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Orthogonal expansions for the generalized Cramér-von Mises statistics

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Abstract. We provide a new proof for the representation of Cramér-von Mises statistics under (known) gamma and normal distributions. The new method uses orthogonal polynomials and provides an explicit form of the statistics from which the asymptotic distribution can be calculated.

Key words: Cramér-von Mises family, goodness of fit, normal distribution, gamma distribution, orthogonal polynomials

1 Introduction

In general, the term goodness of fit is associated with the statistical testing of hypothetical models with data. Examples of such tests abound and are to be found in most discussions on inference, least-squares theory and multivariate analysis. The classical test for the goodness of fit problem is the χ^2 test which is well adapted for the case when $F_0(x)$ represents a discrete distribution. Another important class of goodness of fit statistics are the EDF statistics, so called because they are measures of the discrepancy between the empirical distribution function (EDF) and a given distribution function. They are based on their vertical differences and they are conveniently divided in two classes, the supremum class (the Kolmogorov-Smirnov family) and the quadratic class (the Cramér-von Mises family). In D'Agostino and Stephens (1986) we can find a wide presentation and discussion of goodness of fit techniques.

In this work we will study certain statistics of the Cramér-von Mises family. If $F_n(x)$ is the empirical distribution function and $\psi(t)$ is some non-negative weight function, we consider the generalized Cramér-von Mises statistic:

$$W_n^2(\Psi) = n\int_{-\infty}^{\infty} (F_n(x) - F_0(x))^2 \Psi(F_0(x)) dF_0(x), \tag{1}$$

that includes for different weight functions among others the Cramér-von Mises statistic ($\Psi = 1$), the Anderson Darling statistic ($\Psi(t) = \{t(1-t)\}^{-1}$), etc. When F_0 is completely specified and continuous these statistics are distribution free. Their asymptotic distributions have been obtained in the literature using the theory of empirical processes and the technique of Kac and Siegert (1947), see Durbin (1973) as well as using an approach based on U-statistics, see Gregory (1977).

Pettitt (1978) studied generalized Cramér-von Mises statistics for testing the gamma distribution with known parameters. He showed that the statistics can be written as an infinite sum of uncorrelated components which are polynomial functions of the original function and studied its asymptotic distribution by a method that is analogous to the technique of Kac and Siegert and requiring the solution of integral equations and the theory of differential equations. In a similar way it is studied the case for the normal distribution, see Pettitt (1977) and Gregory (1977).

We will derive the same expressions for the generalized Cramér-von Mises statistics under the null hypothesis in case of gamma and normal by another procedure. This is contained in sections 2 and 3, respectively . The method that we propose consists of expanding the empirical distribution in terms of the orthogonal polynomials with respect to a convenient density function. Hence, we determine certain constants, not given in the literature before, that are necessary to derive their asymptotic distributions and apply these tests.

We also get, in section 4, using the same method a well-known expression for the asymptotic Cramér-von Mises statistic as a weighted infinite sum of χ_1^2 independent variables. This expression was derived, among others, by Durbin and Knott (1972) who used empirical process approaches.

2 Generalized Cramér-von Mises statistics for the gamma distribution

Let $X_1, X_2, \ldots, X_n$ be iid observations from a population with continuous distribution function $F(x)$. To test the null hypothesis that $F(x) = F_0(x)$, where $F_0(x)$ is completely known, the Cramér-von Mises statistic given in (1) can be used.

In this section, we consider F_0 as the gamma distribution with known parameters, α and $1/\theta$, denoted $G_\alpha(x, 1/\theta)$. So we will consider the null hypothesis $H_0 : F(x) = G_\alpha(x/\theta)$ with G_α the gamma distribution with shape parameter α and scale parameter 1, whose density function is

$$g_\alpha(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)} e^{-x}, x > 0.$$

Obviously the density function of $G_\alpha(x, 1/\theta)$, denoted $g_\alpha(x, 1/\theta)$ verifies that $g_\alpha(x, 1/\theta) = 1/\theta g_\alpha(x/\theta)$.

The weight function Ψ that we consider in (1) was suggested by De Wet and Venter (1973), i.e.

$$\Psi(t) = \left[G_\alpha^{-1}(t)(g_\alpha(G_\alpha^{-1}(t)))^2\right]^{-1}.$$

The asymptotic distribution of $W_n^2(\Psi)$ has been studied by diverse authors, De Wet and Venter (1973), Pettitt (1978) using the classic procedure of partitioning the statistic into principal components, see Durbin and Knott (1972), and in a similar way to Kac and Siegert (1947).

