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On Locally Finite Split Lie Triple Systems#

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ABSTRACT

Lie triple systems appear as the natural ternary extension of Lie algebras. The classification in the finite-dimensional setup (over an algebraically closed field of characteristic zero) is well-known [Lister, W. G. (1952). A structure theory of Lie triple systems. *Trans. Amer. Math. Soc.* 72:217–242]. In order to suggest a possible approach to a structure theory of infinite-dimensional Lie triple systems, we introduce and study split and locally finite Lie triple systems, stating that under certain conditions the standard embedding of a split Lie triple system is a split Lie algebra and that the standard embedding of a locally finite Lie triple system is a locally finite Lie algebra. We also give a description of certain locally finite simple split Lie triple systems.

Key Words: Lie triple systems; Split Lie algebras; Locally finite Lie algebras.

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1. INTRODUCTION

The main tool used by Lister (1952) to study finite-dimensional Lie triple systems is the standard embedding, this one being a two-graded Lie algebra $L = L^0 \oplus L^1$, with L^0 the span of $\{\mathscr{L}(x, y) : x, y \in T\}$, where $\mathscr{L}(x, y)$ denotes the left multiplication operator in T, $L^1 := T$, and where the product is given by

$$[(\mathscr{L}(x, y), z), (\mathscr{L}(u, v), w)] \\ := (\mathscr{L}([u, v, y], x) - \mathscr{L}([u, v, x], y) + \mathscr{L}(z, w), [x, y, w] - [u, v, z]).$$

Later, Faulkner (1980) gives an alternative approach to the classification of Lie triple systems, also in the finite-dimensional setup, by introducing a Cartan subalgebra H^0 of the even part of the standard embedding, and a decomposition of T as the direct sum of certain root spaces relative to H^0.

In the framework of infinite-dimensional Lie algebras, Neeb, Stumme and other authors have successfully developed over the recent years a theory of split and locally finite Lie algebras (c.f. Bakhturin and Benkart, 1997; Bakhturin and Strade, 1995a,b; Dimitrov and Penkov, 1999; Neeb, 2000, 2001; Neeb and Stumme, 2001; Stumme, 1999).

In this paper, we combine all the above ideas in order to introduce and study split Lie triple systems of arbitrary dimension in Secs. 2 and 3, the main result being Theorem 3.1 which states that under certain conditions the standard embedding of a split Lie triple system is a split Lie algebra. In Sec. 4 we also state that the standard embedding of a locally finite Lie triple system is a locally finite Lie algebra (Proposition 4.1), and obtain in Theorem 4.1 a description of an important class of locally finite split simple Lie triple systems.

2. BASIC DEFINITIONS

Let $\mathbb{K}$ be a field of characteristic zero and let T be a vector space over $\mathbb{K}$. We say that T is a *triple system* if it is endowed with a trilinear map

$$\langle \cdot, \cdot, \cdot \rangle : T \times T \times T \to T,$$

called the *triple product* of T.

A triple system T is called a *Lie triple system* if its triple product, denoted by $[\cdot, \cdot, \cdot]$, satisfies

(1) $[x, x, y] = 0$.
(2) $[x, y, z] + [y, z, x] + [z, x, y] = 0$ (Jacobi identity).
(3) $[x, y, [a, b, c]] - [a, b, [x, y, c]] = [[x, y, a], b, c] + [a, [x, y, b], c]$.

for any $x, y, z, a, b, c \in T$.

An *ideal* of a Lie triple system T is a subspace I for which $[I, T, T] \subseteq I$. Let us observe that $[I, T, T] \subseteq I$ implies that $[T, I, T] \subseteq I$ and $[T, T, I] \subseteq I$. A Lie triple system T is called *simple* if the product is nonzero and its only ideals are $\{0\}$ and T.

A *two-graded* $\mathbb{K}$*-algebra* A is a $\mathbb{K}$-algebra which splits into the direct sum $A = A^0 \oplus A^1$ of subspaces (called the even and the odd part, respectively) satisfying $A^\alpha A^\beta \subset A^{\alpha+\beta}$ for any α, β in $\mathbb{Z}_2$. A homomorphism f between two-graded algebras A and B is a homomorphism which preserves gradings i.e.,

$$f(A^\alpha) \subset B^\alpha$$

for all $\alpha \in \mathbb{Z}_2$. A *graded ideal* of a two-graded algebra $A = A^0 \oplus A^1$ is an ideal I of A such that there exist two subspaces I^0 and I^1 with $I = I^0 \oplus I^1$ and $I^\alpha \subseteq A^\alpha$ for any $\alpha \in \mathbb{Z}_2$. A two-graded algebra A is *graded-simple* if the product is nonzero and its only graded ideals are $\{0\}$ and A.

