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Irreducible components of the space of foliations associated to the affine Lie algebra

O. CALVO-ANDRADE†, D. CERVEAU‡, L. GIRALDO§ and A. LINS NETO‖

† *CIMAT Apartado Postal 402 36000, Guanajuato, Gto., Mexico*
(e-mail: omegar@fractal.cimat.mx)
‡ *Institut de Recherche Mathématique de Rennes, Campus de Beaulieu, 35042 RENNES Cedex Rennes, France*
§ *Departamento de Matemáticas, Universidad de Cádiz Apartado 40 11510, Puerto Real, Cádiz, Spain*
(e-mail: luis.giraldo@uca.es)
‖ *Instituto de Matemática Pura e Aplicada Estrada Dona Castorina, 110 Horto, Rio de Janeiro, Brazil*
(e-mail: alcides@impa.br)

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Abstract. In this paper, we give the explicit construction of certain components of the space of holomorphic foliations of codimension one, in complex projective spaces. These components are associated to some algebraic representations of the affine Lie algebra $\mathfrak{aff}(\mathbb{C})$. Some of them, the so-called *exceptional* or *Klein–Lie* components, are rigid in the sense that all generic foliations in the component are equivalent (Example 1). In particular, we obtain rigid foliations of all degrees. Some generalizations and open problems are given at the end of §1.

1. *Introduction*

It is known that the space $\mathcal{F}(\nu, n)$ of singular holomorphic codimension one foliations of degree $\nu \geq 0$ on $\mathbb{CP}(n)$, $n \geq 3$, can be considered as an algebraic subset of the space of 1-forms on $\mathbb{C}^{n+1}$ whose coefficients are homogeneous polynomials of degree $\nu + 1$ (cf. [**2, 4–6**]). Some of the irreducible components of this algebraic subset have been described: for example, the *logarithmic components*, which correspond to foliations defined by closed meromorphic 1-forms (cf. [**2**]). Other components are the *rational* (cf. [**5**]) and the *pull-back components* (cf. [**6**]). For $\nu = 0, 1, 2$ the complete decomposition of $\mathcal{F}(\nu, n)$ in irreducible components was obtained in [**5**].

In this paper, we present new components of $\mathcal{F}(\nu, n)$, $n \geq 3$, related with some special representations of the affine Lie algebra $\mathfrak{aff}(\mathbb{C}) := \{\mathbf{e}_1, \mathbf{e}_2, [\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_2\}$ in the algebra of polynomial vector fields of an affine chart $\mathbb{C}^3 \subset \mathbb{CP}(3)$. These new components include as a particular case the 'exceptional component' of $\mathcal{F}(2, n)$, described in [**5**].

To obtain our result we follow three steps.

(1) We construct families of foliations $\mathcal{F}_{\mathfrak{P}} \subset \mathcal{F}(\nu, 3)$, where $\mathfrak{P}$ denotes a discrete invariant, arising from representations of the affine algebra.

(2) We find sufficient conditions in order to prove stability under deformations of some of these families, i.e. we prove that for certain values of $\mathfrak{P}$ the deformation of a generic foliation $\mathcal{F} \in \mathcal{F}_{\mathfrak{P}}$ is still a foliation in $\mathcal{F}_{\mathfrak{P}}$.

(3) We get codimension one foliations in $\mathbb{CP}(n)$, $n \geq 4$, by pull-back of the foliations just constructed and prove that we also have irreducible components in $\mathcal{F}(\nu, n)$.

In the first step the families are geometrically described. To do that, we consider the so-called *Klein–Lie* curves. They are characterized by the fact of being the rational projective curves fixed by an infinite group of projective automorphism. In $\mathbb{CP}(3)$ such curves, up to an automorphism in $\mathbf{PGL}(4, \mathbb{C})$, can be parameterized by $\Gamma(t : s) = (t^p : t^q s^{p-q} : t^r s^{p-r} : s^p)$, where $1 \leq r < q < p$ are positive integers with $\gcd(p, q, r) = 1$.

For each $\ell \neq 0$ such that $\ell + r \in \{0\} \cup \mathbb{N}$, we have a representation of the affine Lie algebra $\rho_\ell : \mathfrak{aff}(\mathbb{C}) \to \mathfrak{X}(\mathbb{C})$, determined by the two vector fields $\mathbf{s}_\ell := (1/\ell)t(\partial/\partial t)$, and $\mathbf{x}_\ell := t^{\ell+1}(\partial/\partial t)$. Consider the linear semi-simple vector field on $\mathbb{C}^3$

$$S = px\frac{\partial}{\partial x} + qy\frac{\partial}{\partial y} + rz\frac{\partial}{\partial z}. \tag{1}$$

Suppose that there is another polynomial vector field X on $\mathbb{C}^3$ such that $[S, X] = \ell X$, and so that

$$\gamma_*(\mathbf{s}_\ell) = \frac{1}{\ell} S(\gamma(t)), \quad \gamma_*(\mathbf{x}_\ell) = X(\gamma(t)),$$

where $\gamma(t) = (t^p, t^q, t^r)$ is the affine curve $\Gamma \cap \mathbb{C}^3$. Then the algebraic foliation $\mathcal{F} = \mathcal{F}(S, X)$ on $\mathbb{C}^3$ defined by the 1-form $\Omega = i_S i_X(dz_1 \wedge dz_2 \wedge dz_3)$ is associated to a representation of the affine algebra in the algebra of polynomial vector fields in $\mathbb{C}^3$, and it can be extended to a foliation on $\mathbb{CP}(3)$ of certain degree ν.

We explicitly give several examples in §2, all in the case $r = 1$. Note also that both $\mathbf{s}_\ell$ and $\mathbf{x}_\ell$ are complete vector fields on $\mathbb{C}$ just in the case $\ell = -1$. This is what happens in Example 1, where S and X are complete and the flow of S is periodic: both are necessary conditions for the existence of an action of the affine group on $\mathbb{C}^3$ associated to the foliation.

We define

$$\mathcal{F}((p, q, r); \ell, \nu) := \{\mathcal{F} \in \mathcal{F}(\nu, 3) \mid \mathcal{F} = \mathcal{F}(S, X) \text{ in some affine chart}\}$$

and we show that they are irreducible subvarieties of $\mathcal{F}(\nu, 3)$. We also show that if $\mathcal{F} \in \mathcal{F}((p, q, r); \ell, \nu)$ then the tangent sheaf $T_{\mathcal{F}}$ is isomorphic to $\mathcal{O} \oplus \mathcal{O}(2 - \nu)$.

In order to carry out the second step, we need some technical results. Let us first give some definitions.

Definition 1. Let ω be an integrable 1-form defined in a neighborhood of $\mathbf{p} \in \mathbb{C}^3$. We say that $\mathbf{p}$ is a *generalized Kupka* (GK) singularity of ω if $\omega_{\mathbf{p}} = 0$ and either $d\omega_{\mathbf{p}} \neq 0$ or $\mathbf{p}$ is an isolated zero of $d\omega$.

The local structure of a foliation near a GK singularity is well known. When $d\omega_{\mathbf{p}} \neq 0$ it is of *Kupka type* and it is locally the product of two foliations: a singular one in dimension two and a non-singular one of dimension one (cf. [**10, 15**]). When $\mathbf{p}$ is an isolated singularity of $d\omega$, the singularity is *logarithmic*, *degenerate* or *quasi-homogeneous* (cf. possibilities 2a, 2b and 2c and Theorem A of §2.1 and [**4**] and [**12**]).

We also prove that GK singularities are stable under deformations (cf. Proposition 1).

Definition 2. A codimension one holomorphic foliation $\mathcal{F}$ in a complex three manifold M is GK if all the singularities of $\mathcal{F}$ are GK.

We show, as a consequence of the stability of GK singularities, that GK foliations are stable under deformations. In fact, we first note that the local structure of GK singularities implies that the analytic tangent sheaf of a GK foliation is locally free. Using well-known results on holomorphic vector bundle theory (Theorem B), we can prove the following theorem.

THEOREM 1. *Suppose that $\mathcal{F}((p, q, r); \ell; \nu)$ contains some GK foliation. Then $\overline{\mathcal{F}((p, q, r); \ell; \nu)}$ is an irreducible component of $\mathcal{F}(\nu, 3)$.*

The families of foliations of Example 1 in §2 provide irreducible components of $\mathcal{F}(\nu, 3)$, $\nu \geq 2$. As we will see, these families correspond to $\mathcal{F}((\nu^2+\nu+1, \nu+1, 1); -1; \nu)$ and all of them contain GK foliations. In fact, any component like that is the closure of an orbit of the natural action of $\mathbf{PGL}(4, \mathbb{C})$ on $\mathcal{F}(\nu, 3)$.

On the other hand, for each $p \geq 3$, the foliations in the family $\mathcal{F}((p, 2, 1); -1; p)$ are never GK, so that Theorem 1 does not hold in this case. In fact, as we will see in §2.2, any foliation in $\mathcal{F}((p, q, 1); -1; p)$ has a meromorphic first integral, which in the case of $\mathcal{F}((p, 2, 1); -1; p)$ can be written in homogeneous coordinates of $\mathbb{CP}(3)$ as f^p/g^2, where f and g are homogeneous polynomials, $\mathrm{dg}(f) = 2$ and $\mathrm{dg}(g) = p$. In the notation of [5], such a foliation belongs to $\mathcal{R}(2, p) \subset \mathcal{F}(p, 3)$, which is an irreducible (rational) component of $\mathcal{F}(p, 3)$ (cf. [**5**]). On the other hand, it is not very difficult to prove that a generic foliation in $\mathcal{R}(2, p)$ has no quasi-homogenous singularity. Hence, $\mathcal{F}((p, 2, 1); -1; p)$ is not an irreducible component of $\mathcal{F}(p, 3)$, if $p \geq 3$ (see also Remark 4).

Theorem 3 states that given (p, q, r) positive integers such that $p > q > r$, the set $\{(\ell, \nu)\}$ such that the family $\mathcal{F}((p, q, r); \ell; \nu)$ contains some GK foliation is finite. This motivates the following problem.

Problem 1. Given three positive integers $p > q > r \geq 1$, are there (ℓ, ν) such that $\mathcal{F}((p, q, r); \ell; \nu)$ contains a GK foliation?

The examples in §2.2 are GK foliations in $\mathbb{CP}(3)$, all of them belonging to some $\mathcal{F}((p, q, r); \ell; \nu)$. Consequently, the tangent sheaf for these examples splits. This motivates the following questions.

Problem 2. Is it true that $T_{\mathcal{F}}$ splits for any GK foliation $\mathcal{F}$ on $\mathbb{CP}(3)$? More generally, let $\mathcal{F}$ be a codimension one foliation on $\mathbb{CP}(3)$ such that for any $p \in \mathbb{CP}(3)$ the sheaf of germs of vector fields at p tangent to $\mathcal{F}$ is free with two generators. Does $T_{\mathcal{F}}$ split?

We observe that all examples that we have of GK foliations on $\mathbb{CP}(3)$ have at most two quasi-homogeneous singularities. A natural question is the following.

Problem 3. Are there GK foliations on $\mathbb{CP}(3)$ with more than two quasi-homogeneous singularities?

Finally, concerning the third step, in §3.2 we consider foliations on $\mathbb{CP}(n)$, $n \geq 4$, which are pull-backs of GK foliations on $\mathbb{CP}(3)$ by a generic linear rational map $f\colon \mathbb{CP}(n) \longrightarrow \mathbb{CP}(3)$. Denote by $\mathcal{F}((p,q,r);\ell;\nu;n) \subset \mathcal{F}(\nu,n)$ the set of foliations so obtained from $\mathcal{F}((p,q,r),\ell,\nu)$,

$$\mathcal{F}((p,q,r);\ell;\nu;n) := \{\mathcal{F} \mid \mathcal{F} = f^*\mathcal{G}, \mathcal{G} \in \mathcal{F}((p,q,r),\ell,\nu)\}.$$

We prove the following.

THEOREM 2. *Let $\mathcal{F}$ be a foliation on $\mathbb{CP}(n)$, $n \geq 4$ and $i\colon \mathbb{CP}(3) \to \mathbb{CP}(n)$ be a linear embedding of a 3-plane in a general position with respect to $\mathcal{F}$. Suppose that $\mathcal{G} = i^*(\mathcal{F})$ is a GK foliation in $\mathcal{F}(\nu,3)$ and that it is generated by two one-dimensional foliations on $\mathbb{CP}(3)$. Then there exists a linear rational map $f\colon \mathbb{CP}(n) \longrightarrow \mathbb{CP}(3)$ such that $\mathcal{F} = f^*(\mathcal{G})$. In particular, $\overline{\mathcal{F}((p,q,r);\ell;\nu;n)}$ is an irreducible component of $\mathcal{F}(\nu,n)$.*

2. *Preliminary results and examples*

Notation. Throughout the paper, we consider $(z_1 : z_2 : z_3 : z_4)$ as homogeneous coordinates in $\mathbb{CP}(3)$. The basic affine open subsets will be

$$E_1 = \{(1 : w : v : u) \mid (u,v,w) \in \mathbb{C}^3\} \quad E_2 = \{(r : 1 : s : t) \mid (r,s,t) \in \mathbb{C}^3\},$$
$$E_3 = \{(r : s : 1 : t) \mid (r,s,t) \in \mathbb{C}^3\} \quad E_0 = \{(x : y : z : 1) \mid (x,y,z) \in \mathbb{C}^3\}.$$

2.1. *Generalized Kupka and quasi-homogeneous singularities.* Let $p \geq q \geq r > 0$ be relatively prime integers and S be the semi-simple vector field on $\mathbb{C}^3$ defined as in (1) by $S = px(\partial/\partial x) + qy(\partial/\partial y) + rz(\partial/\partial z)$. We say that a vector field X, holomorphic in a neighborhood of $0 \in \mathbb{C}^3$, is *S-quasi-homogeneous of weight ℓ*, if we have the following Lie bracket identity: $[S,X] = \ell X$. Remark that necessarily $\ell + r$ is a non-negative integer and X is a polynomial vector field. In fact, if $X = P_1(\partial/\partial x) + P_2(\partial/\partial y) + P_3(\partial/\partial z)$, the condition that X is S-quasi-homogeneous of weight ℓ is equivalent to the fact that, after giving weights p, q and r to the variables x, y and z, respectively, the polynomials P_1, P_2 and P_3 are weighted homogeneous of degrees $\ell + p$, $\ell + q$ and $\ell + r$, respectively.

