^q(\Omega) \text{ such that} \\ \langle Au + Bu, v \rangle = \langle f, v \rangle \quad \text{for all } v \in V_0^q(\Omega)' \,, \end{cases} \tag{4.2}$$

where A and B are given in (3.2). As it has been stated in the former section the operator A and B maps $X_0^q(\Omega)$ onto its dual space $X_0^q(\Omega)'$. Since, (see Sec. 2.3) $X_0^q(\Omega)' = H^{-1}(\Omega)^2 + Y + Z$ and $H^{-1}(\Omega)^2, Y, Z \subset W^{-1,q'}(\Omega)^2$, we also have $Au, Bu \in W^{-1,q'}(\Omega)^2$, so that the condition $f \in W^{-1,q'}(\Omega)^2$ is meaningful.

We begin this analysis with a result which relates certain distributions in Ω with distributions defined in ω.[14]

Lemma 4.1. *The following assertions are equivalent*:

(1) $F \in \mathcal{D}'(\Omega)$ *and* $\frac{\partial F}{\partial z} = 0$;
(2) *there exists a unique distribution* $S \in \mathcal{D}'(\omega)$ *such that*

$$\langle F, \varphi \rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)} = \langle S, M(\varphi) \rangle_{\mathcal{D}'(\omega), \mathcal{D}(\omega)} \,, \quad \textit{for all } \varphi \in \mathcal{D}(\Omega) \,. \tag{4.3}$$

Owing to Lemma 4.1, we may deduce[14] a version of de Rham's lemma useful in the resolution of (4.1). Indeed, it tells us that the resolution of (4.1) reduces to that of (4.2). Also, this result is a generalization of the one due to Lions, Temam and Wang,[13] to more general domains Ω, without the assumption on the existence of a sidewall all along $\partial\omega$.

Lemma 4.2. *Let* $F \in W^{-1,q'}(\Omega)^2$. *Then the following conditions are equivalent*:

(1) $\langle F, \varphi\rangle_{W^{-1,q'}(\Omega)^2, W_0^{1,q}(\Omega)^2} = 0$ *for all* $\varphi \in \mathcal{V}(\Omega)$;

(2) F *does not depend upon the z-variable and there exists* $p_{\rm s} \in L_D^{q'}(\omega)$ (*unique up to an additive constant*) *such that* $F = \nabla p_{\rm s}$.

Remark 4.1. Lemma 4.2 is equivalent to the following reduced inf–sup condition: there exists $\beta > 0$ such that

$$\|p\|_{L_{D,0}^{q'}(\omega)} \le \beta \sup_{v\in W_0^{1,q}(\Omega)} \frac{\int_\omega p\nabla\cdot M(v)}{\|v\|_{W_0^{1,q}(\Omega)}}, \quad \text{for all } p \in L_{D,0}^{q'};$$

this reduced inf–sup condition was already shown by Chacón and Guillén,[7] and its proof is based in the application of the de Rham lemma in Ω. Note that we have applied de Rham's lemma in ω.

Remark 4.2. Obviously, if Ω has a sidewall all along $\partial\omega$, that is $\operatorname{ess\,inf}_\omega D > 0$, then $p_{\rm s} \in L^{q'}(\omega)$ as it was first considered by Lions, Temam and Wang.[13]

Corollary 4.1. *The space* $\mathcal{V}(\Omega)$ *is dense in* $V_0^q(\Omega)$.

Proof. Since $V_0^q(\Omega)$ is a closed subspace of $X_0^q(\Omega)$, we may consider the dual space $V_0^q(\Omega)'$ as a subspace of $X_0^q(\Omega)'$. On the other hand, since $X_0^q(\Omega)' \subset W^{-1,q'}(\Omega)^2$, we also have $V_0^q(\Omega)' \subset W^{-1,q'}(\Omega)^2$. □

Now, let $F \in V_0^q(\Omega)'$ such that $\langle F, \varphi\rangle_{W^{-1,q'}(\Omega)^2, W_0^{1,q}(\Omega)^2} = 0$, for all $\varphi \in \mathcal{V}(\Omega)$. We want to show that $F = 0$. Indeed, owing to Lemma 4.2, there exists $p_{\rm s} \in L_D^{q'}(\omega)$ such that $F = \nabla p_{\rm s}$. Now, let $v \in X_0^q(\Omega)$, then $M(v) \in L^q(\Omega)$ and using the density of $\mathcal{D}(\Omega)^2$ in $X_0^q(\Omega)$ (Lemma 3.2) it is easy to check that

$$\langle \nabla p_{\rm s}, v\rangle_{W^{-1,q'}(\Omega)^2, W_0^{1,q}(\Omega)^2} = -\int_\omega p_{\rm s}\nabla\cdot M(v)\,, \quad \text{for all } v \in X_0^q(\Omega)\,;$$

in particular, this implies that

$$\langle F, v\rangle_{W^{-1,q'}(\Omega)^2, W_0^{1,q}(\Omega)^2} = \langle \nabla p_{\rm s}, v\rangle_{W^{-1,q'}(\Omega)^2, W_0^{1,q}(\Omega)^2} = 0\,, \quad \text{for all } v \in V_0^q(\Omega)\,.$$

The existence result for problem (4.2), or equivalently (4.1), now follows.

Theorem 4.1. *Assume* (2.4), (2.6) *and also* (2.7) *for* Φ_1 *and* Φ_2 *respectively. Then, for every* $f \in W^{-1,q'}(\Omega)^2$ *there exists* $u \in V_0^q(\Omega)$ *solution to problem* (4.2), *and the energy identity* (3.10) *holds.*

Furthermore, for every solution $u \in V_0^q(\Omega)$ *of* (4.2), *there exists* $p_{\rm s} \in L^{q'}(\omega)$, *uniquely determined up to an additive constant, such that the couple* $(u, p_{\rm s})$ *is a solution to problem* (4.1) *in* $W^{-1,q'}(\Omega)^2$.

