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Comparison of Two Procedures for the Estimation of Surface Temperature History Using Function Speci®cation Method

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This work presents a comparative study of two alternative procedures for the estimation of surface temperature of a heated body from transient interior temperature measurements. This Inverse Heat Conduction Problem (IHCP) is solved (in both procedures) by using the Function Speci®cation Method. A numerical test was used in order to compare the best estimation achieved in each procedure. The in¯uences of the time step size, the total number of measurements and the noise level in the measurement have been considered in the estimation. Two criteria (minimization of total error and residual principle [3]) are used to choose the best hyper-parameter (r). The comparisons con®rm that the procedures and criteria used provide similar results, nevertheless this study reveals slight differences with respect to the accurate and the CPU time. [DOI: 10.1115/1.1738420]

Keywords: Conduction, Heat Transfer, Inverse, Numerical Methods

Introduction

As it is well known, the main dif®culty of the Inverse Heat Conduction Problem (IHCP) is the great ampli®cation of the measurement errors. The in¯uence of the more important factors in this problem can be discussed considering the exact solution of Burggraf [1]. Many methods have been reported to solve IHCPs, among the more versatile (applicable to solve multidimensional and non-linear IHCP) the following can be mentioned: Tikonov regularization [2], iterative regularization [3], molli®cation [4] and function speci®cation method [1]. Several types of the unknown boundary conditions are considered in the IHCPs. In many problems, the unknown boundary condition is stated as type 2 (or Newman's condition). This one can be considered the most usual procedure because, once the heat ¯ux has been estimated, the temperature ®eld (including the surface temperature) and heat transfer coef®cient can be calculated in a ''post-processor'' [5,6]. Nevertheless, depending on the methodology and the application considered, the procedures can be different. For example, in reference [7] surface temperature and heat ¯ux are estimated simultaneously in steady-state, and in [8] temperature ®eld and heat transfer coef®cient are estimated simultaneously in a transitory and nonlinear problem. In other cases, the sequence of calculations is the following: ®rst, the surface temperature (or Dirichlet's condition) is estimated, and then heat ¯ux and heat transfer coef®cient are calculated [9,10].

The purpose of this technical note is to present a comparative study of two alternative procedures in order to estimate surface temperature: Procedure IÐFirst, the surface heat ¯ux history is estimated by solving the IHCP, and then the discrete form of Duhamel's integral is used to calculate the surface temperature. Procedure IIÐThe surface temperature is estimated through a direct formulation of the corresponding IHCP. In both procedures, the classic Function Speci®cation Method (FSM) proposed by Beck [1] has been used. It is expected that different procedures provide similar results. Nevertheless, in an IHCP problem, it is suitable to carry out a comparative study. This is due to the ill-posed nature of the IHCP and the different sensitivity coef®cients used. The comparison is made by taking into account the best estimation achieved in each procedure using a numerical test. In order to obtain the best estimation, it is necessary to consider a criterion that permits an adequate choice of the hyperparameter. In the FSM, the hyperparameter is the number of future temperature used (r), and the criteria considered are (1) the minimization of total error, and (2) the residual principle [3]. The two criteria have been applied in both procedures.

Analysis

In order to validate the inverse algorithm, the previous formulation of the direct problem will be necessary. The following one-dimensional problem of heat conduction will be considered: a ¯at plate exposed to a heat ¯ow that varies in time in a triangular fashion (Fig. 1(a)). The opposite face is insulated. The mathematical formulation will be used in dimensionless form in order to simplify the notation. A more detailed description of this linear problem can be found in reference [1]. The problem is governed by the differential equation Eq. (1). The boundary conditions are indicated in Fig. 1(a). It is noted that τ represents the dimensionless duration of the triangular heating (in this case $\tau = 1.2$). Finally, the initial condition is: $T(x,0) = 0$.