Moreover, S and X give a representation of the affine Lie algebra in the algebra of polynomial vector fields. If we suppose that S and X are linearly independent at generic points, then these vector fields generate an algebraic foliation on $\mathbb{C}^3$, which is given by the integrable 1-form $\Omega = i_S i_X(dx \wedge dy \wedge dz)$. Since Ω is a polynomial 1-form, this foliation can be extended to a singular foliation of $\mathbb{CP}(3)$, which will be denoted by $\mathcal{F}(\Omega)$ or by $\mathcal{F}(S,X)$. Observe that S extends to a holomorphic vector field on $\mathbb{CP}(3)$ and that its trajectories are contained in the leaves of $\mathcal{F}(\Omega)$. On the other hand, in general, the vector field X is meromorphic in $\mathbb{CP}(3)$, but the foliation defined by it on $\mathbb{C}^3$ extends to a foliation on $\mathbb{CP}(3)$, which will be denoted by $\mathcal{G}(X)$, whose leaves are also contained in the leaves of $\mathcal{F}(\Omega)$. Remark that the singular set of $\mathcal{F}(\Omega)$, denoted by $\text{sing}(\mathcal{F}(\Omega))$, is invariant under the flow of S, $\exp(tS) := S_t$. This follows from the relation

$$L_S(\Omega) = m\Omega, \quad m = \ell + tr(S) = \ell + p + q + r, \tag{2}$$

as the reader can check. Relation (2) also implies that if $p_0 \notin \text{sing}(S)$, then $\mathcal{F}(\Omega)$ is, in a neighborhood of p_0, equivalent to the product of a foliation in dimension two by a one-dimensional disk. In fact, let $(U, (u, v, w))$ be a holomorphic coordinate system such that $S|_U = \partial/\partial u$. Then it is not difficult to see that the integrability condition and (2) imply that

$$\Omega(u, v, w) = e^{mu}\Omega(0, v, w) = e^{mu}(A(v, w)dv + B(v, w)dw),$$

which proves the assertion.

In the affine chart $\mathbb{C}^3 \subset \mathbb{CP}(3)$, where S is as in (1), the leaves of $\mathcal{F}(\Omega)$ are 'S-cones' with vertex at $0 \in \mathbb{C}^3$, that is, immersed surfaces invariant by the flow of S. If $\text{sing}(\mathcal{F}(\Omega))$ has codimension two, then each of its components is the closure of an orbit of S. Now we impose a condition which implies the local stability of this kind of singularity by small perturbations of the form defining the foliation.

Let ω be an integrable 1-form in a neighborhood of $p_0 \in \mathbb{C}^3$ and μ be a holomorphic 3-form such that $\mu_{p_0} \neq 0$. Then $d\omega = i_Z(\mu)$, where Z is a holomorphic vector field. The integrability of ω is equivalent to $i_Z(\omega) = 0$. It is not difficult to see that if p_0 is a GK singularity of ω, then we have two possibilities as follows.

1. $Z(p_0) \neq 0$. In this case we have a singularity of Kupka type, that is the foliation is locally the product of a singular foliation in dimension two by a non-singular one of dimension one.
2. $Z(p_0) = 0$ and p_0 is an isolated singularity of Z. In this case, there exists a neighborhood U of p_0 such that all singularities of ω in $U \setminus \{p_0\}$ are of Kupka type. Let $L := DZ(p_0)$ be the linear part of Z at p_0 and $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of L. Note that $\lambda_1 + \lambda_2 + \lambda_3 = 0$. This implies that we have three sub-cases.
   - 2a. $\lambda_1, \lambda_2, \lambda_3 \neq 0$. In this case, if we take $p_0 = 0$, the second jet of ω at p_0 is of the form

     $$j^2(\omega)_0 = ayz\,dx + bxz\,dy + cxy\,dz = xyz\left(a\frac{dx}{x} + b\frac{dy}{y} + c\frac{dz}{z}\right),$$

     where $\lambda_1 = c - b$, $\lambda_2 = a - c$ and $\lambda_3 = b - a$. When $a, b, c \neq 0$ it is proven in [**4**] that there exists a germ of vector field X at p_0 such that $[X, Z] = 0$ and

     $$i_X i_Z(dx \wedge dy \wedge dz) = f\omega,$$

     where $f(p_0) \neq 0$, so that the foliation is locally generated by an action of $\mathbb{C}^2$. It is also proven in [**4**] that if the triple (a, b, c) satisfies some conditions of non-resonance, then there exists a local coordinate system (x, y, z) such that $\omega = xyz(a(dx/x) + b(dy/y) + c(dz/z))$. For this reason we say that the singularity is of *logarithmic type* (even if ω is not equivalent to its 2-jet).
   - 2b. One of the eigenvalues, say λ_3, is zero and the other two satisfy $\lambda_1 = -\lambda_2 \neq 0$. We call this type of singularity *degenerate*.

An example of this situation is $\omega = xy\,dz + z^n(ax\,dy + by\,dx)$, where $a \cdot b \cdot (a-b) \neq 0$ and $n \geq 2$. In this case, if we take $\mu = dx \wedge dy \wedge dz$ then we get $d\omega = i_Z\mu$ where

$$Z = x(1 - bnz^{n-1})\frac{\partial}{\partial x} - y(1 - anz^{n-1})\frac{\partial}{\partial y} + (b-a)z^n\frac{\partial}{\partial z}.$$

Note that $0 \in \mathbb{C}^3$ is an isolated singularity of Z with multiplicity $\mathrm{mult}(Z, 0) = n$ and that the eigenvalues of $DZ(0)$ are $1, -1, 0$.
We observe that this case does not happen in the singularities of the examples of §2.2.

2c. $\lambda_1, \lambda_2, \lambda_3 = 0$. In this case, the germ of Z at p_0 is nilpotent (as a derivation in the local ring of formal power series at p_0).

Definition 3. We say that p_0 is a *quasi-homogeneous* singularity of ω if p_0 is an isolated singularity of Z and the germ of Z at p_0 is nilpotent.

This definition is justified by the following result (cf. [**12**]).

THEOREM A. *Let $p_0 \in \mathbb{C}^3$ be a quasi-homogeneous singularity of an integrable 1-form ω. Then there exist two holomorphic vector fields S and Z and a local chart $(U, (x, y, z))$ around p_0 such that $x(p_0) = y(p_0) = z(p_0) = 0$ and:*

(a) *$\omega = \alpha i_S i_Z(dx \wedge dy \wedge dz)$, $\alpha \in \mathbb{Q}_+$ and $d\omega = i_Z(dx \wedge dy \wedge dz)$;*

(b) *$S = px(\partial/\partial x) + qy(\partial/\partial y) + rz(\partial/\partial z)$, where p, q and r are positive integers with* $\gcd(p, q, r) = 1$*;*

(c) *p_0 is an isolated singularity for Z, Z is polynomial in the chart $(U, (x, y, z))$ and $[S, Z] = \ell Z$, where $\ell \geq 1$.*

Definition 4. Let $p_0 \in \mathbb{C}^3$ be a quasi-homogenous singularity of ω. We say that it is of *type $(p, q, r; \ell)$*, if for some local chart and vector fields S and Z, the properties (a), (b) and (c) of Theorem A are satisfied.

Remark 1. Let p_0 be a quasi-homogenous singularity of type $(p, q, r; \ell)$ of an integrable 1-form ω. If S and Z are as in Theorem A, then the multiplicity of Z at the singularity p_0, $\mathrm{mult}(Z, p_0)$ (also called the Milnor number), is given by

$$\mathrm{mult}(Z, p_0) = \frac{(\ell + p)(\ell + q)(\ell + r)}{pqr}. \tag{3}$$

In particular, pqr must divide $(\ell + p)(\ell + q)(\ell + r)$. The proof of this fact can be found in [**12**].

We can now state the stability result.

PROPOSITION 1. *Let $(\Omega_s)_{s\in\Sigma}$ be a holomorphic family of integrable 1-forms defined in a neighborhood of a compact ball $B = \{z \in \mathbb{C}^3; |z| \leq \rho\}$, where Σ is a neighborhood of $0 \in \mathbb{C}^k$. Suppose that all singularities of Ω_0 in B are GK and that* $\mathrm{sing}(d\Omega_0) \subset \mathrm{int}(B)$*. Then there exists $\epsilon > 0$ such that if $s \in B(0, \epsilon)$, then all singularities of Ω_s in B are GK. Moreover, if $0 \in B$ is a logarithmic or quasi-homogenous singularity of type $(p, q, r; \ell)$ then there exists a holomorphic map $B(0, \epsilon) \ni s \mapsto z(s)$, such that $z(0) = 0$ and $z(s)$*

is a GK singularity of Ω_s of the same type (logarithmic or quasi-homogenous of the type $(p, q, r; \ell)$, according to the case).

Proof. Let $\mu = dx \wedge dy \wedge dz$ and Z_s be such that $d\Omega_s = i_{Z_s}\mu$. Since all singularities of Ω_0 in B are GK, we get that the singularities of Z_0 in B are isolated and that the singularities of Ω_0 which are not singularities of Z_0 are of Kupka type. Let $\mathrm{sing}(Z_0) \cap B = \{p_1, \ldots, p_r\} \subset \mathrm{int}(B)$ with the Milnor numbers $m_j = \mathrm{mult}(Z_0, p_j)$, $j = 1, \ldots, r$. It is well known that $\mathrm{mult}(Z_0, p_j) = \mathrm{PH}(Z_0, p_j) > 0$, the Poincaré–Hopf index of Z_0 at p_j. Hence, there exists $\epsilon_1 > 0$ such that if $|s| < \epsilon_1$ then the singularities of Z_s in B are isolated and

$$\sum_{p \in \mathrm{sing}(Z_s)} \mathrm{mult}(Z_s, p) = \sum_{j=1}^{r} m_j.$$

In particular, the singularities of Z_s in B are isolated, so that all singularities of Ω_s in B are GK.

Now, since the integrability condition for Ω_s is equivalent to $i_{Z_s}\Omega_s = 0$ and the singularities of Z_s in B are isolated, it follows from the parametric De Rham division theorem (cf. [**7**] and [**3**]) that there exists a holomorphic family of 2-forms $(\theta_s)_{s \in B(0,\epsilon_1)}$ such that $\omega_s = i_{Z_s}\theta_s$. Since we are in dimension three, we have $\theta_s = -i_{X_s}\mu$, where $(X_s)_{s \in B(0,\epsilon_1)}$ is a holomorphic family of vector fields. Note that

$$\Omega_s = i_{X_s} i_{Z_s}\mu = i_{X_s}(d\Omega_s) \implies L_{X_s}\Omega_s = \Omega_s \implies L_{X_s}(d\Omega_s) = d\Omega_s. \qquad (*)$$

The last relation above implies that, for s fixed, the set $\mathrm{sing}(Z_s) \cap B = \{p \in B \mid d\Omega_s(p) = 0\}$ is invariant under the flow of X_s. Since $\mathrm{sing}(Z_s) \cap B$ is finite, we obtain that $\mathrm{sing}(Z_s) \cap B \subset \mathrm{sing}(X_s) \cap B$, otherwise Z_s would have non-isolated singularities.

Let us suppose that 0 is a logarithmic or quasi-homogenous singularity of type $(p, q, r; \ell)$ of Ω_0. If we can guarantee that 0 is a non-degenerate singularity of X_0, that is such that $\det(DX_0(0)) \neq 0$, then we can assert the existence of an analytic map $B(0, \epsilon) \ni s \mapsto z(s)$ such that $z(0) = 0$ and $z(s)$ is a non-degenerate singularity of X_s for all $s \in B(0, \epsilon)$. Since $\mathrm{sing}(Z_s) \subset \mathrm{sing}(X_s)$, in this case we can assert that all the singularities of Z_s that appear by bifurcation of 0 must be at $z(s)$. This gives the map $s \mapsto z(s)$ of the statement. Before proving this fact, let us observe that X_s and X_s^1 are vector fields satisfying $(*)$ if and only if $X_s^1 = X_s + f_s Z_s$, where f_s is holomorphic. This fact follows from $i_{Z_s} i_{(X_s^1 - X_s)}\mu = 0$ and the division theorem.

Suppose first that 0 is singularity of logarithmic type of Ω_0. We can not assert *a priori* that 0 is a non-degenerate singularity of X_0. However, we can take, instead of X_s, a vector field of the form $X_s^1 = X_s + aZ_s$, $a \in \mathbb{C}$. Since $\det(DZ_0(0)) \neq 0$, it is possible to choose $a \in \mathbb{C}^*$ such that $\det(DX_0^1(0)) \neq 0$. Note that $z(s)$ will be a logarithmic singularity for Ω_s, since $\det(DZ_s(z(s))) \neq 0$ (for small $|s|$).

Suppose now that 0 is a quasi-homogenous singularity of Ω_0 of the type $(p, q, r; \ell)$. Let us prove that $\det(DX_0(0)) \neq 0$. Since Z_0 is nilpotent, by Theorem A there exists a germ of vector field S at $0 \in \mathbb{C}^3$, such that $[S, Z_0] = \ell Z_0$, $\Omega_0 = \alpha i_S i_{Z_0}\mu$, $\alpha \in \mathbb{Q}_+$, $d\Omega_0 = i_{Z_0}(\mu)$ and $DS(0) = px(\partial/\partial x) + qy(\partial/\partial y) + qz(\partial/\partial z) := S_0$. Set $DX_0(0) = A_0$ and $DZ_0(0) = B_0$. Note that $X_0 = \alpha S + fZ_0$, where $f \in \mathcal{O}_3$, so that $A_0 = \alpha S_0 + f(0)B_0$. On the other hand, the relation $[S, Z_0] = \ell Z_0$ implies that $[S_0, B_0] = \ell B_0$.

Consider a basis of $\mathbb{C}^3$ such that the matrices of S_0 and B_0 are $S_0 = \text{diag}(p, q, r)$ and $B_0 = (b_{ij})_{1\le i,j\le 3}$, respectively, and $B_0 S_0 - S_0 B_0 = \ell B_0$, $\ell > 0$. If we assume that $p \ge q \ge r > 0$, then a straightforward calculation gives that $b_{ij} = 0$ for $j \ge i$. Hence, $\det(A_0) = \det(\alpha S_0) \ne 0$. This implies also that the eigenvalues of A_0 are $\alpha p, \alpha q, \alpha r$.

Fix $\delta > 0$ and $\epsilon > 0$ such that, for $|s| < \epsilon$, X_s has a unique singularity $z(s)$, with $|z(s)| < \delta$, where $s \mapsto z(s)$ is analytic and $\det(DX_s(z(s))) \ne 0$. Recall that Z_0 has an isolated singularity at 0. By Remark 1, we have

$$\text{mult}(Z_0, 0) = \frac{(p+\ell)(q+\ell)(r+\ell)}{p.q.r} > 1.$$

Therefore, if ϵ and δ are small, then, for $|s| < \epsilon$, Z_s has at most $\text{mult}(Z_0, 0)$ singularities in the ball $B(0, \delta)$. As we have seen before, $\text{sing}(Z_s) \cap B(0, \delta) \subset \text{sing}(X_s) \cap B(0, \delta)$. This implies that $\text{sing}(Z_s) \cap B(0, \delta) = \{z(s)\}$ and $\text{mult}(Z_s, z(s)) = \text{mult}(Z_0, 0)$.