Proof. Since $\mathcal{V}(\Omega)$ is dense and $V_0^q(\Omega)$ is a separable, reflexive Banach space, we may apply a Faedo–Galerkin procedure as in the proof of Theorem 3.1. We may repeat all the steps of the proof, adapted to this new framework, and obtain the existence of $u \in V_0^q(\Omega)$ solution to (4.2). The energy identity is easily deduced by taking $v = u$ in the variational formulation of (4.2). □

Finally, putting $F = f - Au - Bu \in W^{-1,q'}(\Omega)^2$ we have that F fulfills the assumptions of Lemma 4.2, whence the existence of $p_s \in L^{q'}(\omega)$.

5. Monotone Perturbations of the Hydrostatic Approximation

We finally study the hydrostatic approximation of Navier–Stokes equations with the usual boundary conditions, namely homogeneous Dirichlet boundary conditions on Γ_b and nonhomogeneous Neumann boundary conditions on Γ_s. We also include monotone perturbations like those introduced in the former sections. Thus the problem is the following: let $f \in W^{-1,q'}(\Omega)^2$ and $g_s \in H^{-1/2}(\Gamma_s)^2$ be given, we search for $u: \Omega \mapsto \mathbb{R}^2$ and $p_s: \omega \mapsto \mathbb{R}$ such that

$$\begin{cases} (u \cdot \nabla)u + W(u)\dfrac{\partial u}{\partial z} - \nu_1 \Delta u - \nu_2 \dfrac{\partial^2 u}{\partial z^2} + \gamma u^\perp + \nabla p_s \\ \qquad - \nabla\Phi_1(\nabla \cdot u) + d_b^{-1}\Phi_2(d_b^{-1}u) = f \quad \text{in } \Omega\,, \\ \qquad \nabla \cdot M(u) = 0 \quad \text{in } \omega\,, \\ \qquad u = 0 \quad \text{on } \Gamma_b\,, \quad \nu_2 \dfrac{\partial u}{\partial z} = g_s \quad \text{on } \Gamma_s\,, \end{cases} \tag{5.1}$$

W and M being defined in (1.3) and (2.1), respectively, and $d_b \in W^{1,\infty}(\Omega)$ is the distance to Γ_b. The natural space to search for solutions u is $V_b^q(\Omega)$ defined in (2.3). The formal multiplication by $v \in V_b^q(\Omega)$ of the differential equation of (5.1), followed with an integration over Ω leads to the variational formulation

$$\begin{cases} \text{To find } u \in V_b^q(\Omega) \text{ such that} \\ \langle Au + Bu, v\rangle = \langle \ell, v\rangle \quad \text{for all } v \in V_b^q(\Omega)\,, \end{cases} \tag{5.2}$$

where A and B are given in (3.2), and $\ell \in V_b^q(\Omega)'$ is given by

$$\langle \ell, v\rangle = \langle f, v\rangle_{W_b^{-1,q'}(\Omega)^2, W_b^{1,q}(\Omega)^2} + \langle g_s, v\rangle_{H^{-1/2}(\Gamma_s)^2, H^{1/2}(\Gamma_s)^2}\,. \tag{5.3}$$

As in the preceding sections, we show some intermediate results in order to deduce an existence result for these equivalent formulations. The first thing we note is that $V_b^q(\Omega)$ is a closed subspace of $X_b^q(\Omega)$. The next result tells us that smooth functions are dense in this space.

Lemma 5.1. *The space $\mathcal{D}_b(\Omega)^2$ is dense in $X_b^q(\Omega)$.*

Proof. Consider the set $\widehat{\Omega}$ consisting of Ω together with its reflecting image with respect to the plane $z = 0$ and Γ_s, that is

$$\widehat{\Omega} = \{(x, y, z) \in \mathbb{R}^3\colon (x, y) \in \omega \text{ and } - D(x, y) < z < D(x, y)\}\,. \tag{5.4}$$

A very easy generalization of Lemma 3.2 yields that $\mathcal{D}(\widehat{\Omega})^2$ is dense in the space $X_0^q(\widehat{\Omega})$. □

Now let $v\colon \Omega \mapsto \mathbb{R}^2$ and consider the function $\hat{v}$ defined in $\widehat{\Omega}$ as

$$\hat{v}(x, y, z) = \begin{cases} v(x, y, z)\,, & \text{if } z < 0\,, \\ v(x, y, -z)\,, & \text{if } z > 0\,; \end{cases} \tag{5.5}$$

it is straightforward that $v \in X^q_{\mathrm{b}}(\Omega)$ if and only if $\widehat{v} \in X^q_0(\widehat{\Omega})$ and, in that case, the two following inequalities hold

$$\frac{\sqrt{2}}{2}\|\hat{v}\|_{X^q_0(\widehat{\Omega})} \le \|v\|_{X^q_{\mathrm{b}}(\Omega)} \le \|\hat{v}\|_{X^q_0(\widehat{\Omega})}\,. \tag{5.6}$$

Let $v \in X^q_{\mathrm{b}}(\Omega)$ and $\varepsilon > 0$ be given; then, there exists $\varphi_\varepsilon \in \mathcal{D}(\widehat{\Omega})^2$ such that $\|\hat{v} - \varphi_\varepsilon\|_{X^q_0(\widehat{\Omega})} < \varepsilon$. Then it is an easy task to check that $\varphi_\varepsilon|_\Omega \in \mathcal{D}_{\mathrm{b}}(\Omega)^2$ and $\|v - \varphi_\varepsilon|_\Omega\|_{X^q_{\mathrm{b}}(\Omega)} < \varepsilon$.