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{\partial T(x,t)}{\partial t} \quad 0 \leqslant x \leqslant 1 \tag{1}$$

The solution to this problem is based on the superposition of the fundamental function $\theta(x,t)$, and it can be expressed as

$$\begin{aligned} 0<t<\tau/2 \quad & T(x,t) = \theta(x,t) \\ \tau/2<t<\tau \quad & T(x,t) = \theta(x,t) - 2\,\theta(x,t-\tau/2) \\ t>\tau \quad & T(x,t) = \theta(x,t) - 2\,\theta(x,t-\tau/2) + \theta(x,t-\tau) \end{aligned} \tag{2}$$

where $\theta(x,t)$ represents the analytical solution [11] of the problem when $q(t) = t$ for $t > 0$:

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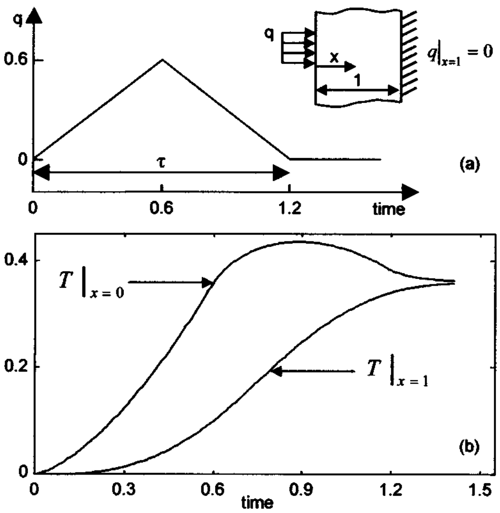

Fig. 1 (*a*) One-dimensional problem and heat ¯ow considered; and (*b*) analytical solution of the problem at $x=0$ and at $x=1$

$$\theta(x,t)=8t^{3/2}\sum_{n=0}^{\infty}\left\{i^3\,\mathrm{erfc}\frac{2n+x}{2t^{1/2}}+i^3\,\mathrm{erfc}\frac{2(n+1)-x}{2t^{1/2}}\right\} \quad (3)$$

For subsequent discussion, the solution, Eq. (2), will be evaluated at two points of special interest: at $x=1$, where a sensor will be placed, and at $x=0$, which corresponds to the surface temperature that will be estimated. Both functions are represented in Fig. 1(*b*).

The previous direct problem will be used as a test for the validation of the inverse problem. In the IHCP, the unknown boundary condition at $x=0$ is stated as type-I (or Dirichlet's condition), and the objective is to estimate the surface temperature using the discrete reading of temperature provided by a sensor located at $x=1$. As the measured temperatures Y_i are affected by errors, they are simulated using the discrete values of the analytical (or exact) temperature $T_i=T(1,t_i)$ (from Eq. (2)) at times $t_i=i\Delta t$ (the time intervals of the measurements). Then, random errors ε_i are added according to: $T_i+\varepsilon_i$, where $\varepsilon_i=Cu_i$. The random numbers u_i have been obtained using a random generator according to a normal (or Gaussian) distribution with zero mean, uncorrelated, and unit standard deviation. The constant C is chosen, so that $C=\sigma_Y$, where σ_Y is the standard deviation of measured temperatures. FSM method is based on the speci®cation of the functional form corresponding to an unknown input Ψ. To avoid unnecessary repetitions, the input denoted as Ψ, can be used in this problem for the surface heat ¯ux q (in procedure I) or the surface temperature T_S (in procedure II). The speci®cation of this method includes only r future steps from the last estimated component (component $M-1$). Then, the future components $\Psi_M, \Psi_{M+1}, \ldots, \Psi_{M+r-1}$, can be written in terms of Ψ_M, and only this component is estimated in each step. The temporary assumption can be made by several ways. In this note the simplest form is used, that being a piecewise constant form, and the r future components being assumed temporarily constant. With this assumption, a particular sequential inverse algorithm is derived from the minimization of the difference (in the least squares sense) between the measured (Y) and calculated (T) temperatures within the interval of future times. Details of this algorithm can be seen in reference [1]. The estimated component, noted as $\tilde{\Psi}_M$ can be expressed as

$$\tilde{\Psi}_M=\frac{\Sigma_{i=1}^{r}(Y_{M+i-1}-\tilde{T}_{M+i-1}|_{\Psi fut.=0})Z_i}{\Sigma_{i=1}^{r}Z_i^2} \quad (4)$$

where subscript i denotes the future times. $\tilde{T}_{M+i-1}|_{\Psi fut.=0}$ represents the calculated temperatures assuming that the future components $\Psi_M, \Psi_{M+1}, \ldots, \Psi_{M+r-1}$ are equal to zero. Z is the ®rst derivative of calculated temperature with respect to Ψ_M. This derivative, represents the sensitivity coef®cient to a unit step change in the input, and can be obtained by analytical [11] or numerical methods. In Eq. (4), Z_i is evaluated at $x=1$ (sensor location) and for the times $t_i=i\Delta t$, $i=1,2,\ldots,r$.

