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On distributions independent of x_N in certain non-cylindrical domains and a de Rham lemma with a non-local constraint

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Abstract

We study distributions not depending on the x_N-variable in an open set $\Omega \subset \mathbb{R}^N$. It is assumed that Ω may be described through a very general function $D: \omega \mapsto \mathbb{R}$, where $\omega \subset \mathbb{R}^{N-1}$ is any open set. We give a representation theorem for this kind of distributions and show how they are related to distributions defined in ω. A direct application of this theorem is the derivation of a de Rham-like lemma with a non-local constraint. These results can be applied to the analysis of hydrostatic approximation of Navier–Stokes equations.

Keywords: Distributions; Non-local constraint; Hydrostatic approximation; de Rham's lemma; Sobolev spaces

1. Introduction

A distribution $F \in \mathscr{D}'(\mathbb{R}^N)$ is said to be independent of x_i, $1 \leqslant i \leqslant N$, $(x_1, \ldots, x_N) \in \mathbb{R}^N$, if $\partial F/\partial x_i = 0$ in $\mathscr{D}'(\mathbb{R}^N)$. The structure of distributions independent of one of the variables is well-known since the time they were introduced by Schwartz at the end of the 1940s [12]. Indeed, if $F \in \mathscr{D}'(\mathbb{R}^N)$ does not depend on, say, the last variable x_N, then there exists $S \in \mathscr{D}'(\mathbb{R}^{N-1})$ such that $F = S \otimes \mathbf{1}_{x_N}$, i.e. for all $\varphi \in \mathscr{D}(\mathbb{R}^{N-1})$, for all $\psi \in \mathscr{D}(\mathbb{R})$,

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and $\phi(x_1,\ldots,x_{N-1},x_N)=\varphi(x_1,\ldots,x_{N-1})\psi(x_N)$, the following equality holds:

$$\langle F,\phi\rangle_{\mathscr{D}'(\mathbb{R}^N)\mathscr{D}(\mathbb{R}^N)}=\langle S,\varphi\rangle_{\mathscr{D}'(\mathbb{R}^{N-1})\mathscr{D}(\mathbb{R}^{N-1})}\int_{\mathbb{R}}\psi(x_N)\,\mathrm{d}x_N$$

and thus, according to Fubini's theorem for distributions, for all $\varphi\in\mathscr{D}(\mathbb{R}^N)$ one has

$$\langle F,\varphi\rangle_{\mathscr{D}'(\mathbb{R}^N)\mathscr{D}(\mathbb{R}^N)}=\left\langle S,\int_{\mathbb{R}}\varphi(x_1,\ldots,x_{N-1},x_N)\,\mathrm{d}x_N\right\rangle_{\mathscr{D}'(\mathbb{R}^{N-1})\mathscr{D}(\mathbb{R}^{N-1})}.$$

This expression also applies to open cylinders $\Omega=\omega\times(a,b)\subset\mathbb{R}^N$, where $\omega\subset\mathbb{R}^{N-1}$ is an open domain and $-\infty\leqslant a<b\leqslant+\infty$: if $F\in\mathscr{D}'(\Omega)$ does not depend on $x_N\in(a,b)$, then there exists $S\in\mathscr{D}'(\omega)$ such that, for all $\varphi\in\mathscr{D}(\Omega)$, one has [8]

$$\langle F,\varphi\rangle_{\mathscr{D}'(\Omega)\mathscr{D}(\Omega)}=\left\langle S,\int_a^b\varphi(x_1,\ldots,x_{N-1},x_N)\,\mathrm{d}x_N\right\rangle_{\mathscr{D}'(\omega)\mathscr{D}(\omega)}. \tag{1}$$

We observe that expression (1) relates a distribution F in the cylinder Ω with a distribution S defined on the base of the cylinder ω under the only assumption that F is independent of the variable describing the axis of the cylinder.

The goal of this paper is to give a version of this result to a more general class of open sets, i.e.

$$\Omega=\{(x',x_N)\in\mathbb{R}^N/x'\in\omega,-D(x')<x_N<0\}, \tag{2}$$

where $\omega\subset\mathbb{R}^{N-1}$ is an arbitrary open domain, $D:\omega\mapsto\mathbb{R}$, $x'=(x_1,\ldots,x_{N-1})$, and $D(x')>0$, for all $x'\in\omega$, and to apply it to the derivation of a de Rham-like lemma which characterizes all distributions vanishing on the space $\mathscr{V}(\Omega)\subset\mathscr{D}(\Omega)^{N-1}$ defined as

$$\mathscr{V}(\Omega)=\left\{\varphi\in\mathscr{D}(\Omega)^{N-1}/\nabla'\cdot\left(\int_{-D(x')}^0\varphi(x',x_N)\,\mathrm{d}x_N\right)=0\text{ in }\omega\right\}, \tag{3}$$

where $\nabla'\cdot$ is the divergence operator with respect to the variables x_1 to x_{N-1}. This kind of results can be applied in the study of some mathematical models arising in oceanography, for instance, in the analysis of the hydrostatic approximation of Navier–Stokes equations [9,10]; in this case, $N=3$, Ω stands for a portion of the ocean or lake, $\omega\times\{0\}$ is the sea surface, and D is the depth function describing the bathymetry of the sea bottom.