Corollary 5.1. *The operator* B: $X^q_{\mathrm{b}}(\Omega) \mapsto X^q_{\mathrm{b}}(\Omega)'$ *is well defined and is continuous* (1) *from* $X^q_{\mathrm{b}}(\Omega)$*-strong to* $X^q_{\mathrm{b}}(\Omega)'$*-strong, and* (2) *from* $X^q_{\mathrm{b}}(\Omega)$*-weak to* $X^q_{\mathrm{b}}(\Omega)'$*-weak. Moreover, the following representation formula holds*:

$$\langle Bu, v\rangle = -\int_\Omega (u\cdot\nabla)vu - \int_\Omega W(u)\frac{\partial v}{\partial z}u\,, \quad \textit{for all } u,\ v \in X^q_{\mathrm{b}}(\Omega)\,, \tag{5.7}$$

in particular

$$\langle Bu, u\rangle = 0\,, \quad \textit{for all } u \in X^q_{\mathrm{b}}(\Omega)\,. \tag{5.8}$$

Proof. Let $u, v \in \mathcal{D}_{\mathrm{b}}(\Omega)^2$; then, using the identity $\frac{\partial W(u)}{\partial z} = -\nabla\cdot u$ in Ω, and the boundary conditions we have

$$\begin{aligned}\langle Bu, v\rangle &= \int_\Omega \left[(u\cdot\nabla)uv + W(u)\frac{\partial u}{\partial z}v\right] \\ &= \int_{\partial\Omega}(u_1n_1 + u_2n_2)uv + \int_{\partial\Omega} W(u)n_3uv \\ &\quad - \int_\Omega \left[(\nabla\cdot u)uv + (u\cdot\nabla)vu + \frac{\partial W(u)}{\partial z}uv + W(u)\frac{\partial v}{\partial z}u\right] \\ &= -\int_\Omega \left[(u\cdot\nabla)vu + W(u)\frac{\partial v}{\partial z}u\right]\end{aligned}$$

and thus (5.7) holds true for smooth functions. From this point on, we can repeat the same arguments as in the proof of Corollary 3.1 to achieve the desired results. □

Remark 5.1. As in the two problems studied in the preceding sections, the expression (5.8) is the key for the derivation of the energy identity for solutions to problem (5.2).

We turn to the space $V^q_{\rm b}(\Omega)$; we introduce the space $V^q_0(\widehat{\Omega})$, $\widehat{\Omega}$ being defined in (5.4), as follows

$$V^q_0(\widehat{\Omega}) = \left\{\phi \in X^q_0(\widehat{\Omega}) \colon \nabla \cdot \widehat{M}(\phi) = 0 \text{ in } \omega\right\}, \quad \widehat{M}(\phi) = \int_{-D(x,y)}^{D(x,y)} \phi(x,y,z)\, dz\,.$$

A straightforward adaptation of Corollary 4.1 yields the density of $\mathcal{V}(\widehat{\Omega})$ in the space $V^q_0(\widehat{\Omega})$. As a result, we have the following consequence.

Corollary 5.2. *The space* $\mathcal{V}_{\rm b}(\Omega)$ *is dense in* $V^q_{\rm b}(\Omega)$.

Proof. For a function $v \in V^q_{\rm b}(\Omega)$, we consider $\hat{v} \in X^q_0(\Omega)$, its extension to $\widehat{\Omega}$ given in (5.5). Then,

$$\widehat{M}(\hat{v}) = \int_{-D(x,y)}^{D(x,y)} \hat{v}\, dz = 2\int_{-D(x,y)}^{0} \hat{v}\, dz = 2M(v)\,,$$

consequently, $\hat{v} \in V^q_0(\widehat{\Omega})$.

Now, let $\varepsilon > 0$. Then, we may find $\phi_\varepsilon \in \mathcal{V}(\widehat{\Omega})$ such that $\|\hat{v} - \phi_\varepsilon\|_{V^q_0(\widehat{\Omega})} < \varepsilon$. In general, $\phi_\varepsilon|_\Omega \notin \mathcal{V}_{\rm b}(\Omega)$ since we cannot guarantee that $\nabla \cdot M(\phi_\varepsilon|_\Omega) = 0$ in ω. For that reason, we consider the symmetric part of ϕ_ε, namely $\psi_\varepsilon = \frac{1}{2}(\phi_\varepsilon + \phi'_\varepsilon)$, where $\phi'_\varepsilon(x,y,z) = \phi_\varepsilon(x,y,-z)$, $(x,y,z) \in \widehat{\Omega}$; it is an easy task to check that $\psi_\varepsilon|_\Omega \in \mathcal{V}_{\rm b}(\Omega)$. Finally, using (5.6) and taking into account that $\widehat{\psi_\varepsilon|_\Omega} = \psi_\varepsilon$, we have

$$\begin{aligned}\|v - \psi_\varepsilon|_\Omega\|_{V^q_{\rm b}(\Omega)} &\le \|\hat{v} - \psi_\varepsilon\|_{V^q_0(\widehat{\Omega})} \\ &\le \frac{1}{2}\|\hat{v} - \phi_\varepsilon\|_{V^q_0(\widehat{\Omega})} + \frac{1}{2}\|\hat{v} - \phi'_\varepsilon\|_{V^q_0(\widehat{\Omega})} \\ &= \|\hat{v} - \phi_\varepsilon\|_{V^q_0(\widehat{\Omega})} < \varepsilon\,.\end{aligned}$$

□

This last result allows us to show the existence result for the hydrostatic approximation with monotone perturbations by means of an implementation of a Faedo–Galerkin procedure like the one described in Sec. 3.

Theorem 5.1. *Assume* (2.4), (2.6) *and also* (2.7) *for* Φ_1 *and* Φ_2 *respectively. Then, for every* $f \in W^{-1,q'}(\Omega)^2$ *and* $g_{\rm s} \in H^{-1/2}(\Gamma_{\rm s})^2$ *there exists* $u \in V^q_{\rm b}(\Omega)$ *solution to problem* (5.2), *and the following energy identity holds*:

$$\nu_1 \int_\Omega |\nabla u|^2 + \nu_2 \int_\Omega \left|\frac{\partial u}{\partial z}\right|^2 + \int_\Omega \Phi_1(\nabla\cdot u)\nabla\cdot u + \int_\Omega \Phi_2(d_{\rm b}^{-1}u)d_{\rm b}^{-1}u = \langle \ell, u\rangle\,, \tag{5.9}$$

where ℓ *is given in* (5.3).

Furthermore, for every solution $u \in V^q_{\rm b}(\Omega)$ *of* (5.2), *there exists* $p_{\rm s} \in L^{q'}_D(\omega)$, *uniquely determined up to an additive constant, such that the couple* $(u, p_{\rm s})$ *is a solution to problem* (5.1), *the differential equation being taken in the sense of* $W^{-1,q'}(\Omega)^2$.

