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Integral Equations and Operator Theory

The Volterra Operator is not Supercyclic

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Abstract. It is shown that the classical Volterra operator, which is cyclic, is not supercyclic on any of the spaces $L^p[0,1]$, $1 \le p < \infty$. This solves a question posed by Héctor Salas. This contrasts with the fact that the derivative operator, the left inverse of the Volterra operator, although unbounded, is hypercyclic on $L^p[0,1]$.

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1. Introduction

The space $L^p[0,1]$, $1 \le p < \infty$, will denote the Banach space of complex measurable functions f on $[0,1]$ for which the norm

$$\|f\|_p = \left(\int_0^1 |f(x)|^p \, dx \right)^{1/p}$$

is finite. For each $f \in L^p[0,1]$ the Volterra operator is defined by

$$Vf(x) = \int_0^x f(t) \, dt.$$

Clearly, V is bounded on each of the $L^p[0,1]$ spaces. It is also compact and quasinilpotent, (see [2], for instance). In addition, since the linear span of $\{V^n 1 = x^n/n!\}$ is dense in $L^p[0,1]$, $1 \le p < \infty$, the Volterra operator is cyclic with cyclic vector the constant function 1. Actually, the functions that are different from zero almost everywhere in a neighborhood of zero are cyclic vectors for V and V is unicellular. Indeed, its only invariant subspaces are $L^p[\beta,1]$ with $0 \le \beta \le 1$ (see the paper by Sarason [13] and [7, p. 199-200]). A bounded linear operator T acting on a Banach space $\mathcal{B}$ is said to be supercyclic if there is a vector f, also called

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supercyclic, such that the scalar multiples of the elements in the orbit $\{T^n f\}$ are dense in $\mathcal{B}$. Salas [12] asked whether the Volterra operator is supercyclic or not. Of course quasinilpotent operators can be supercyclic (see [12], for instance). In this note we prove

Main Theorem. *The Volterra operator is not supercyclic on any of the spaces* $L^p[0,1]$*,* $1 \leq p < \infty$.

Before proving the theorem above we will prove that the left inverse of the Volterra operator, that is, the derivative operator, which is only defined on a dense subset of $L^p[0,1]$, is hypercyclic. This means that the orbit of some vector under the operator, without the help of scalar multiples, is dense. This fact is in a strong contrast with the situation for invertible bounded operators. It is well known that an invertible operator is supercyclic (or hypercyclic) if and only if T^{-1} is (see [5]).

2. The derivative operator

The derivative operator D assigns to each function $f \in \mathcal{C}^1([0,1]) \subset L^p[0,1]$ its derivative $(Df)(x) = f'(x)$, which is in $L^p[0,1]$. We have the following Theorem

Theorem 2.1. *The derivative operator is hypercyclic on* $L^p[0,1]$*,* $1 \leq p < \infty$.

Proof. By a result of Mclane [8] there is an entire function f such that $\{D^n f\}$ is dense in $\mathcal{H}(\mathbb{C})$, the space of all entire functions endowed with the topology of uniform convergence on compact subsets. Obviously, for each n, the restriction of $D^n f$ to $[0,1]$ that we still denote by $D^n f$ is in $L^p[0,1]$, $1 \leq p < \infty$. Since the polynomials are dense in $L^p[0,1]$, $1 \leq p < \infty$, and each polynomial can be approximated in the norm $\|g\|_\infty = \max_{[0,1]} |g(x)|$ by $D^n f$ for some n, the result follows. □

Remark 2.2. We could have also proved Theorem 2.1 following the lines of the proof of Rolewicz [11] in which he showed that there are hypercyclic scalar multiples of the backward shift on ℓ^p, $1 \leq p < \infty$. Indeed, some of the results for bounded hypercyclic or supercyclic operators remain true for unbounded operators. For instance, while for bounded hypercyclic (supercyclic) operators there is a *residual* dense subset of hypercyclic (supercyclic) vectors, unbounded operators may only have just a (non-residual) dense subset of hypercyclic (supercyclic) vectors.

3. Proof of the Main Theorem

Proof of the Main Theorem. Since the topology of $L^p[0,1]$, $1 \leq p < \infty$, is stronger than the topology of $L^1[0,1]$ and the former spaces are dense in the latter, we may apply the comparison principle (see [12]) to conclude that it is enough to prove that V is not supercyclic on $L^1[0,1]$.

In addition, we claim that if V is supercyclic on $L^1[0,1]$, then so is on $L^2[0,1]$. To show this, suppose that f is a supercyclic vector for V acting on $L^1[0,1]$. Then Vf is a continuous function and, in particular, it is in $L^2[0,1]$. Now, we have

$$\|Vf\|_2 \le \|Vf\|_\infty \le \|f\|_1.$$

Therefore, V is a bounded operator from $L^1[0,1]$ into $L^2[0,1]$ and it clearly has dense range. Since the image of a dense set under a bounded operator with dense range is itself dense, it follows that $\{\lambda V^n f : n \ge 1 \text{ and } \lambda \in \mathbb{C}\}$ is dense on $L^2[0,1]$. Thus Vf is a supercyclic vector for V on $L^2[0,1]$ and the claim follows.