2. The representation theorem

We consider $\omega\subset\mathbb{R}^{N-1}$ an arbitrary open domain, and a measurable function $D:\omega\mapsto\mathbb{R}$ such that the set $\Omega\subset\mathbb{R}^N$ as defined in (2) is open. We will make the following assumption on D:

$$D(x')>0\ \text{in}\ \omega,\quad\text{and for every compact set } K\subset\omega,\ \text{ess}\inf_K D>0. \tag{4}$$

Remark. In this setting, the function D need not be continuous in ω. Indeed, it may have jumps along some curves in ω.

Remark. Notice that if $\text{ess inf}_\omega D > 0$, then Ω has a sidewall all along $\partial\omega$. Under this additional assumption, the analysis below simplifies. This was the case considered in [9] in the analysis of the functional spaces involved in the study of the hydrostatic approximation. Observe that (4) does not exclude the case $\text{ess inf}_\omega D = 0$, leading to more general situations.

Let $u \in L^1(\Omega)$. Owing to Fubini's theorem, the function

$$x' \in \omega \mapsto \int_{-D(x')}^{0} u(x', x_N)\, \mathrm{d}x_N$$

is defined almost everywhere in ω and belongs to $L^1(\omega)$. This allows us to introduce the operator $M : L^1(\Omega) \mapsto L^1(\omega)$:

$$M(u)(x') = \int_{-D(x')}^{0} u(x', x_N)\, \mathrm{d}x_N. \tag{5}$$

Obviously, if $\varphi \in \mathscr{D}(\Omega) = \{\phi \in \mathscr{C}^\infty(\Omega)/\text{supp}\, \phi \text{ is compact in } \Omega\}$, then we have $M(\varphi) \in \mathscr{D}(\omega)$.

Now we are ready to state the representation theorem.

Theorem 1. *Assume hypothesis* (4). *Then, the following assertions are equivalent*:

(1) $F \in \mathscr{D}'(\Omega)$ *and* $\partial F/\partial x_N = 0$.
(2) *There exists a unique distribution* $S \in \mathscr{D}'(\omega)$ *such that*

$$\langle F, \varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} = \langle S, M(\varphi)\rangle_{\mathscr{D}'(\omega),\mathscr{D}(\omega)}, \quad \textit{for all } \varphi \in \mathscr{D}(\Omega). \tag{6}$$

Proof. First assume assertion 2. It is straightforward that expression (6) defines a distribution $F \in \mathscr{D}'(\Omega)$. To see that $\partial F/\partial x_N = 0$, observe that for all $\phi \in \mathscr{D}(\Omega)$ one has $M(\partial\phi/\partial x_N) = 0$ in ω. Consequently, (6) gives

$$0 = \left\langle F, \frac{\partial\phi}{\partial x_N}\right\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} = -\left\langle \frac{\partial F}{\partial x_N}, \phi\right\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)},$$

i.e. $\partial F/\partial x_N = 0$ in $\mathscr{D}'(\Omega)$.

The property $1 \Rightarrow 2$ is the tricky part, so we divide its proof into three steps.

Step 1: There exists $S \in \mathscr{D}'(\omega)$ such that for any compact set $K \subset \omega$

$$\begin{cases} \langle F, \varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} = \langle S, M(\varphi)\rangle_{\mathscr{D}'(\omega),\mathscr{D}(\omega)}, & \text{for all } \varphi \in \mathscr{D}(\Omega), \\ \text{with supp}\, \varphi \subset \overset{\circ}{K} \times (-\delta_K, 0), & \delta_K = \text{ess inf}_K D > 0 \end{cases} \tag{7}$$

(here on, $\overset{\circ}{B}$ stands for the interior of a set $B \subset \mathbb{R}^{N-1}$).