6. Convergence of Regularized Solutions to a Solution of the Hydrostatic Approximation

In this section we assume the monotone terms are of the forms $\varepsilon\Phi_1$ and $\varepsilon\Phi_2$, where Φ_1 and Φ_2 verify (2.7), and $\varepsilon > 0$ plays the role of a small parameter. We write $(u^\varepsilon, p_s^\varepsilon)$ a solution to problem (5.1), and we want to analyze the behavior of $(u^\varepsilon, p_s^\varepsilon)$ as ε goes to zero. Indeed, we show that there exists a couple (u, p_s) such that $u^\varepsilon \to u$ and $p_s^\varepsilon \to p_s$, weakly in certain Banach spaces, whereas (u, p_s) is a solution to the hydrostatic approximation of Navier–Stokes equations (1.1). In particular, this gives another proof of the existence of a solution to the hydrostatic approximation with a completely different approach.

The parametrized problem is the following:

$$\begin{cases} (u^\varepsilon \cdot \nabla)u^\varepsilon + W(u^\varepsilon)\dfrac{\partial u^\varepsilon}{\partial z} - \nu_1 \Delta u^\varepsilon - \nu_2 \dfrac{\partial^2 u^\varepsilon}{\partial z^2} + \gamma u^{\varepsilon\perp} + \nabla p_s^\varepsilon \\ \qquad - \varepsilon \nabla \Phi_1(\nabla \cdot u^\varepsilon) + \varepsilon d_b^{-1}\Phi_2(d_b^{-1}u^\varepsilon) = f \quad \text{in } \Omega\,, \\ \qquad \nabla \cdot M(u^\varepsilon) = 0 \quad \text{in } \omega\,, \\ u^\varepsilon = 0 \quad \text{on } \Gamma_b\,, \quad \nu_2 \dfrac{\partial u^\varepsilon}{\partial z} = g_s \quad \text{on } \Gamma_s\,. \end{cases} \tag{6.1}$$

Here we assume a bit more regularity to f, namely $f \in H_b^{-1}(\Omega)^2$. According to Theorem 5.1, if $g_s \in H^{-1/2}(\Gamma_s)^2$, problem (6.1) has a solution $(u^\varepsilon, p_s^\varepsilon)$ such that $u^\varepsilon \in V_b^q(\Omega)$ and $p_s^\varepsilon \in L_{D,0}^{q'}(\omega)$. When ε goes to zero, the monotone nonlinear terms vanish; so, it is quite natural to expect that the regularity in the space $V_b^q(\Omega)$ is lost in the limit.

Theorem 6.1. *Assume* (2.4), (2.6) *and also* (2.7) *for* Φ_1 *and* Φ_2, *respectively, hold. Let* $f \in H_b^{-1}(\Omega)^2$ *and* $g_s \in H^{-1/2}(\Gamma_s)^2$, *and let* $(u^\varepsilon, p_s^\varepsilon) \in V_b^q(\Omega) \times L_{D,0}^{q'}(\omega)$ *be a solution to problem* (6.1).

Then, there exists $(u, p_s) \in H_b^1(\Omega)^2 \times L_{D,0}^r(\omega)$, *for all* $r < 2$, *such that* (u, p_s) *is a solution of the hydrostatic approximation equations* (1.1) *and*

$$u^\varepsilon \to u \quad in\ H^1(\Omega)^2\text{-}weakly\,, \quad p_s^\varepsilon \to p_s \quad in\ L_{D,0}^{q'}(\omega)\text{-}weakly\,. \tag{6.2}$$

Proof. Since $f \in H_b^{-1}(\Omega)^2$, and according to the energy identity (5.9), we have

$$\begin{aligned} \nu_1 \int_\Omega |\nabla u^\varepsilon|^2 + \nu_2 \int_\Omega \left|\frac{\partial u^\varepsilon}{\partial z}\right|^2 + \varepsilon \int_\Omega \Phi_1(\nabla \cdot u^\varepsilon)\nabla \cdot u^\varepsilon + \varepsilon \int_\Omega \Phi_2(d_b^{-1}u^\varepsilon)\, d_b^{-1}u^\varepsilon \\ = \langle f, u^\varepsilon\rangle_{H_b^{-1}(\Omega)^2, H_b^{-1}(\Omega)^2} + \langle g_s, u^\varepsilon\rangle_{H^{-1/2}(\Gamma_s)^2, H^{1/2}(\Gamma_s)^2} \\ \le C(\|f\|_{H_b^{-1}(\Omega)^2}\|u\|_{H_b^1(\Omega)^2} + \|g_s\|_{H^{-1/2}(\Gamma_s)^2}\|u\|_{H_b^1(\Omega)^2})\,. \end{aligned}$$

Using (2.7), we may deduce that

$$(u^\varepsilon) \text{ is bounded in } H_b^{-1}(\Omega)^2\,,$$

$$(\varepsilon^{1/q'}\Phi_1(\nabla \cdot u^\varepsilon)) \text{ is bounded in } L^{q'}(\Omega)\,,$$

$$(\varepsilon^{1/q'}\Phi_2(d_{\rm b}^{-1}u^\varepsilon)) \text{ is bounded in } L^{q'}(\Omega)^2\,.$$

Then, for $v \in \mathcal{V}_{\rm b}(\Omega)$, we deduce, as $\varepsilon \to 0^+$,

$$\varepsilon\int_\Omega \Phi_1(\nabla\cdot u^\varepsilon)\nabla\cdot v = \varepsilon^{1/q}\int_\Omega \varepsilon^{1/q'}\Phi_1(\nabla\cdot u^\varepsilon)\nabla\cdot v \to 0\,,$$

$$\varepsilon\int_\Omega \Phi_2(d_{\rm b}^{-1}u^\varepsilon)\, d_{\rm b}^{-1}v = \varepsilon^{1/q}\int_\Omega \varepsilon^{1/q'}\Phi_2(d_{\rm b}^{-1}u^\varepsilon)\, d_{\rm b}^{-1}v \to 0\,.$$