Let us prove that the germ of Z_s at $z(s)$ is nilpotent for $|s| < \epsilon$. Set $A_s = DX_s(z(s))$ and $B_s = DZ_s(z(s))$. Since $\text{mult}(Z_s, z(s)) > 1$, at least one of the eigenvalues of B_s is 0, and so their eigenvalues are $b(s), -b(s), 0$, where $b(s) \in \mathbb{C}$. On the other hand, for $|s| < \epsilon$, $\det(A_s) \ne 0$ and so all eigenvalues A_s are non-zero. We are going to use $(*)$ to prove that if $\epsilon > 0$ is small then $b(s) = 0$, so that B_s is nilpotent, if $|s| < \epsilon$. Since $i_{Z_s}(\mu) = d\Omega_s$, we get from $(*)$ that

$$\begin{aligned} i_{Z_s}(\mu) &= L_{X_s}(i_{Z_s}(\mu)) = i_{[X_s,Z_s]}(\mu) + i_{Z_s}(L_{X_s}(\mu)) \\ &= i_{[X_s,Z_s]}(\mu) + \text{div}(X_s) i_{Z_s}(\mu) \\ &\implies [X_s, Z_s] = (1 - \text{div}(X_s))Z_s := h_s Z_s \\ &\implies [A_s, B_s] = g(s) B_s, \end{aligned} \tag{$**$}$$

where $g(s) = h_s(z(s))$. Note that $g(0) = h_0(0) = \alpha\ell := \beta \ne 0$, because

$$[X_0, Z_0] = [\alpha S + f Z_0, Z_0] = (\alpha\ell - Z_0(f))Z_0 \implies g(0) = \alpha\ell - Z_0(f)(0) = \alpha\ell = \beta.$$

If we take $\epsilon > 0$ small enough then $g(s) \ne 0$ and $\det(A_s) \ne 0$, for $|s| < \epsilon$. Suppose for a contradiction that $b(s) \ne 0$ for some $|s| < \epsilon$. In this case, we can write A_s and B_s in matrix form, in some basis of $\mathbb{C}^3$, as $B_s = \text{diag}(b(s), -b(s), 0)$ and $A_s = (a_{ij})_{1\le i,j\le 3}$, so that $(**)$ is equivalent to

$$B_s A_s - A_s B_s = g(s) B_s \implies g(s)b(s) = b(s)a_{11} - a_{11}b(s) = 0 \implies g(s) = 0,$$

because $b(s) \ne 0$. This contradicts $g(s) \ne 0$ and shows that $b(s) = 0$. Therefore, B_s is nilpotent for $|s| < \epsilon$.

Now, it follows from Theorem A that $z(s)$ is a quasi-homogenous singularity of Ω_s. It remains to prove that it is of the type $(p, q, r; \ell)$. Let S_s be as in Theorem A, so that $\alpha(s) i_{S_s} i_{Z_s}(\mu) = \Omega_s$, $\alpha(s) \in \mathbb{Q}_+$, $[S_s, Z_s] = \ell(s) Z_s$, $\ell(s) \in \mathbb{Z}_+$, $DS_s(z(s))$ is semi-simple and their eigenvalues are positive integers, say $p(s), q(s), r(s)$, where $\gcd(p(s), q(s), r(s)) = 1$ and $p(s) \ge q(s) \ge r(s)$. With the same argument that we have used for $s = 0$, we have $X_s = \alpha(s) S_s + f_s Z_s$, where $f_s \in \mathcal{O}_3(z(s))$ and $DX_s(z(s)) = A_s$ has the same eigenvalues as $\alpha(s).DS_s(z(s))$. Since the function $s \mapsto A_s$ is analytic, the functions $s \mapsto \alpha(s)p(s) \in \mathbb{Q}_+$, $s \mapsto \alpha(s)q(s) \in \mathbb{Q}_+$ and $s \mapsto \alpha(s)r(s) \in \mathbb{Q}_+$

must be constant. Therefore, $p(s) \equiv p$, $q(s) \equiv q$, $r(s) \equiv r$ and $\alpha(s) \equiv \alpha$ (since $\gcd(p(s), q(s), r(s)) = 1$ and we have chosen $p \geq q \geq r$). Hence, the eigenvalues of S_s are p, q, r. Finally, by (∗∗) we have $[X_s, Z_s] = h_s Z_s$ and, with the same proof as in the case $s = 0$, $h_s(z(s)) = \alpha(s)\ell(s) = \alpha\ell(s) \in \mathbb{Q}_+$. Since $s \mapsto h_s(z(s)) \in \mathbb{Q}_+$ is analytic, we get that $\alpha\ell(s) \equiv \alpha\ell$, and so $\ell(s) \equiv \ell$. This finishes the proof of the proposition. □

Let us state two consequences of Proposition 1. The first follows immediately from the proposition.

COROLLARY 1. *Let $\mathcal{F}_0$ be a codimension one GK foliation on a compact complex threefold M. Then there exists a neighborhood $\mathcal{U}$ of $\mathcal{F}_0$ in the space of codimension one foliations, such that any $\mathcal{F} \in \mathcal{U}$ is GK.*

COROLLARY 2. *If p_0 is a GK singularity of a foliation $\mathcal{F}$, then the sheaf of germs of vector fields at p_0 tangent to $\mathcal{F}$ is locally free and has two generators.*

Proof. In fact, if $\mathcal{F}$ is defined by ω in a neighborhood of p_0, then we can write $d\omega = i_Z\mu$ and $\omega = i_X i_Z(\mu)$, where $\mu_{p_0} \neq 0$ and the germ of Z at p_0 has an isolated singularity at p_0. Let Y be a germ of vector field such that $i_Y(\omega) = i_Y i_X i_Z(\mu) = 0$. This implies that $Y = aX + bZ$ where a and b are holomorphic outside $\mathrm{sing}(\omega)$. Since $\mathrm{sing}(\omega)$ has codimension two, it follows from Hartog's Theorem that a and b can be extended to a neighborhood of p_0. □

Remark 2. Let p_0 be an isolated singularity of a codimension one foliation $\mathcal{F}$ on a threefold (for instance, a Morse singularity). Then the sheaf of germs of vector fields at p_0 tangent to $\mathcal{F}$ is not locally free. In fact, it follows from Malgrange's theorem (cf. [**14**]), that $\mathcal{F}$ has a local holomorphic first integral. This implies the assertion, as the reader can check (see also [**11**]).

Remark 3. If $\mathcal{F}$ is a GK foliation on M, the tangent bundle of $\mathcal{F}$, $T_{\mathcal{F}}$, is a rank two vector bundle over M. Moreover, there is a morphism $\pi : T_{\mathcal{F}} \to TM$ with the following property. If $U \subset M$ is an open set and $\sigma : U \to T_{\mathcal{F}}$ is a holomorphic (respectively meromorphic) section of $T_{\mathcal{F}}|_U$ then $\pi \circ \sigma : U \to TM$ is a holomorphic (respectively meromorphic) vector field tangent to $\mathcal{F}$. Conversely, if X is a holomorphic (respectively meromorphic) vector field on U tangent to $\mathcal{F}$, then there exists a holomorphic (respectively meromorphic) section σ of $T_{\mathcal{F}}$ on U such that $\pi \circ \sigma = X$. Let us also observe that, when $p \in \mathrm{sing}(\mathcal{F})$ then $\dim(\ker(\pi_p)) = 1$ if p is a Kupka singularity, whereas $\dim(\ker(\pi_p)) = 2$ if p is a logarithmic or quasi-homogenous singularity.

This motivates the following definition.

Definition 5. We say that a codimension one foliation $\mathcal{F}$ on a complex threefold M is *generated* by two foliations of dimension one, say $\mathcal{G}_1$ and $\mathcal{G}_2$, if for any $p \in M$ there exists a neighborhood U of p and holomorphic vector fields X_1 and X_2 on U such that the following occur.

(a) $\mathcal{G}_j$ is defined in U by X_j, $j = 1, 2$.

(b) $\mathcal{F}|_U$ is defined by the 1-form $\omega = i_{X_1} i_{X_2} \mu$, where μ is a non-vanishing 3-form on U. In particular, we have that $\mathcal{G}_1$ and $\mathcal{G}_2$ are tangent to $\mathcal{F}$ and that:

 (b1) If $p \in M \setminus (\mathrm{sing}(\mathcal{G}_1) \cup \mathrm{sing}(\mathcal{G}_2))$ and $T_p\mathcal{G}_1 \neq T_p\mathcal{G}_2 \subset T_pM$, then $T_p\mathcal{F} = T_p\mathcal{G}_1 \oplus T_p\mathcal{G}_2$;

 (b2) $\mathrm{sing}(\mathcal{F}) = \mathrm{sing}(\mathcal{G}_1) \cup \mathrm{sing}(\mathcal{G}_2) \cup \mathcal{D}$, where

$$\mathcal{D} = \{p \in M \setminus \mathrm{sing}(\mathcal{G}_1) \cup \mathrm{sing}(\mathcal{G}_2) \mid T_p\mathcal{G}_1 = T_p\mathcal{G}_2\}.$$

PROPOSITION 2. *Let $\mathcal{F}$ be a GK foliation on M and $T_{\mathcal{F}}$ be its tangent bundle. Then the following occur.*

(a) *To any line sub-bundle L of $T_{\mathcal{F}}$ corresponds a foliation by curves $\mathcal{G}_L$ on M with the following properties:*

 (1) *$\mathcal{G}_L$ is tangent to $\mathcal{F}$;*

 (2) $\mathrm{sing}(\mathcal{G}_L) \subset \mathrm{sing}(\mathcal{F})$.

(b) *$T_{\mathcal{F}}$ splits as a sum of two line bundles if and only if $\mathcal{F}$ is generated by two foliations of dimension one.*

The proof of the proposition is straightforward and is left for the reader.

In the next section we will see some examples of GK foliations on $\mathbb{CP}(3)$. In all examples the bundle $T_{\mathcal{F}}$ splits. This motivates Problem 2 in §1.

2.2. *Examples.* This section is devoted to describing some examples of GK foliations on $\mathbb{CP}(3)$. Each example is generated by two foliations of dimension one, $\mathcal{G}_1$ and $\mathcal{G}_2$, in the sense of Definition 5. One of these one-dimensional foliations, say $\mathcal{G}_1$, will be generated by a global vector field S on $\mathbb{CP}(3)$, which in some affine coordinate system $(x, y, z) \in \mathbb{C}^3 \subset \mathbb{CP}(3)$ is like in (1): $S = px(\partial/\partial x) + qy(\partial/\partial y) + rz(\partial/\partial z)$, where $p, q, r \in \mathbb{N}$, $\gcd(p, q, r) = 1$ and $p > q > r$. On the other hand, $\mathcal{G}_2$ will be of degree $d \geq 1$, so that the foliation will be of degree $\nu = d + 1$.

Being foliations in $\mathcal{F}((p, q, r); \ell; d + 1)$, all the examples that we give share a geometrical pattern that we now explain. As the singular locus of the foliation is invariant by a global vector field in $\mathbb{CP}(3)$, it is globally fixed by an infinite group of projective automorphisms: that given by the flow of S. Each curve in the singular locus has to be of a very special type.

Klein and Lie showed (see, e.g., [**9**]) that a curve $\mathbb{CP}(n)$ fixed by the action of an infinite group of projective automorphisms is rational algebraic. If it is of degree $p \geq n$, it is obtained as an adequate linear projection of the rational normal curve $\Gamma_p \subset \mathbb{CP}(p)$, i.e. $\mathbb{CP}(1)$ embedded as $\Gamma_p(s : t) := (t^p : t^{p-1}s : \cdots : ts^{p-1} : s^p)$. For $n = 3$, they showed that the projected curve could be written, after a change of coordinates, as (in the affine open set E_0)

$$\gamma_{p,q,r}(t) := (t^p, t^q, t^r),$$

where $p \geq q \geq r \geq 1$ are positive integers. A curve so parameterized is fixed by the projective transformations $x' = \alpha^p x$, $y' = \alpha^q y$, $z' = \alpha^r z$ that correspond to changing t by αt, and fixing the points $A = (1 : 0 : 0 : 0)$ and $B = (0 : 0 : 0 : 1)$. Finally, note that if the numbers p, q, r admit a greatest common divisor $k > 1$, then the curve (Klein–Lie) is a degree p/k one, counted k times. In this case we can substitute the parameter t by a new parameter t'.

Let us write $\Gamma_{p,q,r} := \overline{\gamma_{p,q,r}} \subset \mathbb{CP}(3)$. When $p > q > r$, $\Gamma_{p,q,r}$ is smooth at B if and only if $r = 1$, whereas it is smooth at A if and only if $p - q = 1$, so that $\Gamma_{p,q,r}$ is smooth if and only if $p - q = r = 1$. Moreover, when $r = 1$ and $p \geq q + 2 \geq 4$, it has the point A as its only (cuspidal) singularity. On the other hand, if $r > 1$, B is also a singular point of $\Gamma_{p,q,r}$.

Let us insist on the fact that *not every cuspidal rational algebraic curve* is a Klein–Lie curve. In particular, not all the cuspidal rational curves with the same degree and number of cusps are projectively equivalent (see, e.g., [**8**]).

Let t be the coordinate on $\mathbb{C}$ and consider the vector field $t(\partial/\partial t)$ on $\mathbb{C}$. The vector field $(\gamma_{p,q,r})_*(t(\partial/\partial t))$ can be extended to $\mathbb{C}^3$ as $S = px(\partial/\partial x) + qy(\partial/\partial y) + rz(\partial/\partial z)$. On the other hand, $(\gamma_{p,q,r})_*(t^{\ell+1}(\partial/\partial t))$, $\ell + r \geq 0$, can be extended as a polynomial vector field X which is S-quasi-homogeneous, if certain arithmetical relations hold among p, q, r and ℓ. When $r = 1$, which is the case that we consider in the examples, this extension can be done so that X is S-quasi-homogeneous of weight ℓ. Thus we can define a foliation generated by the subfoliations given by S and X, which will be of degree ν if the foliation generated by X is of degree $d = \nu - 1$.

Example 1. Klein–Lie foliations with one quasi-homogeneous singularity.
We give examples that extend one found in [5], giving rise to the so-called *exceptional components*. They appear in a family that we denote as Klein–Lie foliations in $\mathbb{CP}(3)$. Klein–Lie foliations are not always GK, but for each degree there is exactly one which is GK, and that has just one quasi-homogenous singularity.