Depending on the procedure used, different inputs and different sensitivity coef®cients are used. Details of the two procedures are described below.

Procedure I. With this procedure, the previous estimation of surface heat ¯ux history $\tilde{q}$ is necessary. Then, the surface temperature $\tilde{T}_S$ is calculated from $\tilde{q}$. Accordingly, the input Ψ in the inverse algorithm, Eq. (4), represents the surface ¯ux history q, and Z represents the respective sensitivity coef®cients. Once the estimation of $\tilde{q}$ has been completed, we can recover the ®eld temperature using the discrete form of Duhamel's integral, according to

$$\tilde{T}(x,t_M)=T_0+\sum_{i=1}^{M}\tilde{q}_i(t_{i+1/2})X_i(x,t_{M-i}) \quad (5)$$

where the initial temperature, noted as T_0, will be $T_0=0$ in accordance with the mathematical formulation of the problem. In agreement with the temporal assumption considered in the inverse algorithm, Duhamel's integral is approximated by a constant piecewise function centered at the middle of the time step $(t_{i+1/2})$. Consequently the sensitivity coef®cientsX_i represent the temperature response to a unit pulse in the input, and hence it is evident that $X_i=Z_{i+1}-Z_i$. The surface temperature is obtained from Eq. (5) by setting $x=0$.

Procedure II. With this procedure, the surface temperature history $\tilde{T}_S$ is directly estimated from Eq. (4). Now, Ψ and Z represent the surface temperature and the respective sensitivity coef®cients. The estimation of surface temperature in this form has been described by Woodbury [9]. Nevertheless, the purpose of Woodbury's study was the surface ¯ux estimation, whereas this study is focused on the surface temperature.

In order to compare the best estimation of surface temperature by the two procedures (I and II), it is necessary to select the optimal value of r for a given time step. In this comparison, the following criteria are considered:

Criterion A. In an IHCP there are two sources of error in the estimation. The ®rst source is the unavoidable bias deviation (or deterministic error) when $r>1$. The second source of error is the variance due to the ampli®cation of measurement errors (stochastic error), which can be very important, especially when the time steps are small. The global effect of deterministic and stochastic errors is considered in the mean squared error or total error. Details of these types of error and the corresponding estimates can be found in reference [1]. The estimates used in this study for the bias (D), the variance (σ_Ψ) and the total error (S) are de®ned by Eqs. (6), (7), and (8), respectively.

$$D=\left[\frac{1}{N-1}\sum_{i=1}^{N}(\tilde{\Psi}_i|_{\sigma_Y=0}-\Psi_i)^2\right]^{1/2} \quad (6)$$

$$\sigma_\Psi=\left[\frac{1}{N-1}\sum_{i=1}^{N}(\tilde{\Psi}_i-\tilde{\Psi}_i|_{\sigma_Y=0})^2\right]^{1/2} \quad (7)$$

$$S=\left[\frac{1}{N-1}\sum_{i=1}^{N}(\tilde{\Psi}_i-\Psi_i)^2\right]^{1/2} \quad (8)$$

where $\tilde{\Psi}_i|_{\sigma_Y=0}$ are the "virtual" estimations using errorless measurements, and Ψ_i are the true values of input. The optimality criterion is based on the minimization of S, which allows determination of the necessary balance between the two error sources. As it is evident, the interest of this criterion is mainly theoretical, and it can only be useful in comparative studies.

Criterion B. This criterion is based on the residual principle [3]. Once the vector of input $(\tilde{\Psi}_1, \cdots \tilde{\Psi}_N)$ has been estimated, the evaluation of the residual obtained from the comparison between the measured temperatures Y_i and the recovered temperatures $\tilde{T}_i(r)$ is possible. The residual R is de®ned as

$$R(r) = \left[\frac{1}{N-1} \sum_{i=1}^{N} (Y_i - \tilde{T}_i(r))^2 \right]^{1/2} \tag{9}$$

For the procedure-I, $\tilde{T}_i(r)$ can be obtained from Eq. (5) by setting $x=1$ (sensor location). As $T_0=0$, the Eq. (5) can be used in same form to obtain $\tilde{T}_i(r)$ in procedure II, nevertheless the $\tilde{q}_i$ components must be replaced by $\tilde{T}_{Si}$, and the sensitivity coef®cients X_i represent the temperature response to a unit pulse of imposed surface temperature. In the function speci®cation method, the residual principle is satis®ed when r is such that the residual R assumes the closest (and superior) value to the standard deviation of measurement [12]. This condition can be expressed as follows: $\min_r \{R(r) \geq \sigma_Y\}$.