Let $u \in H_{\rm b}^{-1}(\Omega)^2$ such that $u^\varepsilon \to u$ in $H_{\rm b}^{-1}(\Omega)^2$-weakly, and in $L^s(\Omega)$-strongly, for all $s < 6$. Then,

$$\nu_1\int_\Omega \nabla u^\varepsilon \nabla v + \nu_2\int_\Omega \frac{\partial u^\varepsilon}{\partial z}\frac{\partial v}{\partial z} + \int_\Omega \gamma u^{\varepsilon\perp} v \to \nu_1\int_\Omega \nabla u\nabla v + \nu_2\int_\Omega \frac{\partial u}{\partial z}\frac{\partial v}{\partial z} + \int_\Omega \gamma u^\perp v\,,$$

$$\int_\Omega \left[(u^\varepsilon\cdot\nabla)vu^\varepsilon + W(u^\varepsilon)\frac{\partial v}{\partial z}u^\varepsilon\right] \to \int_\Omega \left[(u\cdot\nabla)vu + W(u)\frac{\partial v}{\partial z}u\right].$$

Consequently $u \in H_{\rm b}^{-1}(\Omega)^2$ satisfies the variational formulation

$$-\int_\Omega \left[(u\cdot\nabla)vu + W(u)\frac{\partial v}{\partial z}u\right] + \nu_1\int_\Omega \nabla u\nabla v + \nu_2\int_\Omega \frac{\partial u}{\partial z}\frac{\partial v}{\partial z} + \int_\Omega \gamma u^\perp v$$
$$= \langle f, v\rangle + \langle g, v\rangle\,, \quad \text{for all } v \in \mathcal{V}_{\rm b}(\Omega)\,.$$

Putting $A_0u = -\nu_1\Delta u - \nu_2\frac{\partial^2 u}{\partial z^2} + \gamma u^\perp \in H_{\rm b}^{-1}(\Omega)^2$, $Bu = (u\cdot\nabla)u + W(u)\frac{\partial u}{\partial z} \in W^{-1,r}(\Omega)$, for all $r < 2$ (this regularity is due to the anisotropic estimates of Sec. 2.4) and $F = f - A_0u - Bu$, then $F \in W^{-1,r}(\Omega)$, for all $r < 2$, and satisfies the conditions of Lemma 4.2. Consequently, there exists $p_{\rm s} \in L^r_{D,0}(\omega)$, for all $r < 2$ such that $A_0u + Bu + \nabla p_{\rm s} = f$ in $W^{-1,r}(\Omega)$, for all $r < 2$ [that is, $(u, p_{\rm s})$ solves the differential equations of the hydrostatic approximation in $W^{-1,r}(\Omega)$]. The boundary condition on $\Gamma_{\rm s}$ may be deduced in the standard way. □

Finally, since

$$\lim_{\varepsilon\to 0^+}\nabla p_{\rm s}^\varepsilon = \lim_{\varepsilon\to 0^+}\{f - A_0u^\varepsilon - Bu^\varepsilon + \varepsilon\nabla\Phi_1(\nabla\cdot u^\varepsilon) - \varepsilon d_{\rm b}^{-1}\Phi_2(d_{\rm b}^{-1}u^\varepsilon)\}$$
$$= f - A_0u - Bu = \nabla p_{\rm s}\,, \quad \text{in } W^{-1,r}(\Omega)^2\text{-weakly}\,, \quad \text{for all } r < 2\,,$$

we obtain $p_{\rm s}^\varepsilon \to p_{\rm s}$, in $L^r(\Omega)$-weakly, for all $r < 2$, or, equivalently,

$$p_{\rm s}^\varepsilon \to p_{\rm s}\,, \quad \text{in } L^r_{D,0}(\omega)\text{-weakly}\,, \quad \text{for all } r < 2\,.$$

7. Application to a Regularized Hydrostatic Turbulence Model

In this last section we show how the regularizing technique described in the preceding sections can be applied to the resolution of a modified turbulence model. We just consider a simple example of a system of equations involving the mean horizontal velocity field u, the mean surface pressure $p_{\rm s}$ and the mean turbulent kinetic energy k. Keeping the same notations as above, the problem is: to find $u\colon \Omega \mapsto \mathbb{R}^2$, $p_{\mathrm{s}}\colon \omega \mapsto \mathbb{R}$ and $k\colon \Omega \mapsto \mathbb{R}$ such that

$$\left\{\begin{aligned} &(u \cdot \nabla)u + W(u)\frac{\partial u}{\partial z} - \nabla \cdot [(\nu_1 + \bar{\nu}_1(k))\nabla u] - \frac{\partial}{\partial z}\left[(\nu_2 + \bar{\nu}_2(k))\frac{\partial u}{\partial z}\right] \\ &\qquad + \gamma u^{\perp} + \nabla p_{\mathrm{s}} - \nabla \Phi_1(\nabla \cdot u) + d_{\mathrm{b}}^{-1}\Phi_2(d_{\mathrm{b}}^{-1}u) = f \quad \text{in } \Omega\,, \\ &\qquad \nabla \cdot M(u) = 0 \quad \text{in } \omega\,, \\ &-\nabla \cdot [(\nu_1 + \bar{\nu}_1(k))\nabla k] - \frac{\partial}{\partial z}\left[(\nu_2 + \bar{\nu}_2(k))\frac{\partial k}{\partial z}\right] + |k|^{1/2}k \\ &\qquad = \bar{\nu}_1(k)|\nabla u|^2 + \bar{\nu}_2(k)\left|\frac{\partial u}{\partial z}\right|^2 + h \quad \text{in } \Omega\,, \\ &u = 0 \quad \text{on } \Gamma_{\mathrm{b}}\,, \quad (\nu_2 + \bar{\nu}_2(k))\frac{\partial u}{\partial z} = g_{\mathrm{s}}\,, \quad \text{on } \Gamma_{\mathrm{s}}\,, \quad k = 0 \quad \text{on } \partial\Omega\,, \end{aligned}\right. \tag{7.1}$$

where $\bar{\nu}_1(k)$ and $\bar{\nu}_2(k)$ stand, respectively, for the horizontal and vertical turbulent viscosity coefficients (usually, $\bar{\nu}_1(k)$ is taken to be null). Also, the datum function h on the right-hand side of the equation of k takes into account density effects and nonhomogeneous Dirichlet boundary conditions on Γ_{s} for the original unknown (in fact, k is a shifted turbulent kinetic energy).