Let $K \subset \omega$ be a compact set. Consider $\psi \in \mathscr{D}(\omega)$ and $a \in \mathscr{D}(\mathbb{R})$ such that $\text{supp}\, \psi \subset \overset{\circ}{K}$ and $\text{supp}\, a \subset (-\delta_K, 0)$. Thus, the function $\phi(x', x_N) = a(x_N)\psi(x')$ belongs to $\mathscr{D}(\Omega)$. We fix ψ and change a; then, the linear mapping $a \mapsto \langle F, a\psi\rangle$ is a distribution in $\mathscr{D}'(-\delta_K, 0)$, which is constant since F does not depend upon x_N. Noting this constant as $S_K(\psi)$ we have

$$\langle F, a\psi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} = \left(\int_{-\delta_K}^{0} a(x_N)\,\mathrm{d}x_N\right) S_K(\psi).$$

Now, we fix a with $\int_{-\delta_K}^{0} a(x_N)\,\mathrm{d}x_N \neq 0$, and change ψ. This yields $S_K \in \mathscr{D}'(\overset{\circ}{K})$, i.e.

$$\langle F, a\psi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} = \left(\int_{-\delta_K}^{0} a(x_N)\,\mathrm{d}x_N\right) \langle S_K, \psi\rangle_{\mathscr{D}'(\overset{\circ}{K}),\mathscr{D}(\overset{\circ}{K})}.$$

We can then apply Fubini's theorem for distributions [12]: for any $\varphi \in \mathscr{D}(\Omega)$, such that $\operatorname{supp}\varphi \subset \overset{\circ}{K} \times (-\delta_K, 0)$ we have

$$\langle F, \varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} = \left\langle S_K, \left(\int_{-\delta_K}^{0} \varphi(x', x_N)\,\mathrm{d}x_N\right)\right\rangle_{\mathscr{D}'(\overset{\circ}{K}),\mathscr{D}(\overset{\circ}{K})}.$$

It is easy to check that if $K' \subset K$ then $S_K|_{\mathscr{D}(\overset{\circ}{K'})} = S_{K'}$, this allows us to introduce the distribution $S \in \mathscr{D}(\omega)$ defined by $\langle S, \psi\rangle_{\mathscr{D}'(\omega),\mathscr{D}(\omega)} = \langle S_K, \psi\rangle_{\mathscr{D}'(\overset{\circ}{K}),\mathscr{D}(\overset{\circ}{K})}$, where it is assumed that $\operatorname{supp}\psi \subset \overset{\circ}{K}$, and this gives (7).

Step 2: Proof of (6).

By repeating the same arguments given above, we easily find that (6) holds true whenever $\varphi \in \mathscr{D}(\overset{\circ}{K} \times (a, b))$, where $\overset{\circ}{K} \times (a, b) \subset \Omega$, $K \subset \omega$ is a compact set and $-\|D\|_{L^\infty(\omega)} < a < b < 0$ (the case $\|D\|_{L^\infty(\omega)} = +\infty$ being not excluded).

Now, let $\varphi \in \mathscr{D}(\Omega)$ and consider the family $\mathscr{F} \subset \mathscr{P}(\Omega)$ of open cylinders in Ω given by

$$\mathscr{F} = \{B(x'; r) \times (a, b) \subset \Omega / x' \in \omega,\ r > 0,\ -\|D\|_{L^\infty(\omega)} < a < b < 0\},$$

where $B(x'; r)$ stands for the open ball of ω centered at x' with radius r. Since $\Omega = \bigcup_{\mathrm{F}\in\mathscr{F}} \mathrm{F}$, $\mathscr{F}$ is an open covering of the compact set $\operatorname{supp}\varphi$; thus $\operatorname{supp}\varphi \subset \bigcup_{j=1}^{J}\Omega_j$, where $\Omega_j = B(x'_j; r_j) \times (a_j, b_j) \in \mathscr{F}$, for some $J \geqslant 1$, $x'_j \in \omega$, $r_j > 0$ and $-\|D\|_{L^\infty(\omega)} < a_j < b_j < 0$. We also put $\Omega_0 = \mathbb{R}^N \backslash \operatorname{supp}\varphi$. To the finite subfamily $\{\Omega_j\}_{j=0}^{J}$ we associate a partition of unity of Ω, $\{\alpha_j\}_{j=0}^{J}$, i.e. [13]:

(1) $\alpha_j \in \mathscr{D}(\mathbb{R}^N)$, $0 \leqslant \alpha_j \leqslant 1$ in $\mathbb{R}^N$, $j = 0, \ldots, J$;
(2) $\operatorname{supp}\alpha_j \subset \Omega_j$, $j = 0, \ldots, J$;
(3) $\sum_{j=0}^{J} \alpha_j = 1$ in Ω.