Klein–Lie foliations in $\mathbb{C}^3$ and polynomial actions of $\mathfrak{aff}(\mathbb{C}) \times \mathbb{C}^3 \to \mathbb{C}^3$.
We are going to study the families $\mathcal{F}((p, q, 1), -1, d + 1)$ for some d, which we are able to choose. Recall that if t is the coordinate on $\mathbb{C}$, the two basic complete vector fields on $\mathbb{C}$, that are the infinitesimal generators of the action of $\mathfrak{aff}(\mathbb{C})$, are $t(\partial/\partial t)$ and $(\partial/\partial t)$. As noted above, the vector fields $(\gamma_{p,q,1})_*(t(\partial/\partial t))$ and $(\gamma_{p,q,1})_*(\partial/\partial t)$, can be extended as

$$S = px\frac{\partial}{\partial x} + qy\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$$

and

$$X_{\boldsymbol{\tau}} = p\Bigg(\sum_{i+qj=p-1} \tau_{ij} z^i y^j\Bigg)\frac{\partial}{\partial x} + qz^{q-1}\frac{\partial}{\partial y} + \frac{\partial}{\partial z} \quad \text{where} \sum_{i+qj=p-1} \tau_{ij} = 1.$$

The vector fields S and $X_{\boldsymbol{\tau}}$ are complete, are linearly independent outside the curve $\gamma_{p,q,1}$ and they satisfy the relation $[S, X_{\boldsymbol{\tau}}] = -X_{\boldsymbol{\tau}}$, thus they generate an action of $\mathfrak{aff}(\mathbb{C}) \times \mathbb{C}^3 \to \mathbb{C}^3$. To define a foliation associated to it, we consider the polynomial 1-form

$$\begin{aligned}\omega^{\boldsymbol{\tau}}_{p,q,1} = i_S i_{X_{\boldsymbol{\tau}}}\, dz \wedge dy \wedge dx &= q(y - z^{q-1})\, dx \\ &+ p\left(\sum \tau_{ij} z^{i+1} y^j - x\right) dy + pq\left(z^{q-1}x - \sum \tau_{ij} z^i y^{j+1}\right) dz,\end{aligned}$$

which has degree $\mathrm{dg}(\omega^{\tau}_{p,q,r}) = \mathrm{dg}(X_\tau)+1$. The relation $d\omega^{\boldsymbol\tau}_{p,q,1} = (p+q)i_{X_{\boldsymbol\tau}}\, dx\wedge dy\wedge dz$ implies that $\gamma_{p,q,1}$ is the Kupka set of the foliation represented by $\omega^{\boldsymbol\tau}_{p,q,1}$ and it has transversal type $\eta = -pv\,du + qu\,dv$. Moreover, the diffeomorphism

$$\phi_{\boldsymbol\tau}(v,u,t) = \left(v + p\sum \tau_{ij}\int_0^t s^i(u+s^q)^j ds,\, u + t^q,\, t\right)$$

which is the time t of the flow of the vector field $X_{\boldsymbol\tau}$, with initial condition $(v,u,0)$, satisfies the relation $\phi^*_{\boldsymbol\tau}(\omega^{\boldsymbol\tau}_{p,q,1}) = -pv\,du + qu\,dv$. Therefore, the foliation has a rational first integral

$$H_{\boldsymbol\tau} = \frac{(y-z^q)^p}{(x-\psi_{\boldsymbol\tau}(z,y))^q} = \frac{f^p}{g^q}$$

where $\psi_{\boldsymbol\tau}$ is a polynomial of degree p on the variable z and depending on the parameters τ_{ij}.

Now we study the extension to $\mathbb{CP}(3)$ of the foliations obtained above. It is given by the homogeneous 1-form $\overline{\omega}^{\boldsymbol\tau}_{p,q,1} = \omega_1\, dz_1 + \omega_2\, dz_2 + \omega_3\, dz_3 + \omega_4\, dz_4$, obtained from $\omega^{\boldsymbol\tau}_{p,q,1}$. Note that, by means of the action of $\mathbf{PGL}(4,\mathbb{C})$ on $\overline{\omega}^{\boldsymbol\tau}_{p,q,1}$, we get a family of foliations: we refer to all of them as *Klein–Lie* foliations in $\mathbb{CP}(3)$.

The degree of the foliation defined by $\overline{\omega}^{\boldsymbol\tau}_{p,q,1}$ is $d+1 = \max\{q, i+j+1 \mid \tau_{ij}\neq 0\}$. Moreover,

$$\omega_1 = qz_4(z_4^d z_2 - z_4^{d-q+1}z_3^q)$$

$$\omega_2 = pz_4\left(\sum \tau_{ij} z_4^{d-i-j} z_3^{i+1} z_2^j - z_4^d z_1\right)$$

$$\omega_3 = pqz_4\left(z_4^{d-q+1}z_3^{q-1}z_1 - \sum \tau_{ij} z_4^{d-i-j} z_3^i z_2^{j+1}\right)$$

$$\omega_4 = \left(p(q-1)\sum \tau_{ij} z_4^{d-i-j} z_3^{i+1} z_2^{j+1} + (p-q)z_4^d z_2 z_1 - q(p-1)z_4^{d-q+1}z_3^q z_1\right)$$

with $1 < q \le d+1 \le p$.

On the other hand, if $\omega = pG\,dF - qF\,dG$, where $F|_{E_1} = f$ and $G|_{E_1} = g$ are homogeneous of degree q and p, respectively, we obtain

$$\omega = z_4^{p+q-d-2}\omega^{\tau}_{p,q,1}.$$

Remark 4. The hypothesis that $\mathcal{F}((p,q,r),\ell,d)$ contains a GK foliation is actually necessary for the conclusion that it is an irreducible component of $\mathcal{F}(d,3)$. The last equation implies that $\mathcal{F}((p,2,1),-1,p) \subset \mathcal{R}(2,p)$, the *rational component* [**5**], and Theorem 1 is not true for these families, since the foliations in $\mathcal{R}(2,p)$ are not GK.

In order to study the singular set, observe that one of the following possibilities holds:

(1) $q = d+1$ and $a+b < d$, where $p-1 = bq+a$, $\quad 0 \le a < q$;

(2) $q = d+1$ and there is a unique pair (i_0, j_0) with $\tau_{i_0 j_0} \neq 0$ and $j_0 = d - i_0$;

(3) $q < d$ and there is a unique pair (i_0, j_0) with $\tau_{i_0 j_0} \neq 0$ and $j_0 = d - i_0$.

In all cases, the hyperplane $\{z_4 = 0\}$ is invariant by the foliation defined by $\overline{\omega}^{\boldsymbol\tau}_{p,q,1}$. Concerning its singular locus, it is the union of $\Gamma_{p,q,1}$ and the set $\{z_4 = \omega_4(z_1, z_2, z_3, 0) = 0\}$ which, according to the possibilities discussed above, is:

(1) $\{z_3^{d+1} = z_4 = 0\} \cup \{z_1 = z_4 = 0\}$;
(2) $\{z_3^{i_0+1} = z_4 = 0\} \cup \{z_4 = p(q-1)\tau_{i_0,d-i_0} z_2^{d-i_0+1} - q(p-1)z_1 z_3^{d-i_0} = 0\}$;
(3) $\{z_3^{i_0+1} = z_4 = 0\} \cup \{z_2^{j_0+1} = z_4 = 0\}$.

To study the foliation around the point $(1 : 0 : 0 : 0)$, we choose its affine open neighborhood E_1 and calculate the rotational of the form which represents the foliation $\eta^{\boldsymbol\tau}_{p,q,1} := \overline{\omega^{\boldsymbol\tau}_{p,q,1}}|_{E_1}$

$$\eta^{\boldsymbol\tau}_{p,q,1} = -\Big(p(q-1)\sum \tau_{ij} u^{d-i-j} w^{i+1} v^{j+1} + (p-q)u^d v - q(p-1)u^{d-q+1} w^q\Big)\, du$$
$$+ p\Big(\sum \tau_{ij} u^{d-i-j+1} w^{i+1} v^j - u^{d+1}\Big)\, dv$$
$$+ pq\Big(u^{d-q+2} w^{q-1} - \sum \tau_{ij} u^{d-i-j+1} w^i v^{j+1}\Big)\, dw.$$

Its exterior derivative is $d\eta^{\boldsymbol\tau}_{p,q,1} = Q^{(p,q,\boldsymbol\tau)}_{uw}\, du \wedge dw + Q^{(p,q,\boldsymbol\tau)}_{wv}\, dw \wedge dv + Q^{(p,q,\boldsymbol\tau)}_{vu}\, dv \wedge du$, where

$$Q^{(p,q,\boldsymbol\tau)}_{uw} = q(p(d+2)-q)u^{d-q+1} w^{q-1} + p(p-q(d+1))\sum \tau_{ij} u^{d-i-j} w^i v^{j+1},$$
$$Q^{(p,q,\boldsymbol\tau)}_{wv} = p(q+p-1)\sum \tau_{ij} u^{d-i-j+1} w^i v^j,$$
$$Q^{(p,q,\boldsymbol\tau)}_{vu} = (p-q+p(d+1))u^d - \sum p(d-p-q+3)\tau_{ij} u^{d-i-j} w^{i+1} v^j,$$

and the rotational is given by

$$\mathbf{R}_{\eta^{\tau}_{p,q,1}} = Q^{(p,q,\boldsymbol\tau)}_{wv} \frac{\partial}{\partial u} + Q^{(p,q,\boldsymbol\tau)}_{vu} \frac{\partial}{\partial w} + Q^{(p,q,\boldsymbol\tau)}_{uw} \frac{\partial}{\partial v}.$$

The only case in which the rotational above has isolated singularities is when $q = d+1$ and there is just one τ_{ij} different from zero (case 2), that corresponding to $i = 0$ and $j = d$, which is 1. In that case, the *Klein–Lie* foliation is GK and the vector field X is given by

$$X = (d^2+d+1)y^d \frac{\partial}{\partial x} + (d+1)z^d \frac{\partial}{\partial y} + \frac{\partial}{\partial z},$$

and the point $A = (1:0:0:0)$ is a quasi-homogenous point of the type $(d^2+d+1, d^2+d, d^2; d^3)$. By changing to the affine coordinates $E_2 = \{(r : 1 : s : t) | (r,s,t) \in \mathbb{C}^3\}$ and $E_3 = \{(r : s : 1 : t) | (r,s,t) \in \mathbb{C}^3\}$, it can be shown that all points in $\mathbb{CP}(3) \backslash \{(1:0:0:0)\}$ are of Kupka type and that $\operatorname{sing}(\mathcal{F})$ is the union of $\overline{\Gamma_{d^2+d+1,d+1,1}}$ with the two curves $\{z_3 = z_4 = 0\}$ and $\{z_4 = (d(d+1)+1)(d-1)z_2^{d+1} - (d-1)(d(d+1)+1)z_1 z_3^d = 0\}$. We leave the details for the reader.

Recall that the foliation has a meromorphic first integral F, which in the affine chart E_0 can be written as

$$F(x,y,z) = \frac{(y-z^q)^p}{(x+z^p\, h(y/z^q))^q},$$

where

$$h(t) = \sum_{j=0}^{d} h_j t^j$$

is the solution of $q(t-1)h'(t) = p(t^d + h(t))$.

In all the other cases, we can check that there is a one-dimensional set of singular points on which the rotational vanishes, so the corresponding *Klein–Lie* foliation is not GK.

Example 2. Let us consider the curve $\gamma_{3,2,1}$ and the extension of the vector field $(\gamma_{3,2,1})_*(t(\partial/\partial t))$ as $S = 3x(\partial/\partial x) + 2y(\partial/\partial y) + z(\partial/\partial z)$ and the polynomial vector field $X = P + z^3 R$, where $R = x(\partial/\partial x) + y(\partial/\partial y) + z(\partial/\partial z)$ is the radial vector field on $\mathbb{C}^3$ and $P = P_1(\partial/\partial x) + P_2(\partial/\partial y) + P_3(\partial/\partial z)$, with

$$\begin{aligned} P_1(x, y, z) &= ax^2 + bxyz + cy^3 \\ P_2(x, y, z) &= dxy + exz^2 + fy^2 z \\ P_3(x, y, z) &= gxz + hy^2 + iyz^2. \end{aligned} \tag{4}$$

We consider this set of polynomials parameterized by $(a, b, c, d, e, f, g, h, i) \in \mathbb{C}^9$. It is not difficult to see that $[S, X] = 3X$ and so X is a weighted S-quasi-homogeneous degree 3 polynomial vector field extending $(\gamma_{3,2,1})_*(t^4(\partial/\partial t))$. The foliations defined by S and X on $\mathbb{CP}(3)$ generate a codimension one foliation of degree four on $\mathbb{CP}(3)$, which will be denoted by $\mathcal{F}(P)$.

We take P in such a way that $d(i_P(dx \wedge dy \wedge dz)) = 0$, which is equivalent to $\operatorname{div}(P) := P_{1x} + P_{2y} + P_{3z} = 0$, or to $2a + d + g = b + 2f + 2i = 0$. In this case, if $\Omega_P = i_S i_X(dx \wedge dy \wedge dz)$, then Ω_P defines $\mathcal{F}(P)$ in the affine chart E_0. A straightforward calculation (using $\operatorname{div}(P) = 0$), gives $d\Omega_P = i_{Z_P}(dx \wedge dy \wedge dz)$, where

$$Z_P = 9P + z^3(9R - 6S).$$

As the reader can check, the set

$$A_0 = \{P \mid 2a + d + g = b + 2f + 2i = 0 \\ \text{and } Z_P \text{ has a non-isolated singularity at } 0 \in E_0 \simeq \mathbb{C}^3\},$$

is an algebraic subset of codimension three of $\mathbb{C}^9$. Therefore, if $P \notin A_0$ then $\mathcal{F}(P)$ has a quasi-homogenous singularity at $0 \in E_0$. Moreover, $\operatorname{sing}(\mathcal{F}(P)) \cap E_0$ contains seven integral curves of S, say Γ_j, $j = 1, \ldots, 7$, where $\Gamma_6 = (y = z = 0)$, $\Gamma_7 = (x = y = 0)$ and the others are generic trajectories of S of the form $\Gamma_j = \{(\alpha_j t^3, \beta_j t^2, t) \mid t \in \mathbb{C}\}$, $\alpha_j, \beta_j \neq 0$.