In summary, to obtain the statement of our main result it is enough to prove that V is not supercyclic on $L^2[0,1]$.

Now, upon performing a change of variables we may consider that the Volterra operator is defined on $L^2[-1,1]$, that is,

$$(Vf)(x) = \int_{-1}^{x} f(t)\,dt \qquad \text{for each } f \in L^2[-1,1].$$

The proof will be accomplished by applying the Angle Criterion (see [9] and [3]). Assume that f is supercyclic for V. We will find a function $g \in L^2[-1,1]$, with $\|g\|_2 = 1$, and a positive integer n_0 such that

$$\sup_{n \ge n_0} \frac{|\langle V^n f, g\rangle|}{\|V^n f\|_2} < 1, \tag{3.1}$$

which would contradict the supercyclicity of f.

In order to obtain (3.1), we will first obtain a lower estimate for $\|V^n f\|_2$. To this end, recall that the adjoint $V^\star$ of the Volterra operator is given by

$$(V^\star f)(x) = \int_{x}^{1} f(t)\,dt.$$

Now, we will use the Legendre polynomials. They are given by

$$P_n(x) = f_n^{n)}(x) \qquad\qquad n \ge 0,$$

where

$$f_n(x) = \frac{(x^2-1)^n}{2^n n!} \qquad\qquad n \ge 0,$$

(see [10, p. 162]). We set $h_n = (-1)^n f_n$. Since, for $0 \le k \le n-1$, the function $f_n^{k)}$ vanishes at 1, one easily checks that

$$(V^{\star n} P_n)(x) = h_n(x) \qquad\qquad n \ge 0.$$

Thus we have

$$\|V^n f\|_2 \ge |\langle V^n f, P_n\rangle| \|P_n\|_2^{-1} = |\langle f, V^{\star n} P_n\rangle| \|p_n\|_2^{-1} = |\langle f, h_n\rangle| \frac{\sqrt{2n+1}}{\sqrt{2}}. \tag{3.2}$$

See [10, p. 175] for the expression of the norm of the Legendre polynomials.

We claim that

$$\frac{(2n+1)!}{2^{n+1} n!} h_n \qquad (n \ge 0)$$

is a positive summability kernel, see [6, pp. 9-10]. Therefore, if f is a continuous function on $[-1,1]$, then

$$\frac{(2n+1)!}{2^{n+1}n!}\langle f, h_n\rangle = \frac{(2n+1)!}{2^{n+1}n!}\int_{-1}^{1} f(x)h_n(x)\,dt \to f(0) \quad \text{as} \quad n\to\infty. \tag{3.3}$$

It is clear that, $h_n(x) \geq 0$ for each $x \in [0,1]$ and for each $n \geq 0$. In addition, performing the change of variables $\varphi(t) = 2t-1$ in the second equality below the Beta function appears (see [10, p. 18-19]) and we find

$$\begin{aligned}
\frac{(2n+1)!}{2^{n+1}n!}\int_{-1}^{1} h_n(x)\,dx &= \frac{(2n+1)!}{2^{2n+1}(n!)^2}\int_{-1}^{1}(1-x^2)^n\,dx \\
&= \frac{(2n+1)!}{(n!)^2}\int_{0}^{1} t^n(1-t)^n\,dt \\
&= \frac{(2n+1)!}{(n!)^2}\beta(n+1,n+1) \\
&= 1.
\end{aligned}$$

It remains to prove that outside any interval $[-\delta,\delta]$, with $0<\delta<1$,

$$\frac{(2n+1)!}{2^{n+1}n!}\int_{[-1,1]\setminus(-\delta,\delta)} h_n(x)\,dx \to 0 \quad \text{as} \quad n\to\infty.$$

We have,

$$\begin{aligned}
\frac{(2n+1)!}{2^{n+1}n!}\int_{[-1,1]\setminus(-\delta,\delta)} h_n(x)\,dx &\leq (1-\delta^2)^n\frac{(2n+1)!}{2^{n+1}n!}\int_{[-1,1]\setminus(-\delta,\delta)}\frac{1}{2^n n!}\,dx \\
&< (1-\delta^2)^n\frac{(2n+1)!}{2^{n+1}n!}\int_{-1}^{1}\frac{1}{2^n n!}\,dx \\
&= (1-\delta^2)^n\frac{(2n+1)!}{2^{2n}(n!)^2}. \qquad (3.4)
\end{aligned}$$

Upon applying Stirling's formula one easily sees that

$$\frac{(2n+1)!}{2^{2n}(n!)^2\sqrt{n}} \to \frac{2}{\sqrt{\pi}} \qquad \text{as } n\to\infty.$$

Thus (3.4) tends to zero as $n\to\infty$.