Notice that $\alpha_0\varphi = 0$ in Ω. Consequently, by applying expression (6) in every set Ω_j, $j = 1, \dots, J$, it yields

$$\begin{aligned}\langle F, \varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} &= \sum_{j=0}^{J} \langle F, \alpha_j\varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} \\ &= \sum_{j=1}^{J} \langle F, \alpha_j\varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} = \sum_{j=1}^{J} \langle S, M(\alpha_j\varphi)\rangle_{\mathscr{D}'(\omega),\mathscr{D}(\omega)} \\ &= \sum_{j=0}^{J} \langle S, M(\alpha_j\varphi)\rangle_{\mathscr{D}'(\omega),\mathscr{D}(\omega)} = \langle S, M(\varphi)\rangle_{\mathscr{D}'(\omega),\mathscr{D}(\omega)}.\end{aligned}$$

Step 3: The distribution S is unique.

Let $S_1, S_2 \in \mathscr{D}'(\omega)$ be two distributions satisfying (6), and put $S = S_1 - S_2$. Then, $\langle S, M(\varphi)\rangle_{\mathscr{D}'(\omega),\mathscr{D}(\omega)} = 0$ for all $\varphi \in \mathscr{D}(\Omega)$. Take $\psi \in \mathscr{D}(\omega)$ and $a \in \mathscr{D}(\mathbb{R})$ with $\operatorname{supp} a \subset (-\delta, 0)$, $\delta = \operatorname{ess\,inf}_{\operatorname{supp}\psi} D$, and $\int_{\mathbb{R}} a(x_N)\,\mathrm{d}x_N = 1$. Thus, $a\psi \in \mathscr{D}(\Omega)$, $M(a\psi) = \psi$ and

$$0 = \langle S, M(a\psi)\rangle_{\mathscr{D}'(\omega),\mathscr{D}(\omega)} = \langle S, \psi\rangle_{\mathscr{D}'(\omega),\mathscr{D}(\omega)}, \quad \text{for all } \psi \in \mathscr{D}(\omega)$$

and therefore $S = 0$. This ends the proof of Theorem 1. □

Now, we study the regularity of S with respect to F. To do that, we introduce the usual first-order Sobolev spaces $W^{1,q}(\Omega)$, $W_0^{1,q}(\Omega)$ given by

$$\begin{cases} W^{1,q}(\Omega) = \{v \in L^q(\Omega)/\nabla v \in L^q(\Omega)^N\}, \quad 1 \leqslant q \leqslant +\infty, \\ W_0^{1,q}(\Omega) = \overline{\mathscr{D}(\Omega)}^{W^{1,q}(\Omega)}, \quad 1 \leqslant q < +\infty, \\ W^{-1,q'}(\Omega) = \text{dual space of } W_0^{1,q}(\Omega), \quad 1/q + 1/q' = 1, \\ H^1(\Omega) = W^{1,2}(\Omega), \end{cases}$$

where $\nabla v = (\partial v/\partial x_1, \dots, \partial v/\partial x_N)^{\mathrm{T}}$ is the gradient operator, all derivatives being taken in the sense of distributions. It is well-known that $W^{1,q}(\Omega)$, $W_0^{1,q}(\Omega)$ are Banach spaces provided with their standard norms, and also $H^1(\Omega)$, $H_0^1(\Omega)$ are Hilbert spaces [1].

Lemma 1. *Let* $F \in \mathscr{D}'(\Omega)$ *and* $S \in \mathscr{D}'(\omega)$ *satisfy* (6). *Assume* $F \in W^{-1,q'}(\Omega)$. *Then*

$$S \in W^{-1,q'}(\overset{\circ}{K}) \quad \textit{for every compact set } K \subset \omega. \tag{8}$$

Moreover, if ω *is bounded in some direction and* $\operatorname{ess\,inf}_\omega D > 0$, *then* $S \in W^{-1,q'}(\omega)$.

Proof. Let $K \subset \omega$ be a compact set, $\delta_K = \operatorname{ess\,inf}_K D > 0$. Take $a \in \mathscr{D}(\mathbb{R})$ such that $\operatorname{supp} a \subset (-\delta_K, 0)$ and $\int_{-\delta_K}^0 a(x_N)\,\mathrm{d}x_N = 1$. According to (6), for any $\psi \in \mathscr{D}(\omega)$, $\operatorname{supp}\psi \subset \overset{\circ}{K}$, we have

$$\langle F, a\psi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} = \langle S, \psi\rangle_{\mathscr{D}'(\omega),\mathscr{D}(\omega)}, \quad \text{for all } \psi \in \mathscr{D}(\overset{\circ}{K})$$

whence,

$$|\langle S,\psi\rangle_{\mathscr{D}'(\omega),\mathscr{D}(\omega)}|\leqslant \|F\|_{W^{-1,q'}(\Omega)}\|a\psi\|_{W_0^{1,q}(\Omega)}.$$