Now, let us see how $\mathcal{F}_P$ looks in the chart $E_1 = \{(1 : w : v : u) \mid (u, v, w) \in \mathbb{C}^3\}$. In this chart we have $S = -S_1$, where

$$S_1 = 3u\frac{\partial}{\partial u} + 2v\frac{\partial}{\partial v} + w\frac{\partial}{\partial w}. \tag{5}$$

Since X has a pole of order two at $(u = 0)$, the foliation $\mathcal{F}(P)$ is generated in this chart by S_1 and $X_1 := u^2 X$. Observe that

$$[S_1, X_1] = -[S, x^{-2}X] = -S(x^{-2})X - x^{-2}[S, X] = 3X_1.$$

This implies that X_1 is of the same type as X, that is $X_1 = Q + mw^3 R$, where $Q = Q_1(\partial/\partial x) + Q_2(\partial/\partial y) + Q_3(\partial/\partial z)$ and Q_1, Q_2, Q_3 are as in (4) (by changing $x \to u$, $y \to v$, $z \to w$ and the parameters $(a, \ldots, i) \to (a', \ldots, i')$). In other words, the point $(1 : 0 : 0 : 0) \in E_1$ is a quasi-homogenous singularity of $\mathcal{F}(P)$ for a generic P. It is possible to verify, by taking other affine charts, that $\mathcal{F}(P)$ is a GK foliation with two quasi-homogenous singularities, the points $p_0 := (0 : 0 : 0 : 1) \in E_0$

and $p_1 := (1 : 0 : 0 : 0) \in E_1$. Moreover, $\operatorname{sing}(\mathcal{F}(P)) = \bigcup_{j=0}^{7} \overline{\Gamma_j}$, where $\Gamma_0 = \{(1 : w : v : u) \in E_1 \mid u = v = 0\}$ and the points in $\operatorname{sing}(\mathcal{F}(P)) \setminus \{p_0, p_1\}$ are of Kupka type. We leave the details for the reader.

Example 3. In this example we take again the curve $\gamma_{3,2,1}$ and $S = 3x(\partial/\partial x)+2y(\partial/\partial y)+z(\partial/\partial z)$, as in Example 2, and

$$X = (ay^2 + bxz)\frac{\partial}{\partial x} + (cx + dy\,z)\frac{\partial}{\partial y} + (ey + fz^2)\frac{\partial}{\partial z}, \tag{6}$$

so that $[S, X] = X$.

The foliation generated by S and X on $\mathbb{CP}(3)$ has degree three in this case. It is defined in the chart E_0 by the form $\Omega = i_S i_X(dx \wedge dy \wedge dz)$. We will denote this foliation by $\mathcal{F}(S, X)$. If we take X in such a way that $\operatorname{div}(X) = 0$, that is $b + d + 2f = 0$, then $d\Omega = i_Z(dx \wedge dy \wedge dz)$, where $Z = 7X$. As the reader can verify, if we take $X \notin A$, where

$$A = \{X \mid X \text{ is as in (6) and } abcdef(acf + bde) = 0\},$$

then $0 \in E_0 \simeq \mathbb{C}^3$ is an isolated zero of $d\Omega$, that is a quasi-homogenous singularity of $\mathcal{F}(S, X)$. For generic $X \notin A$, $\operatorname{sing}(\mathcal{F}(S, X)) \cap E_0$ has three components: $\Gamma_0 = (x = y = 0)$ and Γ_1, Γ_2, which are the closure of two trajectories of S, not contained in the coordinate planes.

If we change coordinates to the chart $E_1 = \{(1 : w : v : u) \mid (u, v, w) \in \mathbb{C}^3\}$, we find that $\mathcal{F}(S, X)$ is generated in E_1 by $S = -S_1$, where S_1 is as in (5) and

$$X_1 = uX = (-buv - auw^2)\frac{\partial}{\partial u} + (euw + (f - b)v^2 - avw^2)\frac{\partial}{\partial v} + (cu + (d - b)vw - aw^3)\frac{\partial}{\partial w}.$$

Therefore, $\mathcal{F}(S, X)$ is represented in this chart by $\Omega_1 = i_{S_1} i_{X_1}(du\wedge dv\wedge dw)$. On the other hand, we have $d\Omega_1 = i_{Z_1}(du \wedge dv \wedge dw)$, where $Z_1 = 8X_1 - \operatorname{div}(X_1)S_1$. As the reader can check, this implies that under generic assumptions on the coefficients a, b, c, d, e, f, the point $0 = p_1 \in E_1$ is an isolated singularity of Z_1, so that it is a quasi-homogenous singularity of $\mathcal{F}(S, X)$. In this chart, the plane $(u = 0)$ is invariant for $\mathcal{F}(S, X)$ and

$$\operatorname{sing}(\mathcal{F}(S, X)) \cap E_1 = (\overline{\Gamma_1} \setminus \{x = 0\}) \cup (\overline{\Gamma_2} \setminus \{x = 0\}) \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5,$$

where $\Gamma_3 = (u = v = 0)$, $\Gamma_4 = (u = w = 0)$ and Γ_5 is a parabola in the plane $(u = 0)$ of the form $\{(0, \alpha t^2, \beta t) \mid t \in \mathbb{C}\}$.

We observe that the curves $\overline{\Gamma_0}$, $\overline{\Gamma_4}$ and $\overline{\Gamma_5}$ meet at the point $(0 : 0 : 1 : 0)$, which is a singularity of logarithmic type for $\mathcal{F}(S, X)$. It can be proved, by changing variables to other affine charts, that $\operatorname{sing}(\mathcal{F}(S, X)) = \bigcup_{j=0}^{5} \overline{\Gamma}_j$ and all points in $\operatorname{sing}(\mathcal{F}(S, X)) \setminus \{(0 : 0 : 0 : 1), (1 : 0 : 0 : 0), (0 : 0 : 1 : 0)\}$ are of Kupka type.

2.3. *Some remarks about the construction of the examples.* In this section we discuss the possibility of constructing families of foliations GK in $\mathbb{CP}(3)$, generated by two one-dimensional foliations, say $\mathcal{G}_1$ and $\mathcal{G}_2$, as in §2.2. We suppose that $\mathcal{G}_1$ is the foliation defined in the affine chart $E_0 = \{(x : y : z : 1) \mid (x, y, z) \in \mathbb{C}^3\}$ by the linear vector field $S = px(\partial/\partial x) + qy(\partial/\partial y) + rz(\partial/\partial z)$, where $p, q, r \in \mathbb{N}$, $p \geq q \geq r > 0$ and $\gcd(p, q, r) = 1$. If $p = q = r = 1$, then it is possible to construct GK foliations of any degree. Take a homogeneous vector field of degree d on E_0, say X, so that $[S, X] = (d-1)X$. The foliation generated by S and X in $\mathbb{CP}(3)$ is defined on E_0 by the form $\Omega = i_S i_X (dx \wedge dy \wedge dz)$. This type of example is considered in [**3**] and for generic X it is GK. On the other hand, in the case where the integers p, q and r are not equal, the situation is not so clear and we do not have a complete picture of all possibilities if we fix p,q,r. Nevertheless, in the case where $p > q > r$, the number of possible families of foliations is finite, as we will see.

Consider S as in (1) and $p > q > r > 0$. Let us suppose that there is a one-dimensional foliation $\mathcal{G}_2$ of degree d, which in the chart E_0 is defined by a polynomial vector field X such that $[S, X] = \ell X$, where $\ell > 0$. We denote by $\mathcal{F}(S, X)$ the foliation on $\mathbb{CP}(3)$, which in the chart E_0 is generated by S and X. Observe that $\mathcal{F}(S, X) \in \mathcal{F}((p, q, r); \ell; d+1)$.

THEOREM 3. *If $p > q > r > 0$ are fixed, then the set*

$$\mathbb{P} = \{(d, \ell) \mid d \geq 0, \ell > 0 \text{ and } \mathcal{F}(p, q, r; \ell; d+1) \text{ contains a GK foliation}\}$$

is finite.

Proof. Observe that S has four singularities in $\mathbb{CP}(3)$, the points $p_0 = (0 : 0 : 0 : 1) \in E_0$, $p_1 = (1 : 0 : 0 : 0) \in E_1$, $p_2 = (0 : 1 : 0 : 0)$ and $p_3 = (0 : 0 : 1 : 0)$. The eigenvalues of S at these points are, respectively, (p, q, r), $(-p, q-p, r-p)$, $(p-q, -q, r-q)$, $(p-r, q-r, -r)$. Note that only in the first two sets do the eigenvalues have the same sign. As a consequence, the points p_2 and p_3 cannot be quasi-homogeneous singularities for a foliation $\mathcal{F} \in \mathcal{F}((p, q, r); \ell; d+1)$.

The idea is to use (3) for the multiplicity of an isolated singularity of a quasi-homogenous vector field in Remark 1. We prove that the existence of a GK foliation $\mathcal{F} \in \mathcal{F}((p, q, r); \ell; d+1)$ implies the existence of a one-dimensional foliation $\mathcal{G}$ of degree d with the following properties:

(i) p_0 and p_1 are isolated singularities of $\mathcal{G}$;

(ii) $\mathcal{G}$ is defined in the chart E_0 by a vector field Y such that $[S, Y] = \ell Y$.

Let us suppose the existence of $\mathcal{G}$ satisfying properties (i) and (ii) and prove the theorem. Since p_0 is an isolated singularity for Y, it follows from (3) that

$$\mu_0 = \mu_0(d, \ell) := \text{mult}(Y, p_0) = \frac{(\ell + p)(\ell + q)(\ell + r)}{pqr}. \tag{7}$$

On the other hand, $\mathcal{G}$ is defined in the chart $E_1 = \{(1 : w : v : u) \mid (u, v, w) \in \mathbb{C}^3\}$, by the vector field Y_1, where $Y_1 = u^{d-1}Y = x^{-d+1}Y$ in $E_0 \cap E_1$. It follows that

$$[S, Y_1] = S(x^{-d+1})Y + x^{-d+1}[S, Y] = (\ell - p(d-1))Y_1.$$

Note that, in the chart E_1, we have

$$S = -pu\frac{\partial}{\partial u} - (p-r)v\frac{\partial}{\partial v} - (p-q)w\frac{\partial}{\partial w},$$

so that if we set $S_1 = -S$, then $[S_1, Y_1] = (p(d-1) - \ell)Y_1$. Set $q_1 = p - r$, $r_1 = p - q$ and $\ell_1 = p(d-1) - \ell$. We assert that $\ell_1 \geq 0$, unless $Y_1(p_1) \neq 0$.

In fact, suppose by contradiction that $Y_1(p_1) = 0$ and $\ell_1 < 0$. Let $Y_1 = A(\partial/\partial u) + B(\partial/\partial v) + C(\partial/\partial w)$. Since $p_1 = (0, 0, 0)$ is an isolated singularity of $\mathcal{G}$, we must have $C \not\equiv 0$, so that there is a non-zero monomial of the form $u^a v^b w^c$ in C. Now, the relation $[S_1, Y_1] = \ell_1 Y_1$ implies that $S_1(C) = (\ell_1 + r_1)C$ and so

$$pa + q_1 b + r_1 c = \ell_1 + r_1 < r_1.$$

However, the above relation is not possible if $a + b + c \geq 1$ and $p > q_1 > r_1 \geq 1$. This contradiction implies that $\ell_1 \geq 0$, unless C is a constant. Observe that if C is a constant, then $Y_1(p_1) \neq 0$. In fact, if $C \neq 0$ this is clear and if $C = 0$ then p_1 would not be a singularity of Y_1, otherwise it would not be isolated.

In the case where $Y_1(p_1) = 0$, we get from (3) that

$$\mu_1 = \mu_1(d, \ell) := \text{mult}(Y_1, p_1) = \frac{(\ell_1 + p)(\ell_1 + q_1)(\ell_1 + r_1)}{p q_1 r_1}. \tag{8}$$

Note that when $Y_1(p_1) \neq 0$ then $\mu_1 = 0$ and (8) is still true. Since $\mathcal{G}$ has degree d, we must have (cf. [**13**]):

$$\mu_0 + \mu_1 \leq d^3 + d^2 + d + 1. \tag{9}$$

Let us see how [**9**] implies the theorem. First of all we write (9) as a function of ℓ and ℓ_1. Since $\ell + \ell_1 = p(d-1)$ we have

$$\begin{aligned} d^3 + d^2 + d + 1 &= (d-1)^3 + 4(d-1)^2 + 6(d-1) + 4 \\ &= \frac{1}{p^3}[(\ell + \ell_1)^3 + 4p(\ell + \ell_1)^2 + 6p^2(\ell + \ell_1) + 4p^3] := \frac{1}{p^3} G(\ell, \ell_1). \end{aligned}$$

Therefore, (9) is equivalent to $F(\ell, \ell_1) \leq 0$, where

$$F(\ell, \ell_1) = p^2 q_1 r_1 (\ell + p)(\ell + q)(\ell + r) + p^2 q r (\ell_1 + p)(\ell_1 + q_1)(\ell_1 + r_1) - q q_1 r r_1 G(\ell, \ell_1).$$

Let us consider first the case $\ell, \ell_1 \geq 0$. Note that $F(\ell, \ell_1)$ is a degree three polynomial in (ℓ, ℓ_1) and its homogeneous term of degree three is

$$F_3(\ell, \ell_1) = p^2 q_1 r_1 \ell^3 + p^2 q r \ell_1^3 - q q_1 r r_1 (\ell + \ell_1)^3.$$

ASSERTION. *If $\ell, \ell_1 \geq 0$ and $p > q > r > 0$, then there exists $C > 0$ (which depends only on p, q, r) such that $F_3(\ell, \ell_1) \geq C(\ell + \ell_1)^3$.*

Proof. Suppose that $\ell_1 > 0$, $\ell \geq 0$ and set $y = \ell/\ell_1$. Then $F_3(\ell, \ell_1) = \ell_1^3 . f(y)$, where $f(y) = p^2 q_1 r_1 y^3 + p^2 q r - q q_1 r r_1 (y+1)^3$. Observe that $f(0) = qr(p^2 - q_1 r_1) > 0$ and

$$\tfrac{1}{3} f'(y) = p^2 q_1 r_1 y^2 - q q_1 r r_1 (y+1)^2$$

so that $f'(0) < 0$ and $f'(y) = 0$ has a unique positive root $y_0 = \sqrt{qr}/(p - \sqrt{qr})$. As the reader can check, by calculating f'' and f''', the point y_0 is the positive minimum of $f(y)$. Since

$$f(y_0) = \frac{2p^3 qr}{(p - \sqrt{qr})^2} \left(\frac{q + r}{2} - \sqrt{qr} \right) > 0,$$

we have $f(y) \geq f(y_0) = \alpha > 0$ for all $y \geq 0$, so that $F_3(\ell, \ell_1) \geq \alpha \ell_1^3$. Similarly, there exists $\beta > 0$ such that $F_3(\ell, \ell_1) \geq \beta \ell^3$, if $\ell > 0$ and $\ell, \ell_1 \geq 0$. It follows that

$$F_3(\ell, \ell_1) \geq \tfrac{1}{2}\alpha \ell_1^3 + \tfrac{1}{2}\beta \ell^3 \geq C(\ell + \ell_1)^3$$

for some $C > 0$ and $\ell, \ell_1 \geq 0$. □

Now, since $F(\ell, \ell_1) - F_3(\ell, \ell_1)$ is a degree two polynomial in (ℓ, ℓ_1), there exists $\rho > 0$ such that if $\ell, \ell_1 \geq 0$ and $\ell + \ell_1 \geq \rho$, then $|F(\ell, \ell_1) - F_3(\ell, \ell_1)| \leq (C/2)(\ell + \ell_1)^3$, which implies that $F(\ell, \ell_1) \geq (C/2)(\ell + \ell_1)^3$, if $\ell, \ell_1 \geq 0$ and $\ell + \ell_1 \geq \rho$. It follows that the number of pairs $(\ell, \ell_1) \in \mathbb{N}^2$ which are solutions of $F(\ell, \ell_1) \leq 0$ is finite. Since $\ell + \ell_1 = p(d-1)$, the number of pairs $(\ell, d) \in \mathbb{N}^2$ which are solutions of (9) is also finite.