We obtain the same result expanding the expression under the integral sign of (1) in terms of the generalized Laguerre polynomials. These polynomials are orthogonal with respect to the gamma density function. In the next proposition we give some of the properties that these polynomials verify.

Proposition 2.1 *Let* $\mathrm{L}_k^{(\alpha)}$ *be the kth generalized Laguerre polynomial that is given by the following expression:*

$$\mathrm{L}_k^{(\alpha)} = \sum_{m=0}^{k} \binom{k+\alpha}{k-m} \frac{(-x)^m}{m!}, \alpha > 0.$$

The generalized Laguerre polynomials have the following properties:

1. *Recurrence relationship:*
$$j\mathrm{L}_j^{(\alpha)}(x) = (2j+\alpha-1-x)\mathrm{L}_{j-1}^{(\alpha)}(x) - (j+\alpha-1)\mathrm{L}_{j-2}^{(\alpha)}(x), \quad j \geq 1,$$
$$\mathrm{L}_{-1}^{(\alpha)}(x) = 0, \quad \mathrm{L}_0^{(\alpha)}(x) = 1.$$

2. *Orthogonality relationship:*
$$\int_0^\infty \mathrm{L}_k^{(\alpha)}(x)\mathrm{L}_j^{(\alpha)}(x)x^\alpha e^{-x}dx = \delta_{k,j}\frac{\Gamma(k+\alpha+1)}{k!}, \tag{2}$$
with $\delta_{k,j}$ *the Kronecker delta.*

3. *Rodrigues formula:*
$$\mathrm{L}_k^{(\alpha)}(x) = \frac{x^{-\alpha}e^x}{k!}\frac{d^k}{dx^k}(x^{k+\alpha}e^{-x}). \tag{3}$$

4. *Differentiation rule:*
$$\frac{d}{dx}\mathrm{L}_k^{(\alpha)}(x) = -\mathrm{L}_{k-1}^{(\alpha+1)}(x). \tag{4}$$

The above properties can be found in Chihara (1978).

The following result gives us an expression of the generalized Cramér-von Mises statistics for the gamma distribution.

Theorem 2.1 *Under* H_0 *the statistics* $W_n^2(\Psi)$ *given in (1) has the following expansion:*

$$W_n^2(\Psi) = \sum_{j=1}^{\infty} \frac{Z_{n,j}^2}{j}, \tag{5}$$

with

$$Z_{n,j} = \left(\frac{j!\Gamma(\alpha)}{\Gamma(\alpha+j)}\right)^{1/2} \sum_{i=1}^{n} \frac{\mathrm{L}_j^{(\alpha-1)}(X_i/\theta)}{\sqrt{n}}. \tag{6}$$

Proof. Let

$$y_n(x) = \sqrt{n}(F_n(x) - F_0(x)), \tag{7}$$

which we want to expand in terms of the Laguerre polynomials. As

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I_{(-\infty,x]}(X_i), \tag{8}$$

in the first place we will get an expansion for the indicator function $I_{(0,x]}(X_i)$ with X_i a variable from $F_0(\cdot)$. Since it is a bounded function it admits the following Fourier expansion:

$$I_{(0,x]}(s) = \sum_{k=0}^{\infty} a_k(x)\mathrm{L}_k^{(\alpha-1)}\left(\frac{s}{\theta}\right). \tag{9}$$

The Fourier coefficients are:

$$\begin{aligned} a_k(x) &= \frac{\int_0^x \mathrm{L}_k^{(\alpha-1)}(\frac{s}{\theta})g_\alpha(s,1/\theta)ds}{\left\|\mathrm{L}_k^{(\alpha-1)}(\frac{s}{\theta})\right\|_2^2}, k \geq 1, \\ a_0(x) &= F_0(x), \end{aligned} \tag{10}$$

with

$$\begin{aligned} \left\|\mathrm{L}_k^{(\alpha-1)}\left(\frac{s}{\theta}\right)\right\|_2^2 &= \int_0^\infty \left(\mathrm{L}_k^{(\alpha-1)}\left(\frac{s}{\theta}\right)\right)^2 g_\alpha(s,1/\theta)ds \\ &= \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)k!}, \end{aligned} \tag{11}$$

which is deduced from (2).