On the other hand,

$$\|a\psi\|^q_{W_0^{1,q}(\Omega)}\leqslant \int_\Omega |a\nabla\psi|^q+\int_\Omega |a'\psi|^q\leqslant \|a\|^q_{W^{1,\infty}(\mathbb{R})}\left(\int_{\overset{\circ}{K}}|\nabla\psi|^q+\int_{\overset{\circ}{K}}|\psi|^q\right)$$

and owing to Poincaré's inequality, we finally deduce

$$|\langle S,\psi\rangle_{\mathscr{D}'(\omega),\mathscr{D}(\omega)}|\leqslant C_K\|\psi\|_{W_0^{1,q}(\overset{\circ}{K})},\qquad \text{for all } \psi\in\mathscr{D}(\overset{\circ}{K}),$$

where $C_K=\lambda_K\|a\|_{W^{1,\infty}(\mathbb{R})}\|F\|_{W^{-1,q'}(\Omega)}$ (λ_K is a constant coming from Poincaré's inequality in $\overset{\circ}{K}$). From a density argument, we obtain the desired regularity $S\in W^{-1,q'}(\overset{\circ}{K})$.

Finally, if ω is bounded in some direction, then the constants λ_K may be taken independent of K and if we also assume that $\text{ess inf}_\omega D>0$, then the function a can be taken independent of K. These two facts give directly the global regularity $S\in W^{-1,q'}(\omega)$.

Remark. Let $m\geqslant 2$ integer and put

$$\begin{cases} W^{m,q}(\Omega)=\{v\in W^{m-1,q}(\Omega)/\nabla v\in W^{m-1,q}(\Omega)^N\}, & 1\leqslant q\leqslant +\infty,\\ W_0^{m,q}(\Omega)=\overline{\mathscr{D}(\Omega)}^{W^{m,q}(\Omega)}, & 1\leqslant q<+\infty,\\ W^{-m,q'}(\Omega)=\text{dual space of } W_0^{m,q}(\Omega), & 1/q+1/q'=1.\end{cases}$$

It is very easy to check the following generalization of Lemma 1: If $F\in W^{-m,q'}(\Omega)$ then

$$S\in W^{-m,q'}(\overset{\circ}{K})\quad\text{for every compact set } K\subset\omega$$

and if ω is bounded in some direction and $\text{ess inf}_\omega D>0$, then $S\in W^{-m,q'}(\omega)$.

3. Some applications of the representation theorem

There are some mathematical models in which the unknown $u:\Omega\mapsto\mathbb{R}^{N-1}$, Ω like in (2), is subject to a non-local constraint of the form $\nabla'\cdot M(u)=0$ in ω, where $M(u)$ is given in (5), and the operator ∇' refers to the $(N-1)$-gradient operator with respect to the $N-1$ first variables $x_1,\dots,x_{N-1}$, and so $\nabla'\cdot v=0$ is the divergence with respect to these $N-1$ variables. Though this is a non-local constraint, it is still linear and, therefore, it is quite natural to search for solutions of such models in some space of the form $\{v\in X/\nabla'\cdot M(u)=0$ in $\omega\}$, where X is a suitable Banach space. The hydrostatic approximation of Navier–Stokes equations is an example of such a model [3–6,9–11,14]. The hydrostatic approximation is a general model arising in oceanography for the description of the circulation of water in oceans and lakes. Taking into account only the essential unknowns, i.e. the horizontal velocity field $u:\Omega\subset\mathbb{R}^3\mapsto\mathbb{R}^2$ and the surface pressure $p_s:\omega\subset\mathbb{R}^2\mapsto\mathbb{R}$, the model, at climatic time scales, becomes

$$\begin{cases} (u\cdot\nabla')u + W(u)\frac{\partial u}{\partial x_3} - \nu_1\Delta' u - \nu_2\frac{\partial^2 u}{\partial x_3^2} + \gamma u^{\perp} + \nabla' p_{\mathrm{s}} = f \quad \text{in } \Omega,\\ \nabla'\cdot M(u) = 0 \quad \text{in } \omega,\\ u = 0 \quad \text{on } \Gamma_{\mathrm{b}}, \qquad \nu_2\frac{\partial u}{\partial x_3} = g_{\mathrm{s}} \quad \text{on } \Gamma_{\mathrm{s}}. \end{cases} \tag{9}$$

Here $N=3$, $W(u)$ represents the vertical velocity. It is given by

$$W(u)(x_1,x_2,x_3) = \int_{x_3}^{0} \nabla'\cdot u(x_1,x_2,\zeta)\,\mathrm{d}\zeta.$$