Numerical Results

A numerical test was carried out for three cases and 24 subcases. The results are presented in tabular and graphical forms, and they correspond to the optimum r-value obtained by each procedure (I and II), criterion (A and B) and level of noise, respectively, as can be seen in Table 1. Each case corresponds to a particular value of the total number of measurements (N) during the time interval τ and a size of time step (Δt). The parameters Δt and N are modi®ed using the condition: $\Delta t\, N = \tau = 1.2$, so that, N is increased gradually and the time step is reduced. Two levels of noise measurements σ_Y (0.001 and 0.01) are considered in each case. Taking as reference the maximum increase of dimensionless temperature at location sensor (0.3581), which corresponds to the maximum of lower curve of Fig. 1(*b*), and considering (around the exact temperatures) an error range between $\pm 2.576\sigma_Y$ (or 99 percent con®dence interval), these noise levels correspond to percentages error of 0.72 percent and 7.2 percent respectively. Finally it is noted that estimates σ_{TS}, S and R are random variables. Therefore, the choice of optimum value of r needs the application of Monte Carlo method. A total of 30 sets random errors have been generated in each sub-case, and the results presented in Table 1 are the arithmetic mean of the corresponding estimates.

Table 1 Summary of numerical results for optimum estimations

P.	σ_Y	C.	r	σ_{TS}	D	S	R
			CASE-1, Δt=0.12, N=10				
I	0.001	A/B	2	0.0049	0.0090	0.0100	0.0015
	0.01	A/B	3	0.0187	0.0107	0.0209	0.0108
II	0.001	A/B	2	0.0033	0.0060	0.0068	0.0039
	0.01	A/B	3	0.0161	0.0211	0.0267	0.0156
			CASE-2, Δt=0.03, N=40				
	0.001	A	7*(6*)	0.0023	0.0034	0.0041	0.0018
I		B	6	0.0035	0.0021	0.0040	0.0012
	0.01	A	10*(9*)	0.0098	0.0101	0.0141	0.0107
		B	10	0.0098	0.0101	0.0141	0.0107
	0.001	A	6	0.0035	0.0043	0.0056	0.0024
II		B	5	0.0058	0.0023	0.0063	0.0014
	0.01	A	9*(8*)	0.0140	0.0132	0.0193	0.0121
		B	8	0.0179	0.0098	0.0204	0.0108
			CASE-3, Δt=0.01, N=120				
	0.001	A	16*(17*)	0.0023	0.0017	0.0029	0.0011
I		B	16	0.0023	0.0017	0.0029	0.0011
	0.01	A	25*(27*)	0.0077	0.0066	0.0101	$<\sigma_Y$
		B	28	0.0064	0.0091	0.0111	0.0106
	0.001	A	15*(16*)	0.0035	0.0027	0.0043	0.0016
II		B	12	0.0067	0.0012	0.0068	0.0011
	0.01	A	22	0.0126	0.0082	0.0126	0.0147
		B	21	0.0141	0.0072	0.0159	0.0105