The *standard embedding* of a Lie triple system T is the two-graded Lie algebra $L = L^0 \oplus L^1$, L^0 being the $\mathbb{K}$-span of $\{\mathscr{L}(x, y) : x, y \in T\}$, where $\mathscr{L}(x, y)$ denotes the left multiplication operator in T, $\mathscr{L}(x, y)(z) := [x, y, z]$; $L^1 := T$ and where the product is given by

$$\begin{aligned}&[(\mathscr{L}(x, y), z), (\mathscr{L}(u, v), w)]\\ &\quad := (\mathscr{L}([u, v, y], x) - \mathscr{L}([u, v, x], y) + \mathscr{L}(z, w), [x, y, w] - [u, v, z]).\end{aligned}$$

Let us observe that L^0 with the product induced by the one in $L = L^0 \oplus L^1$ becomes a Lie algebra.

By using the same arguments as in the finite dimensional case, (see Lister, 1952, Theorem 2.13), we can prove the following propositions

Proposition 2.1. *A Lie triple system T is simple if and only if its standard embedding is graded-simple.*

Proposition 2.2. *Let $L = L^0 \oplus L^1$ be a graded-simple two-graded Lie algebra. Then either*

(1) *L is simple (in the ungraded sense) and the grading is given by*

$$L = Sym(L, \xi) \oplus Skw(L, \xi)$$

where ξ is an involutive automorphism of L, or

(2) *L is isomorphic to $L' \oplus L'$ where L' is a simple Lie algebra and the product is*

$$[(a, b), (c, d)] = ([a, c] + [b, d], [a, d] + [b, c]).$$

3. SPLIT LIE TRIPLE SYSTEMS

3.1. Definitions and Basic Properties

Given an element x of a Lie algebra L, we denote by $ad(x)$ the *adjoint mapping* defined as $ad(x)(y) = [x, y]$ for any $y \in L$.

Following Neeb (2000), Stumme (1999) and Schue (1960, 1961), we recall that a *splitting Cartan subalgebra H of a Lie algebra L* is defined as a maximal abelian subalgebra, (MASA), of L satisfying that the adjoint mappings $ad(h)$ for $h \in H$ are simultaneously diagonalizable. If L contains a splitting Cartan subalgebra H, then L is called a *split Lie algebra*. This means that we have a *root decomposition* $L = H \oplus (\bigoplus_{\alpha\in\Lambda} L_\alpha)$ where $L_\alpha = \{v_\alpha \in L : [h, v_\alpha] = \alpha(h)v_\alpha \text{ for any } h \in H\}$ for a linear functional $\alpha \in H^*$ and $\Lambda := \{\alpha \in H^*\backslash\{0\} : L_\alpha \neq 0\}$. The subspaces L_α for $\alpha \in H^*$ are called *root spaces* of L (respect to H) and the elements $\alpha \in \Lambda \cup \{0\}$ are called *roots* of L respect to H.

Definition 3.1. Let T be a Lie triple system, let $L = L^0 \oplus L^1$ be its standard embedding, and let H^0 be a MASA of L^0. Following the ideas in Faulkner (1980), for a linear functional $\alpha \in (H^0)^*$ we define the root space *of T* (respect to H^0) associated to α as the subspace $T_\alpha := \{t_\alpha \in T : [h, t_\alpha] = \alpha(h)t_\alpha \text{ for any } h \in H^0\}$. The elements $\alpha \in (H^0)^*$ satisfying $T_\alpha \neq 0$ are called roots of T respect to H^0 and we denote $\Lambda^1 := \{\alpha \in (H^0)^*\backslash\{0\} : T_\alpha \neq 0\}$.

Let us observe that $T_0 = \{t_0 \in T : [h, t_0] = 0 \text{ for any } h \in H^0\}$. In the following, we shall denote by Λ^0 the set of all nonzero $\alpha \in (H^0)^*$ such that $L^0_\alpha := \{v^0_\alpha \in L^0 : [h, v^0_\alpha] = \alpha(h)v^0_\alpha \text{ for any } h \in H^0\} \neq 0$.

Lemma 3.1. *Let T be a Lie triple system, let $L = L^0 \oplus L^1$ be its standard embedding, and let H^0 be a MASA of L^0. If $\alpha, \beta, \gamma \in \Lambda^1 \cup \{0\}$ and $\delta \in \Lambda^0 \cup \{0\}$. Then,*

(1) *If $[T_\alpha, T_\beta] \neq 0$ then $\alpha + \beta \in \Lambda^0 \cup \{0\}$ and $[T_\alpha, T_\beta] \subseteq L^0_{\alpha+\beta}$.*
(2) *If $[L^0_\delta, T_\alpha] \neq 0$ then $\delta + \alpha \in \Lambda^1 \cup \{0\}$ and $[L^0_\delta, T_\alpha] \subseteq T_{\delta+\alpha}$.*
(3) *If $[T_\alpha, T_\beta, T_\gamma] \neq 0$ then $\alpha + \beta + \gamma \in \Lambda^1 \cup \{0\}$ and $[T_\alpha, T_\beta, T_\gamma] \subseteq T_{\alpha+\beta+\gamma}$.*

Proof. (1) For any $x \in T_\alpha$, $y \in T_\beta$ and $h \in H^0$, we have $[h, [x, y]] = -[[h, y], x] + [[h, x], y] = (\alpha + \beta)(h)[x, y]$. Therefore, $[T_\alpha, T_\beta] \subseteq L^0_{\alpha+\beta}$.