The constants $\nu_1>0$ and $\nu_2>0$ are the horizontal and vertical viscosity coefficients, respectively (in practice $\nu_2 \ll \nu_1$). Also, $\gamma u^{\perp}$ stands for the Coriolis acceleration term, γ being a function depending upon the angular velocity of the earth and the latitude, whereas $u^{\perp} = (u_2,-u_1)^{\mathrm{T}}$. The boundary of Ω is split into two parts,

$$\partial\Omega = \Gamma_{\mathrm{s}} \cup \Gamma_{\mathrm{b}}, \quad \Gamma_{\mathrm{s}} = \omega\times\{0\}, \quad \Gamma_{\mathrm{b}} = \partial\Omega\backslash\Gamma_{\mathrm{s}},$$

so that Γ_{s} is the sea surface and Γ_{b} is the bottom basin together with (possible) sidewalls or taluses. The right-hand side f is a forcing term taking into account the effects of salinity, density or temperature, which are considered here decoupled from the governing equations of the flow (9). Finally, g_{s} is the wind stress.

As one can readily see, the natural space to search for the horizontal velocity field u is

$$V = \{v\in H^1(\Omega)^2 / v = 0 \text{ on } \Gamma_{\mathrm{b}},\ \nabla'\cdot M(v) = 0 \text{ in } \omega\},$$

which yields the following regularity for the vertical convection term [4,7,10]:

$$W(u)\,\frac{\partial u}{\partial x_3} \in W^{-1,q'}(\Omega)^2, \quad \text{for all } q'<2.$$

Consequently, if the data are smooth enough, e.g. $f\in H^{-1}(\Omega)^2$ and $g_{\mathrm{s}}\in H^{-1/2}(\Gamma_{\mathrm{s}})^2$, one may expect the regularity $p_{\mathrm{s}}\in L^{q'}(\Omega)$, for all $q'<2$ [6,10]. Indeed, putting

$$F = f - (u\cdot\nabla')u - W(u)\frac{\partial u}{\partial x_3} + \nu_1\Delta' u + \nu_2\,\frac{\partial^2 u}{\partial x_3^2} - \gamma u^{\perp},$$

it can be shown that $F\in W^{-1,q'}(\Omega)^2$ for all $q'<2$ and

$$\langle F,\varphi\rangle_{W^{-1,q'}(\Omega)^2, W_0^{1,q}(\Omega)^2} = 0 \quad \text{for all } \varphi\in\mathscr{V}(\Omega),$$

where $\mathscr{V}(\Omega)$ is defined in (3). Therefore, in order to retrieve the surface pressure term $\nabla' p_{\mathrm{s}}$, we have to show that p_{s} is the Lagrange multiplier related to the non-local constraint $\nabla'\cdot M(\varphi)=0$ in ω. This is equivalent to the derivation of a de Rham-like lemma involving this non-local constraint. The next result gives the answer to this question. It is a generalization of the one given in [9] to more general domains Ω, without the assumption on the existence of a sidewall all along $\partial\omega$, together with a non-Hilbert setting.

We will make use of the space $L_D^{q'}(\omega)$, $1<q'<+\infty$, given as

$$L_D^{q'}(\omega)=\left\{h\colon\omega\mapsto\mathbb{R}\Big/\int_\omega D|h|^{q'}<+\infty\right\}.$$

Observe that $h\in L_D^{q'}(\omega)$ if and only if $\partial h/\partial x_N=0$ and $h\in L^{q'}(\Omega)$ (in this context, h as defined in Ω is understood as the function $(x',x_N)\in\Omega\mapsto h(x')$).

Lemma 2. *Let ω be a connected and bounded set in $\mathbb{R}^{N-1}$, $D\in L^\infty(\omega)$ satisfying* (4), *and $\Omega\subset\mathbb{R}^N$ as given in* (2). *Let $F\in W^{-1,q'}(\Omega)^{N-1}$. Then the following conditions are equivalent*:

(1) $\langle F,\varphi\rangle_{W^{-1,q'}(\Omega)^{N-1},W_0^{1,q}(\Omega)^{N-1}}=0$ *for all* $\varphi\in\mathscr{V}(\Omega)$.