We consider the following assumptions for $\bar{\nu}_1$, $\bar{\nu}_2$:

$$\bar{\nu}_1, \bar{\nu}_2 \in \mathcal{C}(\mathbb{R}) \cap L^{\infty}(\mathbb{R})\,, \quad \bar{\nu}_1(s) \geq 0\,, \quad \bar{\nu}_2(s) \geq 0 \quad \text{for all } s \in \mathbb{R}\,. \tag{7.2}$$

Apart from the nonlinearities of the coupled system (7.1), one of the main theoretical difficulties is the regularity of the two terms appearing on the right-hand side of the equation of k: in fact, we may expect to have $u \in H^1(\Omega)$, and thus $\bar{\nu}_1(k)|\nabla u|^2, \bar{\nu}_2(k)|\frac{\partial u}{\partial z}|^2 \in L^1(\Omega)$. This leads us to search for solutions $u \in V_{\mathrm{b}}^q(\Omega)$, as in the preceding sections, whereas $k \in W_0^{1,r}(\Omega)$ for all $r < 3/2$, according to the analysis of renormalized solutions of elliptic equations with right-hand side in $L^1(\Omega)$.[3,8]

In order to solve (7.1), we introduce the operators $D(u,k)$ and $A(u,k)$, and the function $H(u,k)$ given as

$$\left\{\begin{aligned} &D(u,k) = -\nabla \cdot [(\nu_1 + \bar{\nu}_1(k))\nabla u] - \frac{\partial}{\partial z}\left[(\nu_2 + \bar{\nu}_2(k))\frac{\partial u}{\partial z}\right], \\ &A(u,k) = D(u,k) + \gamma u^{\perp} - \nabla \Phi_1(\nabla \cdot u) + d_{\mathrm{b}}^{-1}\Phi_2(d_{\mathrm{b}}^{-1}u)\,, \\ &H(u,k) = \bar{\nu}_1(k)|\nabla u|^2 + \bar{\nu}_2(k)\left|\frac{\partial u}{\partial z}\right|^2 + h\,, \end{aligned}\right. \tag{7.3}$$

so that system (7.1) becomes

$$\left\{\begin{aligned} &\langle A(u,k) + Bu, v\rangle = \langle \ell, v\rangle \quad \text{for all } v \in V_{\mathrm{b}}^q(\Omega)\,, \\ &\langle D(k,k), \phi\rangle + \int_{\Omega} |k|^{1/2}k\phi = \int_{\Omega} H(u,k)\phi\,, \quad \text{for all } \phi \in \mathcal{D}(\Omega)\,, \end{aligned}\right. \tag{7.4}$$

where B is given in (3.2) and ℓ in (5.3).

Theorem 7.1. *Assume the hypotheses of Theorem* 5.1 *and also* $h \in L^1(\Omega)$. *Then, problem* (7.4) *has a solution* (u, k) *such that* $u \in V_{\rm b}^q(\Omega)$ *and* $k \in W_0^{1,r}(\Omega)$ *for all* $r < 3/2$. *Moreover, for every solution* (u, k) *of* (7.4), *there exists a unique* $p_{\rm s} \in L^{q'}_{D,0}(\omega)$ *such that* $(u, p_{\rm s}, k)$ *is a solution to problem* (7.1) *in the sense of* $W_{\rm b}^{-1,q'}(\Omega)^2$ *for the first equation, in* $\mathcal{D}'(\Omega)$ *for the second one.*

The proof of this existence result is developed along the following sections.

7.1. *Approximate problems*

For every $j \geq 1$, we define T_j to be the truncation function at height j, that is $T_j(s) = \min(j, |s|)\,\mathrm{sign}\ s$, whereas sign s is the standard sign function. Then, we put

$$H_j(u,k) = T_j(\bar{\nu}_1(k)|\nabla u|^2) + T_j\left(\bar{\nu}_2(k)\left|\frac{\partial u}{\partial z}\right|^2\right) + T_j(h)$$

and for $j \geq 1$, we set the approximate problems as

$$\begin{cases} \text{To find } u_j \in V_{\rm b}^q(\Omega) \text{ and } k_j \in H_0^1(\Omega) \text{ such that} \\ \langle A(u_j, k_j) + Bu_j, v\rangle = \langle \ell, v\rangle \quad \text{for all } v \in V_{\rm b}^q(\Omega)\,, \\ \langle D(k_j,k_j), \phi\rangle + \displaystyle\int_\Omega |k_j|^{1/2} k_j \phi = \int_\Omega H_j(u_j,k_j)\phi \quad \text{for all } \phi \in \mathcal{D}(\Omega)\,. \end{cases} \tag{7.5}$$

A straightforward application of Schauder's fixed point theorem leads to the existence of a solution (u_j, k_j) to system (7.5).

Remark 7.1. Observe that, for every fixed $k \in L^1(\Omega)$, the operator $u \in V_{\rm b}^q(\Omega) \mapsto A(u,k)$ is monotone and coercive.

7.2. *Estimates for the approximate solutions*

Putting $v = u_j$ in (7.5) yields the corresponding energy identity, and thus we obtain that (u_j) is bounded in $V_{\rm b}^q(\Omega)$. In particular, (u_j) is bounded in $H^1(\Omega)^2$, and thus $H_j(u_j, k_j)$ is bounded in $L^1(\Omega)$. It can be shown then that the sequence (k_j) verifies the Boccardo and Gallouët estimates,[4] and this implies that (k_j) is bounded in $W_0^{1,r}(\Omega)$ for all $r < 3/2$. Consequently, there exist $u \in V_{\rm b}^q(\Omega)$ and $k \in W_0^{1,r}(\Omega)$ for all $r < 3/2$, and subsequences, still denoted in the same way, such that

$$u_j \to u \begin{cases} \text{in } H^1(\Omega)^2\text{-weakly}\,, \\ \text{in } L^s(\Omega)^2\text{-strongly for all } s < 6\,, \quad \text{and} \\ \text{almost everywhere in } \Omega\,; \end{cases} \tag{7.6}$$

$$k_j \to k \begin{cases} \text{in } W_0^{1,r}(\Omega)\text{-weakly}\,, \quad \text{for all } r < 3/2 \\ \text{in } L^s(\Omega)\text{-strongly for all } s < 3\,, \quad \text{and} \\ \text{almost everywhere in } \Omega\,. \end{cases} \tag{7.7}$$

We can readily realize that (7.6) and (7.7) are not enough to achieve the strong convergence of u_j to u in $H^1(\Omega)$ (this is necessary in order to pass to the limit in the approximate problems). Here is where the monotone properties of $A(\cdot\,, k_j)$ comes into the scene.