(2) and (3) The proof is similar to 1. □

Let us observe that if $[T_\alpha, T_0] \neq 0$ then α is a root of L^0 relative to H^0 and if $[L^0_\alpha, T_0] \neq 0$ then α is a root of T relative to H^0.

Definition 3.2. Let T be a Lie triple system, let $L = L^0 \oplus L^1$ be its standard embedding, and let H^0 be a MASA of L^0. We shall call that T is a split Lie triple system (respect to H^0) if:

(1) $T = T_0 \oplus (\bigoplus_{\alpha\in\Lambda^1} T_\alpha)$.
(2) $[T_0, T_0, T] = 0$.

Proposition 3.1. *Let T be a split Lie triple system respect to H^0, then L^0 is a split Lie algebra respect to the splitting Cartan subalgebra H^0.*

Proof. We first prove that $\sum_{\beta\in\Lambda^0} L^0_\beta$ is a direct sum, (let us note that we can also apply Moody and Pianzola, 1995, 2.1 Prop.1). If $x \in L^0_\alpha \cap \sum_{\beta\neq\alpha} L^0_\beta$, we can write

$x = \sum_{\beta \neq \alpha} x_\beta$ with $x_\beta \in L^0_\beta$, and $x = x_\alpha$ with $x_\alpha \in L^0_\alpha$. From Lemma 3.1, we deduce for any $y \in T_\gamma$ and $\gamma \in \Lambda^1 \cup \{0\}$, that

$$[x, y] \in T_{\alpha+\gamma} \cap \sum_{\beta \neq \alpha} T_{\beta+\gamma}$$

and therefore $[x, y] = 0$ for any $y \in T$. As x is a linear map on T, satisfying $x(y) = [x, y]$, we deduce that $x = 0$. Let us also check that $H^0 + (\bigoplus_{\beta \in \Lambda^0} L^0_\beta)$ is a direct sum. If $x \in H^0 \cap (\bigoplus_{\beta \in \Lambda^0} L^0_\beta)$ we can write $x = \sum_{\beta \in \Lambda^0} x_\beta$, with $x_\beta \in L^0_\beta$, and $x = h^0 \in H^0$. For $h \in H^0$, we have

$$0 = [h, h^0] = [h, x] = \sum_{\beta \in \Lambda^0} [h, x_\beta] = \bigoplus_{\beta \in \Lambda^0} \beta(h) x_\beta,$$

so $\beta(h)x_\beta = 0$ and $x_\beta = 0$, since $\beta(H^0) \neq 0$. We conclude $x = 0$. Thus $H^0 \oplus (\bigoplus_{\beta \in \Lambda^0} L^0_\beta) \subseteq L^0$.

Let us observe that the maximal abelian character of H^0 in L^0 implies $H^0 = L^0_0$. Finally, as

$$L^0 = [T, T] = \sum_{\alpha, \gamma \in \Lambda^1 \cup \{0\}} [T_\alpha, T_\gamma] \subseteq \sum_{\alpha, \gamma \in \Lambda^1 \cup \{0\}} L^0_{\alpha+\gamma} \subseteq H^0 \oplus \left(\bigoplus_{\beta \in \Lambda^0} L^0_\beta \right),$$

the proof is complete. □

3.2. Types of Roots–Integrable Roots

In this subsection we distinguish the various types in the root system of a split Lie triple system, taking particular notice of the integrable roots.

Lemma 3.2. *For nonzero root vectors $t_{\pm\alpha} \in T_{\pm\alpha}$ the subalgebra of the standard embedding,*

$$L(t_\alpha, t_{-\alpha}) := span_{\mathbb{K}}\{t_\alpha, t_{-\alpha}, [t_\alpha, t_{-\alpha}]\}$$

is of one of the following types:

(1) *If $[t_\alpha, t_{-\alpha}] = 0$, then $L(t_\alpha, t_{-\alpha})$ is two dimensional abelian. We say that $L(t_\alpha, t_{-\alpha})$ is of abelian type.*
(2) *If $[t_\alpha, t_{-\alpha}] \neq 0$ but $\alpha([t_\alpha, t_{-\alpha}]) = 0$, then $L(t_\alpha, t_{-\alpha})$ is a three dimensional algebra. We say that $L(t_\alpha, t_{-\alpha})$ is of nilpotent type.*
(3) *If $\alpha([t_\alpha, t_{-\alpha}]) \neq 0$, then $L(t_\alpha, t_{-\alpha}) \approx sl(2, \mathbb{K})$. We say that $L(t_\alpha, t_{-\alpha})$ is of simple type.*

Proof. This is an easy verification (cf. Stumme, 1999, Lemma I.2). □

Definition 3.3. For a root $\alpha \in \Lambda^1$ the subalgebras $L(t_\alpha, t_{-\alpha})$ are called test algebras associated to α. We say that a root $\alpha \in \Lambda^1$ is of nilpotent type if all test algebras associated to α are of abelian or nilpotent type. Note that a root $\alpha \in \Lambda^1$ with $-\alpha \notin \Lambda^1$ is of nilpotent type. We call a root $\alpha \in \Lambda^1$ of simple type if there exists an associated test algebra of simple type. A root $\alpha \in \Lambda^1$ of simple type is called integrable if there exists an associated test algebra $L(t_\alpha, t_{-\alpha})$ of simple type such that the middle multiplication operators of T,

$$\mathscr{M}(t_{\epsilon\alpha}) : T \to T,$$

$\mathscr{M}(t_{\epsilon\alpha})(t) := [t_{\epsilon\alpha}, t, t_{\epsilon\alpha}]$, $\epsilon \in \{+, -\}$, are locally nilpotent. We denote by Λ^1_i the set of all integrable nonzero roots of T relative to H^0.

Lemma 3.3. *Let T be a split Lie triple system, let $L = L^0 \oplus L^1$ be its standard embedding and let $\alpha \in \Lambda^1$. then we have*:

(1) $\mathscr{M}(t_\alpha)(t) = -ad^2(t_\alpha)(t)$, *for any* $t_\alpha \in T_\alpha$ *and for any* $t \in T$.
(2) $ad^{3+2n}(t_\alpha)([t, t']) = (-1)^n \mathscr{M}^{n+1}(t_\alpha)([t, t', t_\alpha])$, *for any* $n \in \mathbb{N}$, $t_\alpha \in T_\alpha$ *and* $t, t' \in T$.

Proof.

(1) It is clear.
(2)
$$\begin{aligned} ad^{3+2n}(t_\alpha)([t, t']) &= -ad^{2+2n}(t_\alpha)([t, t', t_\alpha]) \\ &= -(-\mathscr{M}(t_\alpha))^{n+1}([t, t', t_\alpha])) \\ &= (-1)^n \mathscr{M}^{n+1}(t_\alpha)([t, t', t_\alpha]). \end{aligned}$$ □

Proposition 3.2. *Let T be a split Lie triple system, let $L = L^0 \oplus L^1$ be its standard embedding and let $\alpha \in \Lambda^1$. Then, for any $t_\alpha \in T_\alpha$, $\mathscr{M}(t_\alpha)$ is locally nilpotent on T if and only if $ad(t_\alpha)$ is locally nilpotent on L.*

Proof. Suppose that $\mathscr{M}(t_\alpha)$ is locally nilpotent on T. Let be $x = x^0 + x^1 \in L$, with $x^0 \in L^0$ and $x^1 \in L^1$. There exists $m \in \mathbb{N}$ such that $\mathscr{M}^m(t_\alpha)(x^1) = 0$. By Lemma 3.3-1 we also have $ad^{2m}(t_\alpha)(x^1) = 0$. As $x^0 = \sum_{i=1}^q [t_i, t'_i]$ with $t_i, t'_i \in T$, if we consider for each $[t_i, t'_i]$ the element $[t_i, t'_i, t_\alpha] \in T$, there exists $m_i \in \mathbb{N}$ such that $\mathscr{M}^{m_i}([t_i, t'_i, t_\alpha]) = 0$, and by Lemma 3.3-2, $ad^{3+2(m_i-1)}(t_\alpha)([t_i, t'_i]) = 0$. By denoting $r = \max\{2m, \bigcup_{i=1,\ldots,q} 3 + 2(m_i - 1)\}$, we conclude that $ad^r(t_\alpha)(x) = 0$ and therefore $ad(t_\alpha)$ is locally nilpotent on L.