(2) *F does not depend upon the x_N-variable and there exists $p_s\in L^{q'}_{\mathrm{loc}}(\omega)$ (unique up to an additive constant) such that $F=\nabla' p_s$. Moreover,*

   (a) *If ω is Lipschitz continuous and* ess $\inf_\omega D>0$ *then $p_s\in L^{q'}(\omega)$.*

   (b) *If Ω is Lipschitz continuous then $p_s\in L_D^{q'}(\omega)$.*

Proof. $2\Rightarrow 1$. Let $p_s\in L^{q'}_{\mathrm{loc}}(\omega)$ *and* $\varphi\in\mathscr{V}(\Omega)$, then $p_s\in L^{q'}_{\mathrm{loc}}(\Omega)$. Therefore,

$$\begin{aligned}&\langle\nabla' p_s,\varphi\rangle_{W^{-1,q'}(\Omega)^{N-1},W_0^{1,q}(\Omega)^{N-1}}\\&\quad=-\int_\Omega p_s\nabla'\cdot\varphi=-\int_\omega p_s\int_{-D(x')}^0\nabla'\cdot\varphi(x',x_N)\,\mathrm{d}x_N.\end{aligned}$$

Since $\varphi\in\mathscr{V}(\Omega)$, we have in particular $\varphi=0$ near $\Gamma_b=\partial\Omega\backslash\omega\times\{0\}$, which yields

$$\int_{-D(x')}^0\nabla'\cdot\varphi(x',x_N)\,\mathrm{d}x_N=\nabla'\cdot\left(\int_{-D(x')}^0\varphi(x',x_N)\,\mathrm{d}x_N\right)=\nabla'\cdot M(\varphi)=0$$

and thus

$$\langle\nabla' p_s,\varphi\rangle_{W^{-1,q'}(\Omega)^{N-1},W_0^{1,q}(\Omega)^{N-1}}=0.$$

$1\Rightarrow 2$. First of all, we show that F is independent of x_N. To do that we follow [9]: observe that for all $\phi\in\mathscr{D}(\Omega)^{N-1}$ one has $\nabla'\cdot M(\phi)=M(\nabla'\cdot\phi)$. Now let $\phi\in\mathscr{D}(\Omega)^{N-1}$, then $M(\partial\phi/\partial x_N)=0$ in ω so that $\partial\phi/\partial x_N\in\mathscr{V}(\Omega)$; consequently,

$$0=\left\langle F,\frac{\partial\phi}{\partial x_N}\right\rangle=-\left\langle\frac{\partial F}{\partial x_N},\phi\right\rangle,$$

i.e. $\partial F/\partial x_N=0$. This means that the $N-1$ components of F are under the conditions of Theorem 1 and Lemma 1: there exists a unique $S\in\mathscr{D}'(\Omega)^{N-1}$ satisfying (6) and (8).

Now, we show the following assertion: There exists $p_s\in L^{q'}_{\mathrm{loc}}(\omega)$, uniquely determined up to an additive constant, such that $S=\nabla' p_s$. To do that, we consider the increasing sequence of compact sets

$$C_m = \left\{ x' \in \omega / \mathrm{dist}(x', \partial\omega) \geqslant \frac{1}{m} \right\}, \quad m \geqslant m_0,$$

where $m_0 \geqslant 1$ is chosen so that $C_m \neq \emptyset$ and is connected (this is possible since ω is connected). Since C_m may not bear a regular boundary, we consider another sequence of compact sets $(K_m)_{m \geqslant m_0}$ such that

$$\begin{cases} C_m \subset \overset{\circ}{K}_m \subset C_{m+1}, \text{ and} \\ K_m \text{ is connected and Lipschitz-continuous, for all } m \geqslant m_0. \end{cases}$$

For every $m \geqslant m_0$, we take ψ, supp $\psi \subset \overset{\circ}{K}_m$, then a as in the proof of Lemma 1, and assume also that $\nabla' \cdot \psi = 0$ in ω. Then, $a\psi \in \mathscr{V}(\Omega)$ and owing to (6) and the hypothesis on F, we have

$$0 = \langle F, a\psi \rangle_{\mathscr{D}'(\Omega)^{N-1}, \mathscr{D}(\Omega)^{N-1}} = \langle S, \psi \rangle_{\mathscr{D}'(\omega)^{N-1}, \mathscr{D}(\omega)^{N-1}}$$

and consequently,

$$\langle S, \psi \rangle_{\mathscr{D}'(\omega)^{N-1}, \mathscr{D}(\omega)^{N-1}} = 0, \quad \text{for all } \psi \in \mathscr{D}(\overset{\circ}{K})^{N-1}, \text{ such that } \nabla' \cdot \psi = 0 \text{ in } \overset{\circ}{K}.$$

We can then apply de Rham's lemma to $S \in W^{-1,q'}(\overset{\circ}{K}_m)^{N-1}$ [2]: there exists a function $p_m \in L^{q'}(\overset{\circ}{K}_m)$ such that $S = \nabla' p_m$ in $\overset{\circ}{K}_m$. Since K_m is connected, this p_m is uniquely determined up to an additive constant, and we can choose it so that $p_m = p_{m+1}$ in $\overset{\circ}{K}_m$ for all $m \geqslant m_0$. This allows us to define $p_{\mathrm{s}} \in L^{q'}_{\mathrm{loc}}(\omega)$ so that $p_{\mathrm{s}} = p_m$ in $\overset{\circ}{K}_m$ and clearly $S = \nabla' p_{\mathrm{s}}$ in ω.