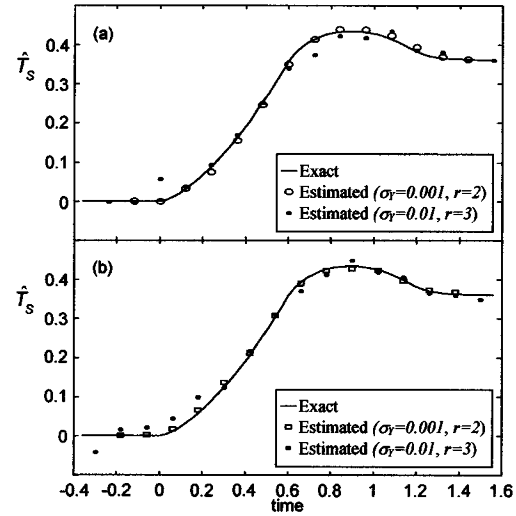

Fig. 2 Case-1: (*a*) Procedure-I; and (*b*) Procedure II

The ®rst case (case-1 in Table 1) considers a relatively large time step $\Delta t = 0.12$. Due to their size, only ten measurements ($N=10$) are included in the interval τ. For all subcases considered, the optimal r-value has been $r=2$ (for $\sigma_Y=0.001$) and $r=3$ (for $\sigma_Y=0.01$), with independence of criterion (A or B). Graphical representations of $\tilde{T}_S$ versus time are plotted in Fig. 2. Considering a visual inspection, the estimations obtained by both procedures are similar. Nevertheless, taking into account the values of S in Table 1, for low noise, procedure II is slightly better than procedure-I, and for high noise, the opposite occurs.

In the second case (case-2 in Table 1), the time step is $\Delta t = 0.03$. This value implies $N=40$, according to the previous considerations. As it is expected, shorter time step requires larger r-value in order to assure the stability. Comparing the tabulated results to the corresponding to previous case, the estimations are now slightly more accurate. In this case, the estimation of $\tilde{T}_S$ by procedure I is slightly better than the one obtained by procedure II. For sub-case procedure I and $\sigma_Y=0.001$, the best estimation corresponds to $r=7$ (criterion A) and $r=6$ (criterion B). Never-

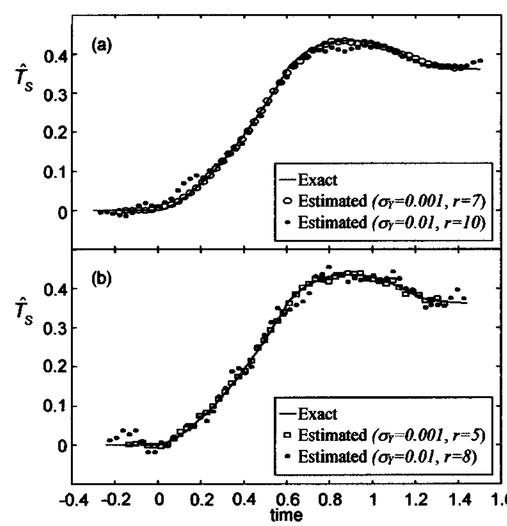

Fig. 3 Case-2: (*a*) Procedure-I; and (*b*) Procedure II

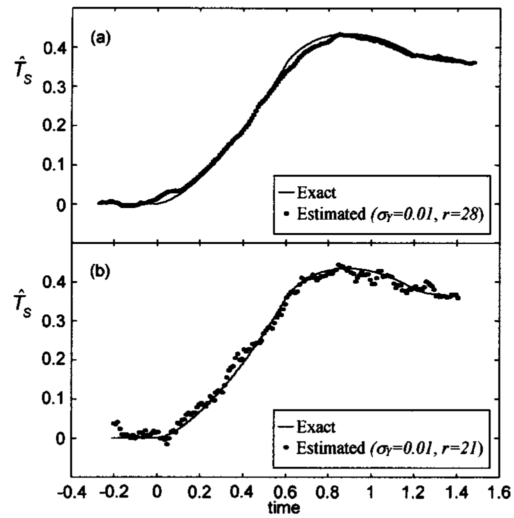

Fig. 4 Case-3: (*a*) Procedure-I; and (*b*) Procedure II

theless, a new statistical analysis reveals that the choice based on criterion A is affected by uncertainty. If the arithmetic mean value of S is recalculated a set of 30 times, according to the central limit theorem, these mean values are nearly normally distributed. This particular point has been corroborated using the Kolmogorov-Smirnov test. This fact permits to establish con®dence intervals. The 95 percent con®dence interval for S with $r=6$ is: (0.0038, 0.0042) and the corresponding interval with $r=7$ is: (0.0039, 0.0041). There is obviously an overlapping, and any of the two values can be considered as the optimum. The cases affected by this type of uncertainty are noted in Table 1 with (*). Similar overlapping exists in the sub-case: procedure I, $\sigma_Y=0.01$. On the other hand, and considering the criterion B corresponding to procedure I and $\sigma_Y=0.01$, the 95 percent con®dence interval for R with $r=9$ is as follows: (0.0097, 0.0105). As $0.0097<\sigma_Y$, this implies that in some samples the residual principle is not statistically satis®ed, consequently the minimum r-value that satis®es (with 95 percent con®dence) the residual principle is $r=10$ (see Table 1). When procedure II is applied, only the estimations corresponding to a high noise level ($\sigma_Y=0.01$) are affected by uncertainty. The estimations obtained by procedure I and II are plotted in Fig. 3. In each case two noise levels are considered. The criteria selected have been: criterion A, in Fig. 3(*a*), and criterion B, in Fig. 3(*b*).