To determine the numerator of (10), we use the Rodrigues fomula given in (3),

$$\int_0^x \mathrm{L}_k^{(\alpha-1)}\left(\frac{s}{\theta}\right)g_\alpha(s,1/\theta)ds = \frac{(k-1)!}{k!\Gamma(\alpha)}(x/\theta)^\alpha e^{-x/\theta}\mathrm{L}_{k-1}^{(\alpha)}(x/\theta), k \geq 1. \tag{12}$$

So, from (12) and (11):

$$a_k(x) = \frac{(k-1)!}{\Gamma(k+\alpha)}(x/\theta)^\alpha e^{-x/\theta}\mathrm{L}_{k-1}^{(\alpha)}(x/\theta), k \geq 1. \tag{13}$$

The indicator function in terms of Laguerre polynomials will be obtained by substituting (13) in (9):

$$I_{(0,x]}(s) = F_0(x) + (x/\theta)^\alpha e^{-x/\theta}\sum_{k=1}^{\infty}\frac{(k-1)!}{\Gamma(k+\alpha)}\mathrm{L}_{k-1}^{(\alpha)}\left(\frac{x}{\theta}\right)\mathrm{L}_k^{(\alpha-1)}\left(\frac{s}{\theta}\right). \tag{14}$$

In this way we get the following expansion for the empirical distribution function from (8) and (14):

$$F_n(x) = F_0(x) + \left(\frac{x}{\theta}\right)^\alpha e^{-x/\theta}\sum_{k=1}^{\infty}\frac{(k-1)!}{\Gamma(k+\alpha)}\mathrm{L}_{k-1}^{(\alpha)}\left(\frac{x}{\theta}\right)\sum_{i=1}^{n}\frac{\mathrm{L}_k^{(\alpha-1)}(X_i/\theta)}{n}.$$

As a consequence from (7) it follows that

$$y_n(x) = \left(\frac{x}{\theta}\right)^\alpha e^{-x/\theta}\sum_{k=1}^{\infty}\frac{(k-1)!}{\Gamma(k+\alpha)}\mathrm{L}_{k-1}^{(\alpha)}\left(\frac{x}{\theta}\right)\sum_{i=1}^{n}\frac{\mathrm{L}_k^{(\alpha-1)}(X_i/\theta)}{\sqrt{n}}.$$

Obviously, $[u = x/\theta]$:

$$W_n^2(\Psi) = \int_{-\infty}^{\infty} (y_n(x))^2 \Psi(F_0(x)) dF_0(x)$$
$$= \int_0^{\infty} \left\{ \sum_{k=1}^{\infty} \frac{\Gamma(\alpha)(k-1)!}{\Gamma(k+\alpha)} \mathrm{L}_{k-1}^{(\alpha)}(u) \sum_{i=1}^{n} \frac{\mathrm{L}_k^{(\alpha-1)}(X_i/\theta)}{\sqrt{n}} \right\}^2 \frac{u^\alpha e^{-u}}{\Gamma(\alpha)} du.$$

Since $\left\{ \mathrm{L}_{k-1}^{(\alpha)}(u) \right\}_{k \geq 1}$ are orthogonal with respect to the function: $u^\alpha e^{-u}$ and due to the relationship (2), we have easily (5), what we wanted to show, with $Z_{n,k}$ given in (6). ■

The random variables $Z_{n,k}$ have the following properties as a consequence of the orthogonality of the Laguerre polynomials:

- $E(Z_{n,k}) = 0$, $\mathrm{var}(Z_{n,k}) = 1$.
- They are uncorrelated variables.
- By the central limit theorem we have:

$$Z_{n,k} \xrightarrow{\mathscr{L}} Z_k, n \to \infty, \tag{15}$$

with Z_k iid random variables $N(0, 1)$.

We are interested in studying the asymptotic behaviour of $W_n^2(\Psi)$. It can be shown that $W_n^2(\Psi)$ given in (5) tends to infinity as $n \to \infty$, see De Wet and Venter (1973), Gregory (1980), and so as suggested Pettitt (1978) the statistic has to be recentered to obtain a limiting distribution.

It is straightforward to show that $W_n^2(\Psi)$ has infinite mean but using the results of Gregory (1977) it is possible to show there exist constants $\{\mu_n\}$ so that

$$\lim_{n\to\infty} (W_n^2(\Psi) - \mu_n) \stackrel{d}{=} \sum_{j=1}^{\infty} \frac{Z_j^2 - 1}{j}, \tag{16}$$

with the distribution of the left hand side of (16) tabulated in De Wet and Venter (1972).