Now let $\varphi \in \mathscr{D}(\Omega)^{N-1}$; owing to (6), we have

$$\begin{aligned} \langle F, \varphi \rangle_{\mathscr{D}'(\Omega)^{N-1}, \mathscr{D}(\Omega)^{N-1}} &= \langle \nabla' p_{\mathrm{s}}, M(\varphi) \rangle_{\mathscr{D}'(\omega)^{N-1}, \mathscr{D}(\omega)^{N-1}} \\ &= -\int_\omega p_{\mathrm{s}} \nabla' \cdot \left(\int_{-D(x')}^0 \varphi(x', x_N) \, \mathrm{d}x_N \right) \\ &= -\int_\omega p_{\mathrm{s}} \int_{-D(x')}^0 \nabla' \cdot \varphi(x', x_N) \, \mathrm{d}x_N \\ &= -\int_\omega \int_{-D(x')}^0 p_{\mathrm{s}} \nabla' \cdot \varphi(x', x_N) \, \mathrm{d}x_N \\ &= -\int_\Omega p_{\mathrm{s}} \nabla' \cdot \varphi(x', x_N) \, \mathrm{d}x_N = \langle \nabla' p_{\mathrm{s}}, \varphi \rangle_{\mathscr{D}'(\Omega)^{N-1}, \mathscr{D}(\Omega)^{N-1}}, \end{aligned}$$

whence $F = \nabla' p_{\mathrm{s}}$ in the sense of distributions.

In order to show 2(a), observe that if ess $\inf_\omega D > 0$ then, according to Lemma 1, $S \in W^{-1,q'}(\omega)^{N-1}$. Therefore, $p_{\mathrm{s}} \in L^{q'}_{\mathrm{loc}}(\omega)$ is such that $S = \nabla' p_{\mathrm{s}} \in W^{-1,q'}(\omega)^{N-1}$; and since we also assume that ω is a connected, bounded and Lipschitz-continuous set, we deduce the global regularity $p_{\mathrm{s}} \in L^{q'}(\omega)$ [2].

Finally, assume that Ω is Lipschitz continuous. Since $p_s \in L^{q'}_{loc}(\omega)$, it is straightforward that $p_s \in L^{q'}_{loc}(\Omega)$. This property together with $\nabla' p_s = F \in W^{-1,q'}(\Omega)^{N-1}$ and the regularity of Ω implies that [2] $p_s \in L^{q'}(\Omega)$, and since p_s does not depend upon x_N, this is equivalent to $p_s \in L^{q'}_D(\omega)$. This shows 2(b) and ends the proof of Lemma 2. □

Remark. If ω is Lipschitz continuous and Ω has a sidewall all along $\partial\omega$, i.e. ess $\inf_\omega D > 0$, then Lemma 2 tells us that $p_s \in L^{q'}(\omega)$, and this is true even if Ω is not Lipschitz continuous. Therefore, Lemma 2 is a generalization of the result appearing in [9] where it was assumed that $q' = 2$ and Ω is Lipschitz continuous.

Remark. In the case of a domain Ω that is Lipschitz continuous (with or without a sidewall), Lemma 2 is equivalent to the following reduced inf-sup condition: there exists $\beta > 0$ such that

$$\|p\|_{L^{q'}_{D,0}(\omega)} \leqslant \beta \sup_{v \in W^{1,q}_0(\Omega)} \frac{\int_\omega p \, \nabla \cdot M(v)}{\|v\|_{W^{1,q}_0(\Omega)}}, \quad \text{for all } p \in L^{q'}_{D,0},$$

where $L^{q'}_{D,0} = \{g \in L^{q'}_D(\omega) / \int_\omega Dg = 0\}$. This reduced inf-sup condition was already shown in [6], and its proof is based in the application of a version of de Rham's lemma in Ω. Notice that we have applied de Rham's lemma in ω after deriving the relation (6) linking a distribution in Ω with one in ω.

Remark. The results described in this work have been applied in [10] in the study of a modified version of the hydrostatic approximation: the differential equations for the horizontal velocity are perturbed with certain monotone expressions; this approach has led to another proof of the existence of a solution to problem (9), and to analyze a one-equation hydrostatic turbulence model.

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