In the third case (case-3 in Table 1) $\tilde{T}_S$ is estimated with a very high temporal resolution. The estimations corresponding to sub-cases of low noise level are the most accurate results, and for the sake of clarity are not plotted. Considering the high noise level and the residual principle (criterion B), the best estimations obtained by procedure I and II are plotted in Fig. 4.

Finally, the computational time has also been evaluated. It is evident that procedure I needs more computation time than the used by procedure II. Moreover, the r-value used by procedure I (in the best estimation) is higher that the one used by procedure II, especially in cases of high temporal resolution. For example, comparing the estimation of surface temperature corresponding to Fig. 4, the CPU time required by procedure I and procedure II has been 11.43 s. and 7.08 s, respectively. This advantage of procedure II can be attractive in an on-line process. All numerical calculations were performed on a personal computer with a Pentium III 700 MHz processor.

Conclusions

Two procedures in conjunction with two possible criteria have been considered in this comparative study in order to estimate the surface temperature. The results obtained in this numerical simulation reveal that, the accuracy of both procedures is similar, nevertheless procedure I provides results slightly more accurate than procedure II. On the other hand, procedure II requires less computational time and needs a smaller number of future temperatures. This fact suggests that procedure II can be more adequate in those applications where the surface temperature must be estimated in an on-line process of long duration.

A comparison has also been made between the criterion of minimum mean squared error and the residual principle. Taking into account the uncertainty in the determination of the optimum r-value, this simulation con®rms that both criteria provide equal or similar values in all the cases considered.

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Nomenclature

C = constant chosen
D = estimate of bias
q = dimensionless heat ¯ux
M = present time step
N = number of measurement during τ
n = integer
T = dimensionless temperature
T_0 = initial temperature
r = number of future time steps
t = dimensionless time
R = estimate of residual
S = estimate of total error
u = Gaussian random numbers (normalized)
x = dimensionless coordinate
X = sensitivity coef®cient, Eq. (5)
Y = measured temperature
Z = sensitivity coef®cient, Eq. (4)

Greek Symbols

Δt = dimensionless time step size
ε = random error
θ = analytical solution, Eq. (3)
σ = standard deviation
τ = dimensionless temporal interval
Ψ = input unknown

Subscripts

i = at time t_i
fut. = future components
S = surface location

Superscripts

Ã = estimated
* = uncertainty

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Critical Values of Gr/Re for Mixed Convection in Vertical Eccentric Annuli With Isothermal/ Adiabatic Walls

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1 Introduction

Mixed (forced-free) convection heat transfer in vertical eccentric annuli can be found in the drilling and cementing operations of oil wells [1], double-pipe heat-exchangers, and cooling of vertical electric motors and generators [2]. Most articles dealing with eccentric annuli treated the fully developed forced ¯ow and the fully developed forced convection [3±9]. Sathymurthy et al. [10] investigated the problem of fully developed mixed (combined forced and free) convection for a Newtonian ¯uid in an eccentric annulus.

The main objective of this paper is to present an analytical solution for the problem of fully developed laminar mixed convection in vertical eccentric annuli under the thermal boundary conditions of one isothermally heated cylinder while the other cylinder is insulated. This analytical solution is used to obtain the critical values of Gr/Re that create buoyancy effects that balance the friction in the annulus. Flows having Gr/Re above these critical values would make the channel, which has a constant cross-sectional area perpendicular to the ¯ow direction, acts as a diffuser with possible incipient ¯ow separation.

2 Problem Description

The geometry under consideration is shown in Fig. 1(*a*). This eccentric geometry can easily be described by the bipolar coordinate system (η, ξ and z) shown in Fig. 1(*b*).