In the literature these constants are only mentioned, they are not explicitly known. In the next, we explicitly determine these constants in a recursive way, using the results given by Gregory (1977) and the weak law of large numbers, see Feller (1966), page 232.

By the weak law of large numbers

$$\mu_n = \int_{-n}^{n} Y_1(s) dF_0(s), \tag{17}$$

with

$$Y_1(s) = \int_{-\infty}^{\infty} \left[I_{(0,x]}(s) - F_0(x) \right]^2 \Psi(F_0(x)) dF_0(x). \tag{18}$$

To calculate the integral (18) we use the expression in terms of Laguerre polynomials for the indicator function given in (9). It follows that

$$Y_1(s) = \sum_{k=1}^{\infty} \frac{1}{k} \frac{\Gamma(\alpha) k!}{\Gamma(\alpha + k)} \left(\mathrm{L}_k^{(\alpha-1)} \left(\frac{s}{\theta} \right) \right)^2,$$

so

$$\mu_n = \sum_{k=1}^{\infty} \frac{1}{k} \frac{k!}{\Gamma(\alpha+k)} I_k(\alpha-1, n/\theta), \tag{19}$$

where

$$I_k(\alpha-1, n) = \int_0^n \left(\mathrm{L}_k^{(\alpha-1)}(x) \right)^2 x^{\alpha-1} e^{-x} dx. \tag{20}$$

Using the relation (4) it is easy to show that:

$$\begin{aligned} I_k(\alpha-1, n) &= \frac{n^\alpha e^{-n}}{k} \mathrm{L}_k^{(\alpha-1)}(n) \mathrm{L}_{k-1}^{(\alpha)}(n) + \frac{1}{k} I_{k-1}(\alpha, n), k \geq 1, \\ I_0(\alpha-1+k, n) &= \Gamma(\alpha+k) G_{\alpha+k}(n). \end{aligned} \tag{21}$$

So we have obtained the constants μ_n that are computable from the recurrence relation (19).

3 The generalized Cramér-von Mises statistic for the normal distribution

In this section, we consider F_0 as the standard normal distribution, denoted Φ, whose density function is given by

$$\varphi(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}, x \in \mathbb{R}.$$

So we will consider the null hypothesis $H_0 : F(x) = \Phi(x)$, and the generalized Cramér-von Mises statistic with the following weight function, suggested by Gregory (1977):

$$\Psi(t) = \left[\varphi\left(\Phi^{-1}(t) \right) \right]^{-2}.$$

We obtain the analogous result expanding the empirical distribution function in terms of the Hermite polynomials.

Some useful properties of Hermite polynomials are given in the following proposition:

Proposition 3.1 *Let* H_{e_k} *be the kth Hermite polynomial that is defined by:*

$$\mathrm{H}_{e_k}(x) = k! \sum_{m=0}^{[k/2]} \frac{(-1)^m x^{k-2m}}{2^m (k-2m)! m!}, \tag{22}$$

The Hermite polynomials verify the following relations, see Chihara (1978):

[1] Recurrence relation:

$$\mathrm{H}_{e_{k+1}}(x) = x \mathrm{H}_{e_k}(x) - k \mathrm{H}_{e_{k-1}}(x), \quad k \geq 0,$$
$$\mathrm{H}_{e_{-1}}(x) = 0, \qquad \mathrm{H}_{e_0}(x) = 1.$$

[2] Orthogonality:

$$\int_{-\infty}^{\infty} \mathrm{H}_{e_k}(x) \mathrm{H}_{e_j}(x) \varphi(x) dx = \delta_{k,j} k!. \tag{23}$$

[3] Rodrigues formula:

$$\mathrm{H}_{e_k}(x) = (-1)^k e^{x^2/2} \frac{d^k}{dx^k} e^{-x^2/2}. \tag{24}$$

[4] Differentiation rule:

$$\frac{d}{dx}\mathrm{H}_{e_k}(x) = k\mathrm{H}_{e_{k-1}}(x).$$

An expression for the generalized Cramér-von Mises statistic for the normal distribution will be given in the following theorem.