Now, for fixed a, with $-1<a\leq -1/2$, we take the normalized characteristic function

$$g(x) = \frac{1}{\sqrt{1+a}}\chi_{[-1,a]}.$$

We have

$$(V^{\star n}g)(x) = \frac{(a-x)^n}{n!\sqrt{1+a}}\chi_{[-1,a]} \quad \text{and} \quad \|V^{\star n}g\|_2 = \frac{(1+a)^n}{n!\sqrt{2n+1}}.$$

Finally, upon replacing f by Vf, if necessary, we may suppose from the beginning that f is continuous. In addition, we claim that we can choose $f(0) \neq 0$. This follows from the facts that the set of supercyclic vectors is dense in $L^2[-1,1]$ and that V defined from $L^2[-1,1]$ into the space of continuous functions on $[-1,1]$ that vanish at -1 endowed with the supremum norm has dense range. An alternative argument is that if $(V^n f)(0) = 0$ for every n, then $V^n f$ would be orthogonal to the characteristic function $\chi_{[-1,0]}$ for every n, that would contradict the supercyclicity of f.

Now, we can apply the Angle Criterion. The first inequality bellow follows from (3.2), the second is Cauchy Schwarz inequality.

$$\frac{|\langle V^n f, g\rangle|}{\|V^n f\|_2} \leq \frac{|\langle f, V^{\star n} g\rangle|}{|\langle f, g_n\rangle| \frac{\sqrt{2n+1}}{\sqrt{2}}} \leq \frac{\|f\|_2 \|V^{\star n} g\|_2}{|\langle f, g_n\rangle| \frac{\sqrt{2n+1}}{\sqrt{2}}} = \frac{\frac{(2n+1)!}{2^{n+1} n!} \frac{(1+a)^n}{n! \sqrt{2n+1}} \|f\|_2}{\frac{(2n+1)!}{2^{n+1} n!} |\langle f, g_n\rangle| \frac{\sqrt{2n+1}}{\sqrt{2}}}.$$

Applying Stirlying's formula and (3.3), we see that the last quantity in the above display is of the same order as

$$\frac{(1+a)^n 2^{n+1/2} \sqrt{n}\, \|f\|_2}{\sqrt{\pi} |f(0)| (2n+1)}.$$

Since $1 + a \leq 1/2$, the last quantity goes to zero as $n \to \infty$. Therefore, f cannot be supercyclic; a contradiction. □

Remark 3.1. It cannot be used a sequence of functions like $h_n = \sin n\pi x$ to obtain a lower estimate for $\|V^n f\|_2$. Although $c_n V^{\star n} \sin n\pi x$ is again a summability kernel for an appropriate sequence $\{c_n\}$, it reproduces the value of f at -1. Thus we would have $(Vf)(-1) = 0$ for any $f \in L^2$, which makes impossible to control $\|V^n f\|_2$. The Legendre polynomials do not play any crucial role in the proof and it could be used other functions. But these polynomials come across in a natural way. Observe that $V^{\star n} 1 = (1-x)^n$, that attains its maximum at -1. If we multiply by $(1+x)^n$, we obtain $(1-x^2)^n$ that attains its maximum at 0. Then one looks for the preimage of $(1-x^2)^n$ under $V^{\star n}$. The proof can, of course, be carried out directly on L^1.

Remark 3.2. Since V and $V^\star$ are similar, the adjoint $V^\star$ is not supercyclic either.

Remark 3.3. Bourdon [1] proved that no hyponormal operator can be supercyclic. The Volterra operator is a perturbation by a one rank operator of a hyponormal operator (see Halmos Problem book [4]). The general question, posed by Salas [12], that remains unsolved, is whether a finite rank perturbation of a hyponormal operator can be supercyclic or not.

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References

[1] P. S. Bourdon, *Orbits of hyponormal operators* Mich. Math. J. **118** (1997), 345–353.
[2] J. B. Conway, *A Course in Functional Analysis*, Springer-Verlag, New York, 1985.
[3] E. A. Gallardo-Gutiérrez and A. Montes-Rodríguez, *The role of the angle in the supercyclic behavior* J. Funct Anal. (to appear).
[4] P. R. Halmos, *A Hilbert space problem book*, Van Nostrand Company, Inc., 1967.
[5] D. A. Herrero and C. Kitai, *On invertible hypercyclic operators* Proc. Amer. Math. Soc. **116** (1992), 873–875.
[6] Y. Katznelson, *An introduction to harmonic Analysis*, Dover Publication, Inc., New York 1976.
[7] B. Ya. Levin *Lecture on entire functions*, Translations of mathematical monographs, 150, American Mathematical Society, RI, Providence, 1996.
[8] G. R. Maclane, *Sequences of derivatives and normal families* J. Analyse Math. **2** (1952), 72–87.
[9] A. Montes-Rodríguez and H. N. Salas, *Supercyclic subspaces: spectral theory and weighted shifts*, Adv. Math. **163** (2001), 74–134.
[10] E. D. Rainville, *Special fuctions*, Chelsea publishing Company, New York, 1971.
[11] S. Rolewicz, *On orbits of elements* Studia Math. **32** (1969), 17–22.
[12] H. N. Salas, *Supercyclicity and weighted shifts*, Studia Math. **135** (1999), 55–74.
[13] D. Sarason, *A remark on the Volterra operator*, J. Math. Anal. Appl. **12** (1965) 244–246.

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