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A Newtonian ¯uid at ambient temperature T_o is forced to ¯ow through this vertical annulus from its bottom. Free convection exists inside this vertical channel as a result of heating one of its walls at a uniform temperature (T_w) different from that of the ambient while keeping the other wall insulated. The ¯uid is assumed to have constant physical properties but obeys the Boussinesq approximation (its density is allowed to vary with temperature in only the gravitational body force (buoyancy) term of the vertical (axial) momentum equation). Body forces in other than the vertical direction, viscous dissipation, internal heat generation, and radiation heat transfer are absent. Using the appropriate coordinate scale factors [11], the governing equations in bipolar coordinates under the above-mentioned assumptions can be obtained.

3 Fully Developed Mixed-Convection Flow

3.1 The Fully Developed Velocity Pro®le. At large values of the dimensionless axial distance Z the ¯ow becomes fully developed with $v=w=0$ and $\partial u/\partial z=0$. Hence the continuity equation and the inertia terms of the axial momentum equation vanish while the ξ and η-momentum equations reduce to $\partial p/\partial\xi=0$ and $\partial p/\partial\eta=0$, respectively. In the axial momentum equation, the gravitational body force per unit volume $F_z=-\rho g$ and according to the Boussinesq approximation: $\rho=\rho_o(1-\beta|T-T_o|)$. Hence, $F_z=-\rho_o g+\rho_o g\beta|T-T_o|$. Using the dimensionless parameters given in the nomenclature, one can write: $F_z=-\rho_o g+(\rho_o \mathrm{Gr}\gamma^2/D_h^3)\theta$. Accordingly, for a hydrodynamic fully developed mixed/forced ¯ow $\partial p/\partial z=[(dp/dz)_{fd}]=$constant and the resulting axial momentum equation reduces to

$$\mu\left(\frac{\partial^2 u_{fd}}{\partial\xi^2}+\frac{\partial^2 u_{fd}}{\partial\eta^2}\right)=h^2\left(\left(\frac{dp'}{dz}\right)_{fd}-\frac{\rho_o\mathrm{Gr}\gamma^2}{D_h^3}\theta_{fd}\right)\tag{1}$$

For values of Pr<1, thermal full-development occurs before the hydrodynamic full development and since the boundary conditions under consideration have one wall isothermal, $\theta_{fd}=1$. With $\theta_{fd}=1$ and using the dimensionless parameters given in the nomenclature the above equation reduces to

$$\frac{\partial^2 U_{fd}}{\partial\xi^2}+\frac{\partial^2 U_{fd}}{\partial\eta^2}=H^2\left(\left(\frac{dP}{dZ}\right)_{fd}-\frac{\mathrm{Gr}}{\mathrm{Re}}\right)=\frac{C^*\left(\left(\frac{dP}{dZ}\right)_{fd}-\frac{\mathrm{Gr}}{\mathrm{Re}}\right)}{(\cosh\eta-\cos\xi)^2}\tag{2}$$

where C^* is a dimensionless constant that depends on the geometry and is given by

$$C^*=(\sinh^2\eta_o)/[4(1-\mathrm{N})^2].\tag{3}$$

It is worth mentioning that Re and Gr cannot, in strict sense, be independently varied for mixed-convection problems in vertical channels. This is because the entrance velocity u_o (which equals to $\bar{u}$, under the steady-state steady-¯ow conditions) is physically in¯uenced by the value of Gr, in vertical duct ¯ows. However, forced and free convection effects can be comparable when an external ¯ow is superimposed on a buoyancy-driven ¯ow. In such a case, ``there exists a well-de®ned forced convection velocity' [12]. Nonetheless, in the case under investigation, the governing Eq. (2) will be handled for given values of the parameter Gr/Re, rather than independent values of each.

Let $C^{**}=-\{(dP/dZ)_{fd}-\mathrm{Gr}/\mathrm{Re}\}$, which is a dimensionless quantity, and dividing both sides of (2) by C^{**}

$$\frac{\partial^2(U_{fd}/C^{**})}{\partial\xi^2}+\frac{\partial^2(U_{fd}/C^{**})}{\partial\eta^2}=\frac{-C^*}{(\cosh\eta-\cos\xi)^2}\tag{4}$$

De®ne the variable U_{fd}/C^{**} as the modi®ed velocity pro®le (U_{fdm}) then the above equation becomes

$$\frac{\partial^2 U_{\mathrm{fdm}}}{\partial\xi^2}+\frac{\partial^2 U_{\mathrm{fdm}}}{\partial\eta^2}=\frac{-C^*}{(\cosh\eta-\cos\xi)^2}\tag{5}$$