By Lemma 3.3-1, it is clear that if $ad(t_\alpha)$ is locally nilpotent on L, then $\mathscr{M}(t_\alpha)$ is locally nilpotent on T. □

Proposition 3.3. *Let T be a split Lie triple system, let $L = L^0 \oplus L^1$ be its standard embedding and let $L(t_\alpha, t_{-\alpha})$ be a test algebra associated to $\alpha \in \Lambda^1$ such that $\mathscr{M}(t_{\pm\alpha})$ are locally nilpotent. Then L is a locally finite $L(t_\alpha, t_{-\alpha})$-module with respect to the adjoint representation.*

Proof. By Proposition 3.2, the adjoint mappings $ad(t_{\pm\alpha})$ are locally nilpotent on L. By using now the same arguments as for $sl(2,\mathbb{K})$ (cf. Moody and Pianzola, 1995, Proposition 2.4.7) we complete the proof. □

Proposition 3.4. *Let T be a split Lie triple system, let $L = L^0 \oplus L^1$ be its standard embedding and let $\alpha \in \Lambda_i^1$. Then we have:*

(1) $dim\, T_\alpha = 1$.
(2) $T_{\pm(2n-1)\alpha} = L^0_{\pm 2m\alpha} = 0$ *for* $n > 1$ *and* $m \geq 1$.
(3) *If* $\alpha \in \Lambda_0$, *then* $-\alpha \in \Lambda_0$ *and* $dim\, L^0_\alpha = dim\, L^0_{-\alpha}$.
(4) *If* $\alpha \notin \Lambda_0$, *then* $\mathbb{Z}\alpha \cap \Lambda^1 = \pm\alpha$ *and* $\mathbb{Z}\alpha \cap \Lambda^0 = \emptyset$.
(5) *There exists a unique element* $h_\alpha = [t_\alpha, t_{-\alpha}]$ *with* $\alpha(h_\alpha) = 2$.
(6) *For each* $\beta \in \Lambda^1$ *we have* $\beta(h_\alpha) \in \mathbb{Z}$.
(7) $\mathbb{K}\alpha \cap \Lambda^1 \subseteq \left(\frac{1}{2}\mathbb{Z}\right)\alpha$.
(8) $\mathbb{K}\alpha \cap \Lambda_i^1 \subseteq \left\{\pm\frac{1}{2}\alpha, \pm\alpha, \pm 2\alpha\right\}$.

Proof. Since α is integrable, we find $t_{\pm\alpha} \in T_{\pm\alpha}$ such that

$$L(t_\alpha, t_{-\alpha}) \approx sl(2,\mathbb{K})$$

and $\mathscr{M}(t_{\pm\alpha})$ are locally nilpotent. We may w.l.o.g. assume that $\alpha([t_\alpha, t_{-\alpha}]) = 2$, and denote $h_\alpha := [t_\alpha, t_{-\alpha}]$.

(1) and (2) We consider the $L(t_\alpha, t_{-\alpha})$-submodule of L

$$V := \mathbb{K}t_{-\alpha} + H^0 + \sum_{n=1}^{\infty} T_{(2n-1)\alpha} + \sum_{n=1}^{\infty} L^0_{2n\alpha}.$$

By Proposition 3.3, L is a locally finite $L(t_\alpha, t_{-\alpha})$-module and, as a submodule of a locally finite module, V is also locally finite. In particular, V is a sum of finite dimensional simple $L(t_\alpha, t_{-\alpha})$-submodules by Weyl's theorem. Hence the representation theory of $sl(2,\mathbb{K})$ implies that the set of h_α-eigenvalues on V is symmetric with

$$dim\, V^\mu(h_\alpha) = dim\, V^{-\mu}(h_\alpha)$$

for each $\mu \in \mathbb{K}$. Now $V^{-2}(h_\alpha) = \mathbb{K}t_{-\alpha}$ implies that $dim\, V^2(h_\alpha) = dim\, T_\alpha = 1$ and furthermore that

$$dim\, V^{2(2n-1)}(h_\alpha) = dim\, T_{(2n-1)\alpha} = 0$$

for $n > 1$, and $dim\, V^{2(2n)}(h_\alpha) = dim\, L^0_{2n\alpha} = 0$ for $n \geq 1$. Likewise we obtain $dim\, T_{-(2n-1)\alpha} = 0$ for $n > 1$, and $dim\, L^0_{-2n\alpha} = 0$ for $n \geq 1$.

(3) We have

$$V' := T_\alpha + T_{-\alpha} + H^0 + \sum_{n=0}^{\infty} T_{\pm 2n\alpha} + \sum_{n=1}^{\infty} L^0_{\pm(2n-1)\alpha}$$

is an $L(t_\alpha, t_{-\alpha})$-submodule of L. By arguing as in (1), we prove (3).

(4) Since by (1), $dim\, T_\alpha = dim\, T_{-\alpha} = 1$, and by (3) $L^0_\alpha = L^0_{-\alpha} = 0$, we can check that

$$V'' := T_\alpha + T_{-\alpha} + H^0 + \sum_{n=0}^{\infty} T_{-2n\alpha} + \sum_{n=2}^{\infty} L^0_{-(2n-1)\alpha}$$

is an $L(t_\alpha, t_{-\alpha})-$submodule of L. By arguing as in (1), we obtain

$$T_{\pm 2n\alpha} = L^0_{\pm(2n-1)\alpha} = 0$$

for $n \geq 1$. From (2) we complete the proof.