Let us consider now the case $Y_1(p_1) \neq 0$ and $\ell_1 < 0$. In this case, we have $(\ell_1 + p)(\ell_1 + q_1)(\ell_1 + r_1) = 0$ and so the inequality $F(\ell, \ell_1) \leq 0$ is equivalent to $H(\ell, \ell_1) \leq 0$, where

$$H(\ell, \ell_1) = p^2(\ell + p)(\ell + q)(\ell + r) - qrG(\ell, \ell_1).$$

Note that the homogeneous part of degree three of H is

$$H_3(\ell, \ell_1) = p^2\ell^3 - qr(\ell+\ell_1)^3 \geq (p^2 - qr)\ell^3 \geq (p^2 - qr)(\ell+\ell_1)^3 = (p^2 - qr)p^3(d-1)^3.$$

Since $C = p^2 - qr > 0$ we can apply the same argument as before to conclude that the number of pairs (ℓ, d) which are solutions of (9) is finite.

It remains to prove the existence of a foliation $\mathcal{G}$ satisfying (i) and (ii). We prove that there are two foliations $\mathcal{G}_0$ and $\mathcal{G}_1$ of degree d such that:

(iii) p_j is an isolated singularity of $\mathcal{G}_j$, $j = 0, 1$;

(iv) $\mathcal{G}_j$ is defined in the chart E_j by a vector field X_j such that $[S_j, X_j] = \ell_j X_j$, where $S_0 = S$ and $\ell_0 = \ell$.

If we have two foliations like above, then the generic foliation in the pencil $\mathcal{G}_\alpha = \mathcal{G}_0 + \alpha \mathcal{G}_1$ satisfies (i) and (ii), as the reader can check. Recall that $\mathcal{G}_\alpha$ is the foliation that in the chart E_0 is defined by $X_\alpha = X_0 + \alpha x^{d-1} X_1$.

Let us construct $\mathcal{G}_0$. Consider a foliation $\mathcal{F} \in \mathcal{F}((p, q, r); \ell; d + 1)$. Then it has degree $d + 1$ and is defined in the chart E_0 by an integrable 1-form Ω such that $d\Omega = i_Z(dx \wedge dy \wedge dz)$, $p_0 = 0$ is an isolated singularity of Z and $[S, Z] = \ell Z$. Since $\mathcal{F}$ has degree $d + 1$, the form Ω has degree $d + 2$, so that $d \leq \mathrm{dg}(Z) \leq d + 1$. If $\mathrm{dg}(Z) = d$, then the foliation $\mathcal{G}(Z)$ on $\mathbb{CP}(3)$ defined in the chart E_0 by Z has degree d and we take $\mathcal{G}_0 = \mathcal{G}(Z)$. Let us suppose that $\mathrm{dg}(Z) = d + 1$. In this case we must have $\mathrm{div}(Z) = 0$, so that if Z_{d+1} is the homogeneous part of Z of degree $d + 1$, then $\mathrm{div}(Z_{d+1}) = 0$ and $[S, Z_{d+1}] = \ell Z_{d+1}$. As the reader can check, these relations imply that $Z_{d+1} = g(mR - nS)$, where R is the radial vector field on $\mathbb{C}^3$, $m = \ell + p + q + r$, $n = d + 3$ and g is a homogeneous polynomial of degree d such that $S(g) = \ell g$. Let us write $Z = P + g(mR - nS)$, where $\mathrm{dg}(P) \leq d$, $P = A(\partial/\partial x) + B(\partial/\partial y) + C(\partial/\partial z)$ and

$$Z = (A + (m - np)xg)\frac{\partial}{\partial x} + (B + (m - nq)yg)\frac{\partial}{\partial y} + (C + (m - nr)zg)\frac{\partial}{\partial z}.$$

Observe that if λ is small then 0 is an isolated singularity of $Z + \lambda g R$. Take λ in such a way that $m - np + \lambda, m - nq + \lambda, m - nr + \lambda \neq 0$. In this case, the vector field

$$X_0 = \left(\frac{A}{m - np + \lambda} + gx\right)\frac{\partial}{\partial x} + \left(\frac{B}{m - nq + \lambda} + gy\right)\frac{\partial}{\partial y} + \left(\frac{A}{m - nr + \lambda} + gz\right)\frac{\partial}{\partial z}$$

has an isolated singularity at 0. Moreover, $[S, X_0] = \ell X_0$ and the foliation defined by X_0 on $\mathbb{CP}(3)$ has degree d. The construction of $\mathcal{G}_1$ is similar and this finishes the proof of Theorem 3. □

Remark 5. When $p = 3$, $q = 2$ and $r = 1$, then the unique possibilities are those of Examples 1 (with $d = 1$), 2 and 3. In fact, in this case, if we set $k = d - 1 \geq 0$, we have $\ell_1 = 3k - \ell$ and

$$F(\ell, 3k - \ell) = 3[A(k)\ell^2 - B(k)\ell + C(k)], \tag{10}$$

where $A(k) = 3k + 4$, $B(k) = 12k + 9k^2$ and $C(k) = 7k^3 + 10k^2 - k - 4$. On the other hand, the inequality $F(\ell, 3k - \ell) \leq 0$ implies that for a solution (k, ℓ) we must have $B^2 - 4AC \geq 0$. Since

$$B^2 - 4AC = -(k - 2)(k + 2)(k + 4)(3k + 4)$$

we get that the unique possible solutions are $k \in \{0, 1, 2\}$, that is $d \in \{1, 2, 3\}$. If we substitute these values of k in (10) we get the following possibilities for ℓ and ℓ_1

$$\begin{aligned} k = 0 &\implies \ell = 1, \ell_1 = -1 \\ k = 1 &\implies \ell, \ell_1 \in \{1, 2\} \\ k = 2 &\implies \ell = \ell_1 = 3 \end{aligned}$$

which give exactly the values of (d, ℓ, ℓ_1) of the examples.

The above result has motivated Problem 1 in §1.

3. *Proofs of Theorems 1 and 2*

3.1. *Proof of Theorem 1.* In this proof we assume that $(p, q, r) \neq (1, 1, 1)$. We observe that the case $p = q = r = 1$ is essentially proven in [**3**], as was remarked in Example 3 of [**4**]. Let $\mathcal{F} \in \mathcal{F}((p, q, r); \ell; \nu)$ be a GK foliation on $\mathbb{CP}(3)$. Observe that $\mathcal{F}$ is generated by two one-dimensional foliations of $\mathbb{CP}(3)$, say $\mathcal{G}_1$ and $\mathcal{G}_2$, the foliations defined in the chart E_0 by the vector fields S and X, respectively. As we have seen in Proposition 2, this implies that its tangent bundle $T_{\mathcal{F}}$ splits as the sum of two line bundles $T_{\mathcal{F}} = L_1 \oplus L_2$, where L_1 corresponds to the foliation $\mathcal{G}_1$ and L_2 to $\mathcal{G}_2$. Moreover, the corollaries of Proposition 1 imply that there exists a neighborhood $\mathcal{U}$ of $\mathcal{F}$ such that any foliation in $\mathcal{U}$ is GK, so that its tangent bundle splits locally.

Remark 6. Since $(p, q, r) \neq (1, 1, 1)$, S is a global vector field in $\mathbb{CP}(3)$ with singular set of codimension greater or equal than two. Therefore, L_1 is a trivial line bundle, that is $L_1 \simeq \mathbb{CP}(3) \times \mathbb{C} = \mathcal{O}(0) = \mathcal{O}$. On the other hand, if d is the degree of $\mathcal{G}_2$, we have that $L_2 \simeq \mathcal{O}(1 - d)$ (cf. [**1**]) and that the degree of $\mathcal{F}$ is $\nu = d + 1$.

Since $\mathcal{F}(d+1,3)$ is finite dimensional, it is sufficient to prove that for any holomorphic curve $\Sigma \ni t \mapsto \mathcal{F}_t \in \mathcal{F}(d+1,3)$, such that $0 \in \Sigma \subset \mathbb{C}$ and $\mathcal{F}_0 = \mathcal{F}$, then $\mathcal{F}_t \in \mathcal{F}((p,q,r);\ell;d+1)$ for small $|t|$.

Let $(\mathcal{F}_t)_{t\in\Sigma}$ be a holomorphic family of foliations on $\mathcal{F}(d+1,3)$, parameterized in an open set $0 \in \Sigma \subset \mathbb{C}$, where $\mathcal{F}_0 = \mathcal{F}$. We take Σ so small that for any $t \in \Sigma$, $\mathcal{F}_t$ is GK and $T_{\mathcal{F}_t}$ splits locally. Moreover, $(T_{\mathcal{F}_t})_{t\in\Sigma}$ is a holomorphic family of rank two vector bundles over $\mathbb{CP}(3)$. We prove first that $T_{\mathcal{F}_t}$ is isomorphic to $T_{\mathcal{F}} = T_{\mathcal{F}_0}$, if $|t|$ is small. To do that, we essentially use the following theorem.

THEOREM B. (Kuranishi [**17**]) *Let $E \to X$ be a holomorphic vector bundle over a complex compact manifold X. Then there exists a versal deformation space $\mathcal{S}$ of E. Moreover, the tangent space of $\mathcal{S}$ at E is isomorphic to $H^1(X, \mathrm{End}(E))$, where $\mathrm{End}(E)$ is the sheaf of endomorphisms of E.*

In order to conclude that for small $|t|$, it is $T_{\mathcal{F}_t} \simeq T_{\mathcal{F}_0}$ by theorem B, it is sufficient to prove that $H^1(\mathbb{CP}(3), \mathrm{End}(T_{\mathcal{F}_0}))$ vanishes. However, the dimension of that vector space is zero, as $\mathrm{End}(T_{\mathcal{F}_0}) = T^*_{\mathcal{F}_0} \otimes T_{\mathcal{F}_0}$, where $T^*_{\mathcal{F}_0} = \mathcal{O} \oplus \mathcal{O}(d-1)$ is the dual bundle of $T_{\mathcal{F}_0}$ (cf. [**16**]).

Now, let $(\mathcal{F}_t)_{t\in\Sigma}$ be a holomorphic family of foliations such that $\mathcal{F} = \mathcal{F}_0 \in \mathcal{F}((p,q,r);\ell;d+1)$ is GK. It follows from Remark 6 and the results above that if Σ is a small neighborhood of $0 \in \mathbb{C}$, then $T_{\mathcal{F}_t} \simeq \mathcal{O} \oplus \mathcal{O}(1-d)$ for all $t \in \Sigma$. On the other hand, Proposition 2 (b) implies that $\mathcal{F}_t$ is generated by two foliations of dimension one, say $\mathcal{G}_1(t)$ and $\mathcal{G}_2(t)$, where $\mathcal{G}_1(t)$ corresponds to the factor $\mathcal{O}$ and $\mathcal{G}_2(t)$ to the factor $\mathcal{O}(1-d)$. As a consequence, $\mathcal{G}_1(t)$ is generated by a global vector field $S(t)$ on $\mathbb{CP}(3)$. Now, Proposition 1 implies that $S(t)$ has a singularity whose eigenvalues, say $\lambda_1, \lambda_2, \lambda_3$, are multiples of p, q, r, so that we can suppose without loss of generality that $\lambda_1 = p$, $\lambda_2 = q$ and $\lambda_3 = r$. Consider an affine coordinate system $(U(t) = \mathbb{C}^3, (x,y,z))$, where $S(t) = px(\partial/\partial x) + qy(\partial/\partial y) + rz(\partial/\partial z)$. Let $\Omega(t)$ be a polynomial integrable 1-form which defines $\mathcal{F}_t$ in this chart. We assert that

$$L_{S(t)}\Omega(t) = (\ell + p + q + r)\Omega(t). \tag{11}$$

In fact, since $\mathcal{G}_1(t)$ is tangent to $\mathcal{F}_t$, we have $i_{S(t)}\Omega(t) = 0$. This implies that $L_{S(t)}\Omega(t) = i_{S(t)}\, d\Omega(t)$. On the other hand, it follows from the integrability condition, $\Omega(t) \wedge d\Omega(t) = 0$, that $\Omega(t) \wedge i_{S(t)}\, d\Omega(t) = 0$, which implies that $L_{S(t)}\Omega(t) = \lambda(t)\Omega(t)$, where $\lambda\colon \mathbb{C}^3 \to \mathbb{C}^*$ is holomorphic. Now, the eigenvalues of the operator $\omega \mapsto L_{S(t)}\omega$ are integers, so that $\lambda(t)$ is a constant. Since $\Omega(0) = \Omega = i_S i_X(dx \wedge dy \wedge dz)$, where $[S,X] = \ell X$, we have $L_S\Omega = (\ell + tr(S))\Omega = (\ell + p + q + r)\Omega$, which proves that $\lambda(0) = \ell + p + q + r \equiv \lambda$ and the assertion.

Now, let $X(t)$ be the vector field in $\mathbb{C}^3 = U(t)$ defined by $i_{X(t)}(dx \wedge dy \wedge dz) = d\Omega(t)$. It follows from (11) that

$$\begin{aligned}
\lambda i_{X(t)}(dx \wedge dy \wedge dz) &= \lambda\, d\Omega(t) = L_{S(t)}\, d\Omega(t) = L_{S(t)}(i_{X(t)}(dx \wedge dy \wedge dz)) \\
&= i_{[S(t),X(t)]}(dx \wedge dy \wedge dz) + i_{X(t)}(L_{S(t)}(dx \wedge dy \wedge dz)) \\
&= i_{[S(t),X(t)]}(dx \wedge dy \wedge dz) + tr(S(t))\, d\Omega(t) \\
&\implies [S(t), X(t)] = (\lambda - tr(S(t)))X(t) = \ell X(t).
\end{aligned}$$

This implies that $\mathcal{F}_t \in \mathcal{F}((p,q,r);\ell,d+1)$ for small $|t|$ and finishes the proof of Theorem 1 as $\overline{\mathcal{F}((p,q,r);\ell;d+1)}$ is an irreducible algebraic subset of $\mathcal{F}(d+1,3)$. Indeed, recall from the description of the foliations in $\mathcal{F}((p,q,r);\ell;d+1)$ that in order to define such a foliation, we need to choose an affine open $\mathbb{C}^3 \subset \mathbb{CP}(3)$ (or equivalently a point in the dual projective space $\mathbb{CP}^*(3)$), fixing linear coordinates on it and choosing (up to multiplication by the same constant) the coefficients of the vector field X. This shows that there is a surjective map from a dense open subset $U \subset \mathbb{CP}^*(3) \times GL(3,\mathbb{C}) \times \mathbb{C}^N$ onto $\mathcal{F}((p,q,r);\ell;d+1)$, for a certain N. So the irreducibility of the last algebraic subset follows from that of U.