7.3. *Strong convergence of the derivatives of the velocity field and conclusion*

Owing to corollary (5.1) and (7.6), we also have

$$Bu_j \to Bu \quad \text{in } V^q_{\rm b}(\Omega)\text{-weakly}\,, \tag{7.8}$$

and, it is easy to check that (7.6)–(7.8) imply that $(A(u_j, k_j))$ is bounded in $V^q_{\rm b}(\Omega)'$. Thus, for a subsequence,

$$A(u_j, k_j) \to \chi \quad \text{in } V^q_{\rm b}(\Omega)'\text{-weakly}\,. \tag{7.9}$$

From (7.5), we have $A(u_j, k_j) = \ell - Bu_j \to \ell - Bu$, and so $\chi = \ell - Bu$. Finally, taking $v = u_j$ in (7.5), it yields

$$\langle A(u_j, k_j), u_j\rangle = \langle \ell, u_j\rangle \to \langle \ell, u\rangle = \langle \ell - Bu, u\rangle\,. \tag{7.10}$$

But now, in order to deduce that $A(u, k) = \ell - Bu$, we cannot apply the monotonicity trick to the monotone operators $A_j = A(\cdot\,, k_j)$ (since they depend on j). To overcome this difficulty, we use the following result.

Lemma 7.1. *Let X be a Banach space, X' its dual and $A_j\colon X \mapsto X'$, $j \geq 1$, a sequence of monotone operators* [*see* (3.4)]. *Assume that the sequence $(u_j) \subset X$ fulfills the following conditions*:

(a) *$u_j \to u$ in X-weakly, for some $u \in X$;*
(b) *$A_j u_j \to \chi$ in X'-weakly, for some $\chi \in X'$;*
(c) $\langle A_j u_j, u_j\rangle \to \langle \chi, u\rangle$;
(d) *there exists an operator $A\colon X \mapsto X'$ such that*

$$\langle A_j v, u_j\rangle \to \langle Av, u\rangle \quad \textit{for all } v \in X\,;$$

(e) *the operator A is hemicontinuous, that is, for all $u, v, w \in X$ the mapping $t \in \mathbb{R} \mapsto \langle A(u + tv), w\rangle$ is continuous.*

Then $Au = \chi$.

Proof. Owing to the monotone character of A_j, we have, for all $v \in X$,

$$\langle A_j u_j, u_j\rangle - \langle A_j v, u_j\rangle - \langle A_j u_j - Av, v\rangle = \langle A_j u_j - A_j v, u_j - v\rangle \geq 0\,;$$

according to the assumptions (a)–(d), the passing to the limit yields

$$\langle \chi, u\rangle - \langle Av, u\rangle - \langle \chi - Av, v\rangle = \langle \chi - Av, u - v\rangle \geq 0\,, \quad \text{for all } v \in X\,;$$

putting $v = u \pm tw$, for $t > 0$ and $w \in X$, we obtain

$$\pm\langle \chi - A(u \pm tw), w\rangle \geq 0\,, \quad \text{for all } w \in X\,;$$

and letting $t \to 0^+$, using the hemicontinuity of A, we finally deduce $\langle \chi - Au, w\rangle = 0$ for all $w \in X$, that is $Au = \chi$. □

Remark 7.2. Taking $A_j = A$ for all $j \geq 1$, we find the standard monotonicity trick. Thus, Lemma 7.1 is a generalization of that result.

We can apply Lemma 7.1 to the sequence of monotone operators $A_j = A(\cdot\,,\,k_j)$ together with the sequence u_j. It is very easy to check that the limit operator A of Lemma 7.1 is $A(\cdot\,,\,k)$. It is enough to argue with (7.2) and use the almost everywhere convergence of the sequence k_j to k, also $A(\cdot\,,\,k)$ verifies conditions (d) and (e) of Lemma 7.1 (in fact, A is a continuous operator). Summing up, we have shown that $u \in V_{\mathrm{b}}^q(\Omega)$ is such that $A(u,k) + Bu = \ell$ in $V_{\mathrm{b}}^q(\Omega)'$. In particular, remembering (7.5), we have

$$\langle A(u_j, k_j) - A(u,k), v\rangle = -\langle Bu_j - Bu, v\rangle\,, \quad \text{for all } v \in V_{\mathrm{b}}^q(\Omega)\,; \tag{7.11}$$

now, we take $v = u_j - u$ in this expression. The right-hand side reads

$$-\langle Bu_j - Bu, u_j - u\rangle = \langle Bu_j, u\rangle + \langle Bu, u_j\rangle\,,$$

and by virtue of (7.8) and (7.6), we readily obtain that

$$\lim_{j\to\infty} \langle Bu_j - Bu, u_j - u\rangle = 0\,. \tag{7.12}$$

We turn to (7.11) for $v = u_j - u$:

$$\begin{aligned}&\langle A(u_j, k_j) - A(u, k_j), u_j - u\rangle\\ &\quad = -\langle A(u, k_j) - A(u,k), u_j - u\rangle\langle Bu_j - Bu, u_j - u\rangle\,,\end{aligned}$$