Theorem 3.1 *The statistic $W_n^2(\Psi)$ given in (1), with the above notation, admits the following expression under the null hypothesis:*

$$W_n^2(\Psi) = \sum_{k=1}^{\infty} \frac{Z_{n,k}^2}{k}, \tag{25}$$

with

$$Z_{n,k} = -\sum_{i=1}^{n} \frac{\mathrm{H}_{ek}(X_i)}{\sqrt{k!}\sqrt{n}}. \tag{26}$$

Proof. The proof is analogous to the above case. ■

Hence the random variables $Z_{n,k}$ verify the same properties of those in the gamma case. Of course the limiting distribution of $W_n^2(\Psi)$ presents the same problem, so it is necessary to determine the constants μ_n, so that

$$\lim_{n\to\infty}\left(W_n^2(\Psi) - \mu_n\right) \stackrel{d}{=} \sum_{j=1}^{\infty} \frac{Z_j^2 - 1}{j}.$$

Again, these constants are not known explicitely in the literature, but we can obtain them using similar arguments as in the gamma case.

So, from (17) we have

$$\mu_n = \sum_{k=1}^{\infty} \frac{1}{kk!} I_k(n), \tag{27}$$

with

$$I_k(n) = \int_{-n}^{n} H_{ek}^2(x)\varphi(x)dx,$$

that satisfies

$$I_k(n) = -2H_{ek}(n)H_{ek-1}(n)\varphi(n) + kI_{k-1}(n), k \geq 1,$$
$$I_0(n) = 2\Phi(n) - 1.$$

4. The Cramér-von Mises statistic

In this section we get, using the same technique, a very well-known expression for the asymptotic Cramér-von Mises statistic. The classic Cramér-von Mises statistic is obtained from (1) with $\Psi = 1$. It has been studied by many authors, such as Anderson and Darling (1952), Durbin and Knott (1972). They expressed this statistic as a weighted infinite sum of χ_1^2 variables, using the stochastic process theory. We will get the same result expanding the empirical distribution function in terms of Jacobi polynomials.

We present in the following result some properties of these polynomials, see Chihara (1978):

Proposition 4.1 *Consider the Jacobi polynomials explicitly for* $a, b > -1$,*:*

$$P_n^{(a,b)}(x) = 2^{-n} \sum_{j=0}^{n} \binom{n+a}{j} \binom{n+b}{n-j} (x-1)^{n-j}(x+1)^j, n \geq 0, \tag{28}$$

These polynomials satisfy the following properties:

1. *Rodrigues formula:*

$$P_n^{(a,b)}(x) = \frac{(-1)^n}{2^n n!} (1-x)^{-a}(1+x)^{-b} \frac{d^n}{dx^n} \left[(1-x)^{n+a}(1+x)^{n+b} \right]. \tag{29}$$

2. *Orthogonality:*

$$\int_{-1}^{1} P_n^{(a,b)}(x) P_m^{(a,b)}(x) w_{(a+1,b+1)}(x) dx = \delta_{n,m} \frac{(a+1)_n (b+1)_n (n+a+b+1)}{n!(a+b+2)_n (2n+a+b+1)}, \tag{30}$$

with $w_{(a+1,b+1)}(x)$ *the beta density with parameters* $a+1, b+1$, *in the interval* $(-1, 1)$ *given by*

$$w_{(a+1,b+1)}(x) = \frac{2^{-1-a-b}}{\beta(a+1, b+1)} (1-x)^a (1+x)^b, x \in (-1, 1), \tag{31}$$

$(a)_0 = 1, (a)_n = a(a+1)\ldots(a+n-1)$ *and* $\delta_{n,m}$ *the Kronecker delta.*

3. *Special case: for* $a = b = -1/2$ *the right hand side of (30) is*

$$\frac{n \left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n}{n!(1)_n 2n} = \frac{1}{2} \left\{ \frac{1}{4^n} \binom{2n}{n} \right\}^2, \tag{32}$$

and for $a = b = 1/2$*:*

$$\frac{(n+2)\left(\frac{3}{2}\right)_n \left(\frac{3}{2}\right)_n}{n!(3)_n (2n+2)} = \left[\frac{1}{4^n} \binom{2n+1}{n} \right]^2. \tag{33}$$

Consider the classical Cramér-von Mises statistic:

$$W_n^2 = n \int_{-\infty}^{\infty} (F_n(x) - F_0(x))^2 dF_0(x). \tag{34}$$

As another application of the technique that we present in this work, the following theorem shows an expansion for the statistic W_n^2 as well as for the asymptotic statistic denoted W_∞^2.