Furthermore, to parameterize $\mathcal{F}((p,q,r);\ell;d+1)$, we should analyze the map above in order to detect which elements in U give rise to the same foliation. Note that for a fixed affine open subset, a linear change of coordinates of the form $x' = \alpha x$, $y' = \beta y$, $z' = \gamma z$ takes S to $S' = px'(\partial/\partial x') + qy'(\partial/\partial y') + rz'(\partial/\partial z')$ and X to an S'-quasi-homogeneous vector field X' of weight ℓ. As the open affine $\mathbb{C}^3$, the coordinates (x',y',z') and the vector fields S', X' define the same foliation, we should factor the group $GL(3,\mathbb{C})$ by the subgroup of diagonal invertible matrices. □

For Klein–Lie foliations we have the following result, extending the existence of the *exceptional component* in [**5**], that corresponds to the case $d = 1$.

COROLLARY 3. *Let $d \geq 1$ be an integer. There is an N-dimensional irreducible component*

$$\overline{\mathcal{F}(d^2+d+1, d+1, 1; d+1; -1)}$$

of the space $\mathcal{F}(d+1,3)$ whose general point corresponds to a GK Klein–Lie foliation with exactly one quasi-homogenous singularity, where $N = 13$ if $d = 1$ and $N = 14$ if $d > 1$. Moreover, this component is the closure of a $\mathbf{PGL}(4,\mathbb{C})$ *orbit on $\mathcal{F}(d+1,3)$.*

Proof. This is an immediate consequence of Theorem 1, the study of Klein–Lie foliations in Example 1 and the analysis of the parameterizations of the sets $\mathcal{F}((p,q,r);\ell;d+1)$. Indeed, if $\mathcal{F}$ is a foliation in $\mathcal{F}(d(d+1)+1, d+1, 1; d+1; -1)$, then in an affine open subset we have that it is determined by the vector fields

$$S = (d^2+d+1)x\frac{\partial}{\partial x} + (d+1)y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} \quad \text{and}$$
$$X_{\alpha\beta} = \alpha(d^2+d+1)y^d\frac{\partial}{\partial x} + \beta(d+1)z^d\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

Note that X, the S-quasi-homogeneous vector field of weight -1, is uniquely defined up to the choice of the non-zero constants α and β (we take the last coordinate, which is necessarily a constant, to be 1). The dependence locus of S and X, which is the singular set of the foliation $\mathcal{F}$ in $\mathbb{C}^3$, is the Klein–Lie curve $(\alpha t^{d^2+d+1}, \beta t^{d+1}, t)$. After the linear change of coordinates given by $x = \alpha x'$, $y = \beta y'$, $z = z'$, the foliation in $\mathbb{C}^3$ is exactly that described in Example 1, whose singular locus is the curve $\gamma_{d^2+d+1,d+1,1}(t) = (t^{d^2+d+1}, t^{d+1}, t)$. The extended foliation in $\mathbb{CP}(3)$ is GK and was studied in Example 1: it has just one quasi-homogenous singularity, an invariant hyperplane (that at infinity, $\mathbb{CP}(3) \setminus \mathbb{C}^3$) and we also know its singular locus. □

3.2. *Proof of Theorem 2.* We observe that the second statement of the theorem is a direct consequence of the first and of Theorem 1, so that we only prove the first.

We perform the arguments in homogeneous coordinates. Let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}(n)$ be the natural projection. Given a codimension one holomorphic foliation $\mathcal{F}$ on $\mathbb{CP}(n)$ of degree d, then the foliation $\mathcal{F}^* = \pi^*(\mathcal{F})$, on $\mathbb{C}^{n+1} \setminus \{0\}$, extends to a foliation on $\mathbb{C}^{n+1}$, which can be defined by a polynomial 1-form $\Omega = \sum_{j=0}^{n} A_j(z)\, dz_j$ satisfying the following properties (cf. [**5**]).

(i) A_j is a homogeneous polynomial of degree $\nu = d + 1$ for all $j = 0, \ldots, n$.
(ii) $\sum_{j=0}^{n} z_j . A_j(z) \equiv 0$.
(iii) $\Omega \wedge d\Omega = 0$ (integrability condition).
(iv) $\pi(\mathrm{sing}(\Omega)) = \mathrm{sing}(\mathcal{F})$ and $\mathrm{cod}_{\mathbb{C}}(\mathrm{sing}(\Omega)) \geq 2$.
(v) If U_α is the affine chart $(z_\alpha = 1)$, then $\mathcal{F}|_{U_\alpha}$ is defined by $\Omega_\alpha = \Omega|_{U_\alpha}$.

Moreover, if $\mathbb{CP}(k) \simeq E \subset \mathbb{CP}(n)$ is a linearly embedded k-plane, $2 \leq k < n$, non-invariant for $\mathcal{F}$, where $\pi^{-1}(E) = E^*$, then:
(vi) $\pi^*(\mathcal{F}|_E) = \mathcal{F}^*|_{E^*}$ is defined by $\Omega|_{E^*}$.

Now, suppose that $n = 3$ and that $\mathcal{F}$ is generated by two one-dimensional foliations, say $\mathcal{G}_j$ of degree d_j, $j = 1, 2$. We have the following.

LEMMA 1. *In the above hypothesis, let Ω be as before. Then there exist polynomial vector fields X_j on $\mathbb{C}^4$, $j = 1, 2$, with the following properties.*
(a) *The components of X_j are homogeneous of degree d_j.*
(b) *The two-dimensional foliation on $\mathbb{C}^4 \setminus \{0\}$, $\pi^*(\mathcal{G}_j)$, extends to $\mathbb{C}^4$ and is generated by X_j and the radial vector field on $\mathbb{C}^4$: $R = \sum_{j=0}^{3} z_j(\partial/\partial z_j)$.*
(c) $\Omega = i_R i_{X_1} i_{X_2}(dz_0 \wedge dz_1 \wedge dz_2 \wedge dz_3)$.

Proof. The existence of vector fields X_j, $j = 1, 2$, satisfying (a) and (b), is well known (cf. [**13**]). Since $\mathcal{G}_1$ and $\mathcal{G}_2$ generate $\mathcal{F}$, we must have $i_{X_j}\Omega = 0$, $j = 1, 2$. We also have $i_R(\Omega) = 0$ (from (ii)). Let $\Theta = i_R i_{X_1} i_{X_2}(dz_0 \wedge dz_1 \wedge dz_2 \wedge dz_3)$. It follows from Definition 5 and (b), that $\mathrm{cod}_{\mathbb{C}}(\mathrm{sing}(\Theta)) \geq 2$ and that for any $p \in \mathbb{C}^4 \setminus \mathrm{sing}(\Theta)$ we have $T_p(\mathcal{F}^*) = \ker(\Theta(p)) = \ker(\Omega(p))$, where $T_p(\mathcal{F}^*)$ denotes the tangent space to the leaf of $\mathcal{F}^*$ through p. This implies that $\Theta = \lambda\Omega$ outside $\mathrm{sing}(\Theta)$, where $\lambda \not\equiv 0$ is some holomorphic function on $\mathbb{C}^4 \setminus \mathrm{sing}(\Theta)$. Since $\mathrm{cod}(\mathrm{sing}(\Theta)) \geq 2$, λ extends to a holomorphic function on $\mathbb{C}^4$, which of course is a homogeneous polynomial. Now, it follows from $\mathrm{dg}(\mathcal{G}_j) = d_j$, that $\mathrm{dg}(\mathcal{F}) = d_1 + d_2$, and so $\mathrm{dg}(\Omega) = d_1 + d_2 + 1 = \mathrm{dg}(\Theta)$. This implies that λ is a constant. Now, if $\tilde{X}_1 = \lambda^{-1} X_1$, then $\Omega = i_R i_{\tilde{X}_1} i_{X_2}(dz_0 \wedge dz_1 \wedge dz_2 \wedge dz_3)$, which proves the Lemma. □

We have the following consequences.

COROLLARY 4. *Let $\mathcal{F}$, $\mathcal{F}^*$ and $\Omega = i_R\, i_{X_1}\, i_{X_2}(dz_0 \wedge dz_1 \wedge dz_2 \wedge dz_3)$ be as in Lemma 1. Then for any $p \in \mathbb{C}^4$ the sheaf of germs of holomorphic vector fields at p which are tangent to $\mathcal{F}^*$ is free and generated by the germs of R, X_1 and X_2 at p.*

The proof is similar to the proof of Corollary 2 and is left for the reader.

COROLLARY 5. *Let $\mathcal{F}$, $\mathcal{F}^*$ and Ω be as in Lemma 1. Let $(V_\alpha)_{\alpha \in A}$ be a covering of $\mathbb{C}^4 \setminus \{0\}$ by Stein open sets and $(X_{\alpha\beta})_{V_{\alpha\beta} \neq \emptyset}$ be an additive cocycle of holomorphic vector fields* *such that for any* $V_{\alpha\beta} \neq \emptyset$*,* $X_{\alpha\beta}$ *is tangent to* $\mathcal{F}^*$*; that is,* $i_{X_{\alpha\beta}}\Omega = 0$*. Then for any* $\alpha \in A$ *there exists a holomorphic vector field* X_α *on* V_α *such that* X_α *is tangent to* $\mathcal{F}^*$ *and* $X_{\alpha\beta} = X_\beta - X_\alpha$ *on* $V_\alpha \cap V_\beta := V_{\alpha\beta} \neq \emptyset$*.*

Proof. Let X_1 and X_2 be as in Lemma 1, so that $\Omega = i_R\, i_{X_1}\, i_{X_2}(dz_0 \wedge dz_1 \wedge dz_2 \wedge dz_3)$. It follows from Corollary 4 that if $V_{\alpha\beta} \neq \emptyset$ then there exist $f^j_{\alpha\beta} \in \mathcal{O}(V_{\alpha\beta})$, $j = 0, 1, 2$, such that

$$X_{\alpha\beta} = f^0_{\alpha\beta} R + f^1_{\alpha\beta} X_1 + f^2_{\alpha\beta} X_2.$$

Clearly, $(f^j_{\alpha\beta})_{V_{\alpha\beta}\neq\emptyset}$ is an additive cocycle for $j = 0, 1, 2$. Since $H^1(\mathbb{C}^4 \setminus \{0\}, \mathcal{O}) = 0$, there exist collections $(f^j_\alpha)_{\alpha\in A}$, where $f^j_\alpha \in \mathcal{O}(V_\alpha)$, $j = 0, 1, 2$, such that $f^j_{\alpha\beta} = f^j_\beta - f^j_\alpha$ on $V_{\alpha\beta} \neq \emptyset$. If we set $X_\alpha = f^0_\alpha R + f^1_\alpha X_1 + f^2_\alpha X_2$, then X_α is tangent to $\mathcal{F}^*$ and $X_{\alpha\beta} = X_\beta - X_\alpha$. □

Now, we consider the case in which $\mathcal{F}|_E$ is GK.

LEMMA 2. *Let* $\mathcal{F}$ *be a codimension one foliation of degree* d *on* $\mathbb{CP}(n)$*. Suppose that there exists a 3-plane* E *as in (vi) before Lemma 1 and that* $\mathcal{F}|_E$ *is GK. Let* $\mathcal{F}^*$*,* E^* *and* Ω *be as before. Then, for any* $p \in E^* \setminus \{0\}$*, there exists a local coordinate system around* p*, say* $(U, (t, u, v))$*, where* $t\colon U \to \mathbb{C}$*,* $u = (u_1, u_2, u_3)\colon U \to \mathbb{C}^3$ *and* $v = (v_1, \ldots, v_{n-2})\colon U \to \mathbb{C}^{n-3}$*, such that* $t(p) = 0$*,* $u(p) = 0$*,* $v(p) = 0$ *and:*

(a) $E^* = (v = 0)$*;*

(b) $\Omega|_U = e^{t(d+2)} \sum_{j=1}^{3} \alpha_j(u)\, du_j$*.*

In particular, $\mathcal{F}^*|_U$ *is locally equivalent to the product of a codimension one foliation on* $\mathbb{C}^4$ *by a non-singular foliation, say* $\mathcal{P}$*, of dimension* $n - 3$*, which is given in this chart by* $(t, u) = const.$

Proof. The lemma is a consequence of [**10**] and [**3**]. First of all, observe that $L_R(\Omega) = (d + 2)\Omega$, because Ω is homogeneous of degree $d + 1$. This implies that

$$R^*_s(\Omega) = e^{s(d+2)}\Omega, \tag{12}$$

where $R_s(q) = e^s q$ is the flow of R. Let $p = (p_0, \ldots, p_n) \in E^* \setminus \{0\}$. After a linear change of variables in $\mathbb{C}^{n+1}$, we can suppose that $E^* = (z_4 = \cdots = z_n = 0)$ and $p = (1, 0, \ldots, 0) \in E^*$. Let H be the hyperplane $(z_0 = 1)$ of $\mathbb{C}^{n+1}$. Since R is transversal to H, there exists coordinate system $(t, x)\colon V \to \mathbb{D} \times \mathbb{C}^n$, where $V = \{R_s(q) \mid s \in \mathbb{D}, q \in H\}$, such that $R = (\partial/\partial t)$, $H = (t = 0)$ and $p = 0$, in this chart. It follows from (12) that

$$\Omega(t, x) = e^{t(d+2)}\omega, \tag{13}$$

where

$$\omega = \sum_{j=1}^{n} \omega_j(x)\, dx_j$$

depends only on $x = (x_1, \ldots, x_n)$. We can suppose also that $E \cap H = E^* \cap H$ is the plane $E_0 = (x_4 = \cdots = x_n = 0)$. Note that (v) and the hypothesis imply that all singularities of $\omega|_{E_0}$ are generalized Kupka. We have three possibilities.

(I) $\Omega(p) = \omega(0) \neq 0$. In this case, we have $\omega|_{E_0}(0) \neq 0$; that is, $\mathcal{F}^*$ is transversal to E_0 at 0. In fact, since $\omega(0) \neq 0$, $\mathcal{F}$ has a holomorphic first integral in a neighborhood of 0, say f, so that $\omega = g\,df$, where $g(0) \neq 0$. Now, $\omega|_{E_0}(0) = 0$ implies that $df|_{E_0}(0) = 0$, and so $f|_{E_0}$ has an isolated singularity at 0, which is not possible (see Remark 2). As the reader can check, this implies the lemma in this case.