(5) Since both spaces $T_{\pm\alpha}$ are one dimensional and do not commute, the space $[T_\alpha, T_{-\alpha}]$ is one dimensional. Hence the element h_α is uniquely determined by $\alpha(h_\alpha) = 2$.

(6) Since L is a locally finite $L(t_\alpha, t_{-\alpha})-$module and $\beta(h_\alpha)$ is the eigenvalue of $ad(h_\alpha)$ on the root space T_β, this is a consequence of the finite dimensional representation theory of $sl(2, \mathbb{K})$.

(7) Let $\beta = c\alpha \in \Lambda^1$ with $c \in \mathbb{K}$. Then 5. and 6. imply that $c \in \frac{1}{2}\mathbb{Z}$.

(8) If, in addition, β is integrable, then we also have that $1/c \in \frac{1}{2}\mathbb{Z}$ and thus $4/2c \in \mathbb{Z}$. Since $2c$ divides 4 it equals 1, 2 or 4, so that we may have $c \in \{\pm\frac{1}{2}, \pm 1, \pm 2\}$. □

We have seen that for integrable roots the root spaces $T_{\pm\alpha}$ are one dimensional, showing that the test algebras $L(t_\alpha, t_{-\alpha})$ do not depend on the choice of $t_{\pm\alpha}$.

Definition 3.4. We say that a simple root α of T is abelian simple if $L^0_\alpha = 0$. If $L^0_\alpha \neq \{0\}$, (and by Proposition 3.4-3, $L^0_{-\alpha} \neq \{0\}$), α is called strongly simple if there exist nonzero elements $e^0_{\pm\alpha} \in L^0_{\pm\alpha}$ such that, for any $\epsilon \in \{+, -\}$, $[t_{\epsilon\alpha}, e^0_{\epsilon\alpha}] = 0$ for some $0 \neq t_{\epsilon\alpha} \in T_{\epsilon\alpha}$, and $[e^0_\alpha, e^0_{-\alpha}] \neq 0$. Finally, we say that a simple root α is weakly simple if $L^0_\alpha \neq \{0\}$ and it is not a strongly simple root. We call an abelian simple root α abelian integrable if α is also an integrable root. A strongly simple root α is said to be strongly integrable if α is also an integrable root and the adjoint mappings $ad(e^0_{\pm\alpha})$ are locally nilpotent on L^0. We denote by $\Lambda^1_{a,i}$ (resp. $\Lambda^1_{s,i}$) the set of all abelian integrable (resp. strongly integrable) nonzero roots of T relative to H^0.

It is clear that if $\alpha \in \Lambda^1_{a,i}$ (resp. $\alpha \in \Lambda^1_{s,i}$), then $-\alpha \in \Lambda^1_{a,i}$ (resp. $-\alpha \in \Lambda^1_{s,i}$).

Proposition 3.5. *Let $\alpha \in \Lambda^1_{s,i}$. Then there exist nonzero elements $e^0_{\pm\alpha} \in L^0_{\pm\alpha}$ such that $\alpha([e^0_\alpha, e^0_{-\alpha}]) \neq 0$ and the adjoint mappings $ad(e^0_{\pm\alpha})$ are locally nilpotent on L^0.*

Proof. Let us fix nonzero elements $e^0_{\pm\alpha} \in L^0_{\pm\alpha}$ such that, for any $\epsilon \in \{+, -\}$, $[t_{\epsilon\alpha}, e^0_{\epsilon\alpha}] = 0$ for some $0 \neq t_{\epsilon\alpha} \in T_{\epsilon\alpha}$, $[e^0_\alpha, e^0_{-\alpha}] \neq 0$, and the adjoint mappings $ad(e^0_{\pm\alpha})$ are locally nilpotent on L^0. Since

$$[[e^0_{-\alpha}, t_\alpha], [e^0_\alpha, t_{-\alpha}]] \in [T_0, T_0] = 0,$$

the Jacobi identity gives

$$\begin{aligned}[[[t_\alpha, t_{-\alpha}], e^0_\alpha], e^0_{-\alpha}] &= [[t_\alpha, [t_{-\alpha}, e^0_\alpha]], e^0_{-\alpha}] \\ &= [t_\alpha, [[t_{-\alpha}, e^0_\alpha], e^0_{-\alpha}]] \\ &= [t_\alpha, [t_{-\alpha}, [e^0_\alpha, e^0_{-\alpha}]]]\end{aligned}$$

so

$$\alpha([t_\alpha, t_{-\alpha}])[e^0_\alpha, e^0_{-\alpha}] = \alpha([e^0_\alpha, e^0_{-\alpha}])[t_\alpha, t_{-\alpha}]$$

and the proposition follows from here. □

Proposition 3.6. *If* $\alpha \in \Lambda^1_{s,i}$, *then*

(1) $dim\, L^0_{\pm\alpha} = 1$.
(2) $\mathbb{Z}\alpha \cap \Lambda^1 = \{\pm\alpha\}$.
(3) $\mathbb{Z}\alpha \cap \Lambda^0 = \{\pm\alpha\}$.