that is

$$\begin{aligned}&\int_\Omega \nu_1|\nabla(u_j - u)|^2 + \int_\Omega \nu_2\left|\frac{\partial(u_j - u)}{\partial z}\right|^2\\
&\quad\leq \int_\Omega (\nu_1 + \bar\nu_1(k_j))|\nabla(u_j - u)|^2 + \int_\Omega (\nu_2 + \bar\nu_2(k_j))\left|\frac{\partial(u_j - u)}{\partial z}\right|^2\\
&\qquad + \int_\Omega \left(\Phi_1(\nabla\cdot u_j) - \Phi_1(\nabla\cdot u)\right)(\nabla\cdot u_j - \nabla\cdot u)\\
&\qquad + \int_\Omega \left(\Phi_2(d_{\mathrm{b}}^{-1}u_j) - \Phi_2(d_{\mathrm{b}}^{-1}u)\right)(d_{\mathrm{b}}^{-1}u_j - d_{\mathrm{b}}^{-1}u)\\
&\quad = -\left\{\int_\Omega (\nu_1 + \bar\nu_1(k_j))\nabla u\nabla(u_j - u) + \int_\Omega (\nu_2 + \bar\nu_2(k_j))\frac{\partial u}{\partial z}\frac{\partial(u_j - u)}{\partial z}\right\}\\
&\qquad + \left\{\int_\Omega (\nu_1 + \bar\nu_1(k))\nabla u\nabla(u_j - u) + \int_\Omega (\nu_2 + \bar\nu_2(k))\frac{\partial u}{\partial z}\frac{\partial(u_j - u)}{\partial z}\right\}\\
&\qquad - \langle Bu_j - Bu, u_j - u\rangle\end{aligned}$$

and letting $j \to \infty$ we finally deduce that $u_j \to u$ in $H^1(\Omega)$-strongly. Now, we can pass to the limit in (7.5) and obtain that (u, k) is a solution to problem (7.4).

Finally, the pressure $p_s \in L^{q'}_{D,0}(\omega)$ is obtained as in Theorem 5.1. This ends the proof of Theorem 7.1.

Remark 7.3. If Φ_1 and Φ_2 verify (3.13), then it is easy to see that $u_j \to u$ in $V^q_b(\Omega)$-strongly.

Remark 7.4. The solution k obtained in Theorem 7.1 satisfies some more interesting properties. Indeed, it can be shown that k is a renormalized solution,[8] so that it also bears the following conditions:

(1) $T_j(k) \in H^1_0(\Omega)$, for all $j > 0$;
(2) $\lim_{j\to\infty} \frac{1}{j} \int_{|k|\le j} (|\nabla k|^2 + \left|\frac{\partial k}{\partial z}\right|^2) = 0$.

Remark 7.5. We may also consider the presence of convection terms of the form $u\nabla k + W(u)\frac{\partial k}{\partial z}$ in the equation for k. In this case, it can be shown that the existence result given in Theorem 7.1 still holds.

Acknowledgments

The author wishes to thank Prof. T. Chacón and Prof. F. Guillén for fruitful discussions and comments. This research was partially supported by Ministerio de Ciencia y Tecnología under DGICYT grant REN2000-1162-C02-01 MAR, and by Junta de Andalucía, research group FQM-315.

References

1. P. Azérad and F. Guillén González, Équations de Navier–Stokes en bassin peu profond: l'approximation hydrostatique, *C. R. Acad. Sci. Paris* **329**, Série I (1999) 961–966.
2. O. Besson and M. R. Laydi, Some estimates for the anisotropic Navier–Stokes equations and for the hydrostatic approximation, *Modél. Math. Anal. Numér.* **26** (1992) 855–865.
3. L. Boccardo, I. Díaz, D. Giachetti and F. Murat, *Existence of a Solution for a Weaker Form of a Nonlinear Elliptic Equation*, in Recent Advances in Nonlinear Elliptic and Parabolic Problems. Notes in Math., Vol. 208 (Longman, 1989).
4. L. Boccardo and T. Gallouët, Non-linear elliptic and parabolic equations involving measure data, *J. Funct. Anal.* **87** (1989) 149–169.
5. T. Chacón Rebollo, E. Chacón Vera and R. Lewandowski, Analysis of the hydrostatic approximation in oceanography with convection term, *Math. Model. Numer. Anal.* **34** (2000) 525–537.
6. T. Chacón Rebollo, M. Gómez Mármol and F. Ortegón Gallego, Estudio matemático de algunos modelos de turbulencia en oceanografía, in *Actas de las II Jornadas de Análisis de Variables y Simulación Numérica del Intercambio de Masas de Agua a través del Estrecho de Gibraltar*, eds. J. J. Alonso del Rosario and F. Ortegón Gallego, Servicio de Publicaciones de la Universidad de Cádiz, 2001, pp. 187–203.

7. T. Chacón Rebollo and F. Guillén González, An intrinsic analysis of existence of solutions for the hydrostatic approximation of Navier–Stokes equations, *C. R. Acad. Sci. Paris* **330**, Série I (2000) 841–846.
8. M. Gómez Mármol and F. Ortegón Gallego, Existencia y unicidad de solución renormalizada para la ecuación de Kolmogórov, in *Actas de la Jornada Científica en homenaje al Profesor Antonio Valle Sánchez*, Servicio de Publicaciones de la Universidad de Sevilla, 1997, pp. 167–183.
9. F. Guillén González, N. Masmoudi and R. Bellido, Anisotropic estimates and strong solutions of the primitive equations, *Diff. Integral Eqns.* **14** (2001) 1381–1408.
10. M. A. Krasnosel'skii, *Topological Methods in the Theory of Nonlinear Integral Equations* (Pergamon Press, 1964).
11. R. Lewandowski, *Analyse Mathématique et Océanographie* (Masson, 1997).
12. J. L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires* (Dunod, 1969).
13. J. L. Lions, R. Temam and S. Wang, On the equations of large scale ocean, *Nonlinearity* **5** (1992) 1007–1053.
14. F. Ortegón Gallego, On distributions independent of x_N in certain non-cylindrical domains and a de Rham lemma with a non-local constraint, *Nonlin. Anal.: Theory, Meth. Appl.* **59** (2004) 335–345.
15. J. Pedlosky, *Geophysical Fluid Dynamics* (Springer-Verlag, 1997).
16. E. A. H. Zuur, The use of a reformulation of the hydrostatic approximation Navier–Stokes equations for geophysical fluid problems, *Int. J. Numer. Meth. Fluids* **21** (1995) 549–570.