(II) $\omega|_{E_0}(0) = 0$ and $d\omega|_{E_0}(0) \neq 0$. In this case, 0 is a Kupka singularity of $\omega|_{E_0}$ and of ω. The Lemma follows from the arguments in [**10**] or [**15**] in this case.

(III) $\omega|_{E_0}(0) = 0$, $d\omega|_{E_0}(0) = 0$ and 0 is an isolated zero of $d\omega|_{E_0}$. In this case, the lemma follows from Theorem 4 of [**3**]. □

Now, Lemma 2 implies that there exists an open covering $(U_\alpha)_{\alpha\in A}$ of $E^* \setminus \{0\}$ with the following properties.

(vii) $U_\alpha = V_\alpha \times W_\alpha$, where V_α is a Stein open subset of E^* and W_α is a polydisk in $\mathbb{C}^{n-3}$.

(viii) $\mathcal{F}^*|_{U_\alpha}$ is the product of a codimension one foliation on V_α by a non-singular foliation $\mathcal{P}_\alpha$ of dimension $n-3$, transversal to E^*.

We suppose that $E^* = (z_4 = \cdots = z_n = 0)$ and use the notation $z = (x, y)$, where $x = (x_1, \ldots, x_4) = (z_0, \ldots, z_3)$ and $y = (y_1, \ldots, y_{n-3}) = (z_4, \ldots, z_n)$. Since $\mathcal{P}_\alpha$ is non-singular of dimension $n-3$ and transversal to E^*, by taking a smaller U_α if necessary, we can suppose that it is generated by $n-3$ holomorphic vector fields, say $Y_\alpha^1, \ldots, Y_\alpha^{n-3}$, of the form

$$Y_\alpha^j(x, y) = \frac{\partial}{\partial y_j} + X_\alpha^j(x, y), \quad \text{where}$$

$$X_\alpha^j(x, y) = \sum_{i=1}^{4} A_{\alpha,i}^j(x, y)\frac{\partial}{\partial x_i} \quad \text{and} \quad A_{\alpha,i}^j \in \mathcal{O}(U_\alpha). \tag{14}$$

LEMMA 3. *For any $j = 1, \ldots, n-3$, there exists a constant vector field Z_j on $\mathbb{C}^{n+1}$ of the form*

$$Z_j = \frac{\partial}{\partial y_j} + \sum_{i=1}^{4} a_i^j \frac{\partial}{\partial x_i} \tag{15}$$

such that $i_{Z_j}\Omega(q) = 0$ for any $q \in E^*$ and any $j \in \{1, \ldots, n-3\}$.

Proof. Fix $j \in \{1, \ldots, n-3\}$ and consider the covering $(U_\alpha = V_\alpha \times W_\alpha)_{\alpha\in A}$ and the vector fields Y_α^j as in (14). Consider the additive cocycle of vector fields $(X_{\alpha,\beta})_{V_{\alpha\beta}\neq\emptyset}$ on $E^* \setminus \{0\}$, where $X_{\alpha,\beta}(x) = Y_\beta^j(x, 0) - Y_\alpha^j(x, 0) = X_\beta^j(x, 0) - X_\alpha^j(x, 0)$. Clearly, $X_{\alpha\beta}$ is tangent to $\mathcal{F}^*|_{E^*}$ if $V_{\alpha\beta} \neq \emptyset$. It follows from Corollary 5 that we can write $X_{\alpha,\beta} = T_\beta - T_\alpha$, where T_α is holomorphic on V_α and tangent to $\mathcal{F}^*|_{E^*}$. Since $Y_\alpha^j(x, 0) + T_\alpha(x) = Y_\beta(x, 0) + T_\beta(x)$ on $V_{\alpha\beta} \neq \emptyset$, there exists a holomorphic vector field Z along $E^* \setminus \{0\}$, such that $Z(x) = Y_\alpha^j(x, 0) + T_\alpha(x)$ if $x \in V_\alpha$. It follows from Hartog's theorem that we can extend Z to a vector field on E^*, which we again denote by Z. Let $Z(x) = \sum_{k=0}^{\infty} Z^k(x)$ be Taylor series of Z at $0 \in E^*$, where $Z^k(x)$ is a vector field with polynomial coefficients homogeneous of degree k. Since Y_α^j is tangent to $\mathcal{F}^*$ and Z_α is tangent to $\mathcal{F}^*|_{V_\alpha}$, we have $i_{Z(q)}\Omega(q) = 0$ for any $q \in E^*$. Now, since the coefficients of Ω are homogeneous of the same degree, we get that $i_{Z^0}\Omega(q) = 0$ for any $q \in E^*$. Finally, observe that Z^0 is a constant vector field as in (15), which proves the lemma. □

Let us finish the proof of the first part of Theorem 2. We prove that there exists a linear change of variables on $\mathbb{C}^{n+1}$ of the form $(x, y) = L(u, v) = (u + b(v), v)$ such that

$$L^*(\Omega) = \sum_{j=1}^{4} \omega_j(u)\, du_j.$$

This clearly implies the first part of Theorem 2.

Let Z_j, $j = 1, \ldots, n-3$, be as in (15). Consider the linear change of variables $(x, y) = L(u, v)$ as above, given by $y = v$ and $x_j = u_j + \sum_{i=1}^{n-3} a_j^i v_i$, $j = 1, \ldots, 4$. As the reader can check, we have $L^*(Z_j) = \partial/\partial v_j$ for all $j = 1, \ldots, n-3$. Therefore, returning to the old notation, we can suppose that $Z_j = \partial/\partial y_j$.

ASSERTION. *Let $(x, y) \in \mathbb{C}^4 \times \mathbb{C}^{n-3}$ be a linear coordinate system such that $E^* = (y = 0)$ and $Z_j = \partial/\partial y_j$, $j = 1, \ldots, n-3$. Then $\Omega = \sum_{j=1}^{4} \omega_j(x)\, dx_j$ in this coordinate system.*

Proof. Let us suppose first that $n = 4$, so that $y \in \mathbb{C}$ and $Z_1 = \partial/\partial y$. Write

$$\Omega(x, y) = \sum_{k=0}^{\nu} y^k \Omega_k(x)$$

where ν is the degree of Ω and the coefficients of Ω_k are homogeneous polynomials of degree $\nu - k$ in x. We can write

$$\Omega_k(x) = \Omega_k^0(x) + f_k(x)\, dy,$$

where

$$\Omega_k^0(x) = \sum_{i=1}^{4} g_k^i(x)\, dx_i$$

and f_k, g_k^i are homogeneous polynomials of degree $\nu - k$, $i = 1, \ldots, 4$. We want to prove that $\Omega = \Omega_0^0$. First of all, observe that $f_0 = 0$, because $f_0(x) = i_{Z_1} \Omega(x, 0) = 0$. Let us suppose by induction that $\Omega_j = 0$ for $j = 1, \ldots, k-1$, $k < \nu$, and prove that $\Omega_k = 0$. In this case, we have

$$\Omega = \Omega_0^0 + y^k(\Omega_k^0 + f_k\, dy) \pmod{y^{k+1}} \quad \text{and} \quad d\Omega = d\Omega_0^0 + k y^{k-1} dy \wedge \Omega_k^0 \pmod{y^k},$$

so that the integrability condition gives us

$$0 = \Omega \wedge d\Omega = \Omega_0^0 \wedge d\Omega_0^0 + k y^{k-1} \Omega_0^0 \wedge dy \wedge \Omega_k^0 \pmod{y^k}.$$

Since $\Omega_0^0 = \Omega|_{E^*}$, it is integrable, $\Omega_0^0 \wedge d\Omega_0^0 = 0$, and we get $\Omega_0^0 \wedge dy \wedge \Omega_k^0 = 0$. However, the forms Ω_j^0 do not contain terms in dy, and so $\Omega_0^0 \wedge \Omega_k^0 = 0$. This implies that $\Omega_k^0 = \lambda \Omega_0^0$, where λ is holomorphic, because $\operatorname{cod}(\operatorname{sing}(\Omega_0^0)) \geq 2$. On the other hand, the fact that the coefficients of Ω_k^0 are homogeneous polynomials of degree $\nu - k$, while the coefficients of Ω_0^0 are of degree $\nu > \nu - k$, implies that $\lambda = 0$ and so $\Omega_k^0 = 0$.

Let us prove that $f_k = 0$. We use the vector fields $Y_\alpha^1 = \partial/\partial y + X_\alpha^1$, $\alpha \in A$, as in (14). We can write for $(x, y) \in V_\alpha \times W_\alpha$ that

$$Y_\alpha^1(x, y) = Z_1 + \sum_{j=0}^{\infty} y^j X_{\alpha, j}(x)$$

where the vector fields $X_{\alpha,j}$ contain only terms in $\partial/\partial x_i$, $i = 1, \ldots, 4$. Since $i_{Y_\alpha^1}\Omega = 0$ and $i_{Z_1}\Omega_0^0 = 0$, we get

$$0 \equiv i_{Y_\alpha^1(x,y)}\Omega(x, y) = i_{Z_1}\Omega(x, y) + \sum_{j=0}^{\infty} y^j i_{X_{\alpha,j}(x)}\Omega(x, y)$$
$$= y^k f_k(x) + \sum_{j=0}^{k} y^j\, i_{X_{\alpha,j}(x)}\Omega_0^0(x) \pmod{y^{k+1}},$$

as the reader can check. This implies that $i_{X_{\alpha,j}}\Omega_0^0 = 0$ for $j = 0, \ldots, k-1$ and $f_k + i_{X_{\alpha,k}}\Omega_0^0 = 0$. For $V_{\alpha\beta} \neq \emptyset$, set $X_{\alpha\beta}(x) = X_{\beta,k}(x) - X_{\alpha,k}(x)$. Clearly, $(X_{\alpha\beta})_{V_{\alpha\beta}\neq\emptyset}$ is an additive cocycle of vector fields. Moreover, $i_{X_{\alpha\beta}}\Omega_0^0 = 0$, so that we can apply Corollary 5 to obtain vector fields T_α on V_α such that $X_{\alpha\beta} = T_\beta - T_\alpha$ on $V_{\alpha\beta} \neq \emptyset$ and $i_{T_\alpha}\Omega_0^0 = 0$ for all $\alpha \in A$. This implies that there exists a vector field X on $E^* \setminus \{0\}$ such that $X|_{V_\alpha} = -(X_{\alpha,k} + T_\alpha)$ for all $\alpha \in A$. By Hartog's theorem, X can be extended to E^*. On the other hand, as the reader can check,

$$i_X\Omega_0^0 = f_k. \tag{16}$$

However, f_k is homogeneous of degree $\nu - k$ and Ω_0^0 is homogeneous of degree $\nu > \nu - k$, so (16) implies that $f_k = 0$. This finishes the case $n = 4$.

The general case can be reduced to the above by taking sections. In fact, since $i_{Z_j}\Omega(x, 0) = 0$, $j = 1, \ldots, n-3$, we can write

$$\Omega(x, y) = \Omega_0^0(x) + \sum_{1\le|\sigma|\le\nu} y^\sigma \Omega_\sigma^0(x) + \sum_{i=1}^{n-3}\sum_{1\le|\sigma|\le\nu} y^\sigma f_\sigma^i(x)\, dy_i,$$

where $\sigma = (\sigma_1, \ldots, \sigma_{n-3})$, $y^\sigma = y_1^{\sigma_1}\ldots y_{n-3}^{\sigma_{n-3}}$, $|\sigma| = \sigma_1 + \cdots + \sigma_{n-3}$, f_σ^i and the coefficients of Ω_σ^0 are homogeneous polynomials of degree $\nu - |\sigma|$ and Ω_σ^0 contains only terms in $dx_1, \ldots, dx_4$. Let $v = (v_1, \ldots, v_{n-3})$ be a non-zero vector of $\mathbb{C}^{n-3}$ and consider the linear immersion $L\colon E^* \times \mathbb{C} \to E^* \times \mathbb{C}^{n-3} \simeq \mathbb{C}^{n+1}$ given by $L(x, w) = (x, wv)$. We have

$$L^*(\Omega) = \Omega_0^0(x) + \sum_{k=1}^{\nu} w^\nu\left[\sum_{|\sigma|=k} v^\sigma\Omega_\sigma^0(x) + \left(\sum_{i=1}^{n-3}\sum_{|\sigma|=k} v^\sigma v_i f_\sigma^i(x)\right) dw\right].$$

It follows from the case $n = 4$ that

$$\sum_{|\sigma|=k} v^\sigma\, \Omega_\sigma^0(x) = 0; \quad \forall v \in \mathbb{C}^{n-3}; \quad \forall 1 \le k \le \nu \implies \Omega_\sigma^0 = 0; \quad \forall\sigma \neq 0.$$

This implies that

$$\Omega(x, y) = \Omega_0^0(x) + \sum_{i,\sigma} y^\sigma f_\sigma^i(x)\, dy_i \implies d\Omega(x, y) = d\Omega_0^0(x)$$
$$+ \sum_{i,\sigma} y^\sigma\, df_\sigma^i(x) \wedge dy_i + \sum_{i<j}\omega_{i,j}\, dy_i \wedge dy_j.$$

Now, by using the integrability condition and collecting in $\Omega \wedge d\Omega = 0$ the coefficients of the terms containing only the factors $dx_i \wedge dx_j \wedge dy_\ell$, we get that

$$\sum_{i,\sigma} y^\sigma (\Omega_0^0 \wedge df_\sigma^i + f_\sigma^i \, d\Omega_0^0) \wedge dy_i = 0$$
$$\implies df_\sigma^i \wedge \Omega_0^0 = f_\sigma^i d\Omega_0^0; \quad \forall i, \sigma; \quad 1 \le |\sigma| \le \nu, 1 \le i \le n-3.$$

The last relation implies that $f_\sigma^i = 0$, for all i, σ. In fact, we have seen in the proof of Lemma 2 that $L_R(\Omega_0^0) = (\nu+1)\Omega_0^0$, so that $i_R(d\Omega_0^0) = i_R(d\Omega_0^0) + d(i_R \, \Omega_0^0) = L_R(\Omega_0^0) = (\nu+1)\Omega_0^0$. Hence,

$$i_R(df_\sigma^i \wedge \Omega_0^0) = i_R(f_\sigma^i d\Omega_0^0) \implies (\nu - |\sigma|) f_\sigma^i = (\nu+1) f_\sigma^i \implies f_\sigma^i = 0,$$

because f_σ^i is homogeneous of degree $\nu - |\sigma|$. This finishes the proof of the assertion and the theorem. □

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