Proof. (1) From Proposition 3.5, α is an integrable root of the split Lie algebra L^0, (see Proposition 3.1), in the sense of Stumme (1999, Definition I.3). Then, by Stumme (1999, Proposition I.6) we have $dim\, L^0_{\pm\alpha} = 1$.

(2) and (3) As by (1) and Proposition 3.4-1, $dim\, T_{\pm\alpha} = dim\, L^0_{\pm\alpha} = 1$, we have $[T_\alpha, L^0_\alpha] = [T_{-\alpha}, L^0_{-\alpha}] = 0$.

Let us consider

$$V := T_0 + H^0 + T_\alpha + T_{-\alpha} + L^0_\alpha + L^0_{-\alpha} + \sum_{n=1}^{\infty} T_{-2n\alpha} + \sum_{n=2}^{\infty} L^0_{-(2n-1)\alpha}.$$

We can verify that V is an $L(t_\alpha, t_{-\alpha})$-module, and by arguing as in Proposition 3.4-1, we conclude $T_{-2n\alpha} = L^0_{-(2m-1)\alpha} = 0$, for $n \geq 1$ and $m \geq 2$. Proposition 3.4-2 completes the proof. □

Corollary 3.1. *Let* $\alpha \in \Lambda^1_{s,i}$. *Then we have*:

(1) $[T_\alpha, L^0_{-\alpha}] \neq 0$
(2) $[T_\alpha, T_0] \neq 0$
(3) $[L^0_\alpha, T_0] \neq 0$

Proof. (1) and (2) Let us fix $t_{\pm\alpha} \in T_{\pm\alpha}$ such that $\alpha([t_\alpha, t_{-\alpha}]) = 1$, and $0 \neq e^0_\alpha \in L^0_\alpha$ satisfying $[t_\alpha, e^0_\alpha] = 0$.

Let us consider

$$V := \mathbb{K}[[e^0_\alpha, t_{-\alpha}], t_{-\alpha}] \oplus \mathbb{K}[e^0_\alpha, t_{-\alpha}] \oplus \mathbb{K}e^0_\alpha.$$

We assert that V is an $L(t_\alpha, t_{-\alpha})$-submodule of L. Indeed, this can be verified taking into account the following identities:

Since $[H_0, T_0] = 0$,

$$\begin{aligned}[[[e^0_\alpha, t_{-\alpha}], t_{-\alpha}], t_\alpha] &= -[[t_{-\alpha}, t_\alpha], [e^0_\alpha, t_{-\alpha}]] - [[t_\alpha, [e^0_\alpha, t_{-\alpha}]], t_{-\alpha}] \\ &= -[[t_\alpha, [e^0_\alpha, t_{-\alpha}]], t_{-\alpha}] \\ &= [[[t_\alpha, t_{-\alpha}], e^0_\alpha], t_{-\alpha}] - [[[t_\alpha, e^0_\alpha], t_{-\alpha}], t_{-\alpha}] \\ &= [[[t_\alpha, t_{-\alpha}], e^0_\alpha], t_{-\alpha}] = [e^0_\alpha, t_{-\alpha}].\end{aligned}$$

As $[t_\alpha, e^0_\alpha] = 0$,

$$[[e^0_\alpha, t_{-\alpha}], t_\alpha] = e^0_\alpha,$$

and finally, by Proposition 3.6-2,

$$[[[e^0_\alpha, t_{-\alpha}], t_{-\alpha}], t_{-\alpha}] \subset T_{-2\alpha} = 0.$$

By arguing as in Proposition 3.4-1, we conclude that $[[e^0_\alpha, t_{-\alpha}], t_{-\alpha}] \neq 0$, and from here (1) and (2)

(3) From Proposition 3.5, we can fix $e^0_{\pm\alpha} \in L^0_{\pm\alpha}$ such that $[t_{\epsilon\alpha}, e^0_{\epsilon\alpha}] = 0$ for some $0 \neq t_{\epsilon\alpha} \in T_{\epsilon\alpha}$, $\epsilon \in \{+,-\}$, and satisfying $\alpha([e^0_\alpha, e^0_{-\alpha}]) \neq 0$. By Jacobi identity, we obtain $[[e^0_{-\alpha}, t_\alpha], e^0_\alpha] = -\alpha([e^0_\alpha, e^0_{-\alpha}])t_\alpha \neq 0$, and (3) is proved. □