Theorem 4.1 *Under the null hypothesis the statistics W_n^2 satisfies:*

$$W_n^2 = \frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{Z_{n,k}^2}{k^2}, \tag{35}$$

with

$$Z_{n,k} = \sqrt{\frac{2}{n}}\sum_{i=1}^{n}\frac{P_k^{(-1/2,-1/2)}(X_i)4^k}{\binom{2k}{k}}. \tag{36}$$

Hence

$$W_\infty^2 = \frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{Z_k^2}{k^2}, \tag{37}$$

where the Z_k are iid random variables $N(0,1)$.

Proof. As it is known the Cramér-von Mises statistic is distribution free when the distribution function F_0 is continuous. So we can consider $F_0(x) = F_{1/2,1/2}(x)$ the beta distribution with parameters $1/2, 1/2$, $(a = b = -1/2)$.

In a similar way to the above cases, we expand the indicator function in terms of the Jacobi polynomials:

$$I_{(-1,x]}(s) = \sum_{k\geq 0}\alpha_k(x)P_k^{(-1/2,-1/2)}(s),$$

with

$$\begin{aligned}\alpha_k(x) &= 2\left\{\frac{4^k}{\binom{2k}{k}}\right\}^2\int_{-1}^{x}P_k^{(-1/2,-1/2)}(s)w_{(1/2,1/2)}(s)ds = \\ &= -\frac{16^k}{\binom{2k}{k}^2}\frac{1}{k\beta(1/2,1/2)}(1-x)^{1/2}(1+x)^{1/2}P_{k-1}^{(1/2,1/2)}(x),\end{aligned} \tag{38}$$

for $k \geq 1$ and $\alpha_0(x) = F_{(1/2,1/2)}(x)$.

As a consequence, we have an orthogonal expansion for the empirical distribution function:

$$\begin{aligned}F_n(x) =& F_{(1/2,1/2)}(x) - \frac{(1-x)^{1/2}(1+x)^{1/2}}{\beta(1/2,1/2)}\cdot \\ &\cdot\sum_{k=1}^{\infty}\frac{16^k}{\binom{2k}{k}^2}\frac{1}{k}P_{k-1}^{(1/2,1/2)}(x)\sum_{i=1}^{n}\frac{P_k^{(-1/2,-1/2)}(X_i)}{n}.\end{aligned}$$

In this way

$$\sqrt{n}\left(F_n(x) - F_{(-1/2,-1/2)}(x)\right) = -\frac{(1-x)^{1/2}(1+x)^{1/2}}{\beta(1/2,1/2)} \cdot \cdot \sum_{k=1}^{\infty} \frac{4^k}{\binom{2k}{k}} \frac{1}{\sqrt{2}k} P_{k-1}^{(1/2,1/2)}(x) Z_{n,k}, \quad (39)$$

with the $Z_{n,k}$ given in (36) having the same properties as the variables above: they have null mean, unit variance, they are uncorrelated and by the central limit theorem, satisfy

$$Z_{n,k} \xrightarrow{\mathscr{L}} Z_k, n \to \infty,$$

with Z_k iid $N(0,1)$.

Following with the proof, from (39) we easily get

$$W_n^2 = \frac{1}{\beta^3(1/2,1/2)} \int_{-1}^{1} \left\{ \sum_{k=1}^{\infty} \frac{4^k}{\binom{2k}{k}} \frac{1}{\sqrt{2}k} Z_{n,k} P_{k-1}^{(1/2,1/2)}(x) \right\}^2 \cdot \cdot (1-x)^{1/2}(1+x)^{1/2} dx.$$

Since the Jacobi polynomials $\left\{P_{k-1}^{(1/2,1/2)}\right\}_{k\geq 1}$ are orthogonal with respect to the weight function $w_{(3/2,3/2)}$,

$$w_{(3/2,3/2)} = \frac{2^{-2}}{\beta(3/2,3/2)}(1-x)^{1/2}(1+x)^{1/2},$$

we arrive at the orthogonality relation given in (33). So

$$W_n^2 = \frac{4\beta(3/2,3/2)}{\beta^3(1/2,1/2)} \sum_{k=1}^{\infty} \frac{16^k}{\binom{2k}{k}^2} \frac{\binom{2k-1}{k-1}^2}{2k^2 16^{k-1}} Z_{n,k}^2$$

$$= \frac{1}{\pi^2} \sum_{k=1}^{\infty} \frac{Z_{n,k}^2}{k^2}.$$

Taking limit in the above expression when $n \to \infty$ we obtain (37). ■

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