Lemma 3.4. *Let $\alpha \in \Lambda^1_{s,i}$. Then there exist nonzero elements $t_\alpha \in T_\alpha$, $t_0 \in T_0$, $e^0_\alpha \in L^0_\alpha$ such that $[e^0_\alpha, t_0] = t_\alpha$ and $[t_\alpha, t_0] = e^0_\alpha$.*

Proof. By Corollary 3.1 there exist $t'_\alpha \in T_\alpha$, $t'_0, t''_0 \in T_0$ and $f^0_\alpha \in L^0_\alpha$ such that $[t'_\alpha, t'_0] \neq 0$ and $[f^0_\alpha, t''_0] \neq 0$. If $[f^0_\alpha, t'_0] \neq 0$ or $[t'_\alpha, t''_0] \neq 0$ the proof is clear taking into account that $dim\, T_\alpha = dim\, L^0_\alpha = 1$.

If $[f^0_\alpha, t'_0] = 0$ and $[t'_\alpha, t''_0] = 0$, let consider $t'''_0 := t'_0 + t''_0$. As $[t'_\alpha, t'''_0] = \lambda f^0_\alpha$ and $[f^0_\alpha, t'''_0] = \beta t'_\alpha$ with α, β nonzero elements of $\mathbb{K}$, it is easy to check that $t_0 := \sqrt{(1/\lambda\beta)}t'''_0$, $t_\alpha := \sqrt{(\beta/\lambda)}t'_\alpha$ and $e^0_\alpha := f^0_\alpha$ complete the proof. □

Proposition 3.7. *Let T be a split Lie triple system respect to H^0 and let $L = L^0 \oplus L^1$ be its standard embedding. Then $H^0 \oplus T_0$ is a MASA of L.*

Proof. It is easy to check, taking into account the definition of split Lie triple system and Proposition 3.1. □

Proposition 3.8. *Let T be a split Lie triple system respect to H^0, let $L = L^0 \oplus L^1$ be its standard embedding and let $\alpha \in \Lambda^1_{s,i}$. Then there exist two nonzero linear functionals $\alpha_1, \alpha_2 : H^0 \oplus T_0 \to \mathbb{K}$ such that $L^0_\alpha \oplus T_\alpha = L_{\alpha_1} \oplus L_{\alpha_2}$.*

Proof. By Lemma 3.4 we can choose nonzero elements $t_\alpha \in T_\alpha$, $t_0 \in T_0$, $e^0_\alpha \in L^0_\alpha$ such that $[e^0_\alpha, t_0] = t_\alpha$ and $[t_\alpha, t_0] = e^0_\alpha$. Since $dim\, T_\alpha = dim\, L^0_\alpha = 1$, we can determine, for any $t'_0 \in T_0$, the unique $\lambda_{t'_0}, \beta_{t'_0} \in \mathbb{K}$ such that $[t_\alpha, t'_0] = \lambda_{t'_0} e^0_\alpha$ and $[e^0_\alpha, t'_0] = \beta_{t'_0} t_\alpha$.

As $[T_0, T_0] = 0$ we have

$$0 = [[t_0', t_0], t_\alpha] = -[[t_0, t_\alpha], t_0'] - [[t_\alpha, t_0']], t_0] = [e_\alpha^0, t_0'] - \lambda_{t_0'}[e_\alpha^0, t_0] = \beta_{t_0'} t_\alpha - \lambda_{t_0'} t_\alpha,$$

and therefore $\beta_{t_0'} = \lambda_{t_0'}$.

Thus, we can define the mapping

$$\delta_\alpha : T_0 \to \mathbb{K} \quad \text{by} \quad \delta_\alpha(t_0') := \lambda_{t_0'}.$$

Now, if we also define

$$\alpha_1, \alpha_2 : H^0 \oplus T_0 \to \mathbb{K}$$

by

$$\alpha_1(h^0 + t_0') := \alpha(h^0) - \delta_\alpha(t_0') \quad \text{and} \quad \alpha_2(h^0 + t_0') := \alpha(h^0) + \delta_\alpha(t_0'),$$

we assert that $L_{\alpha_1} \oplus L_{\alpha_2} = L_\alpha^0 \oplus T_\alpha$. Indeed,

$$\begin{aligned}[h^0 + t_0', e_\alpha^0 + t_\alpha] &= \alpha(h^0)(e_\alpha^0 + t_\alpha) - \lambda_{t_0'} t_\alpha - \lambda_{t_0'} e_\alpha^0 \\ &= (\alpha(h^0) - \delta_\alpha(t_0'))(e_\alpha^0 + t_\alpha) = \alpha_1(h^0 + t_0')(e_\alpha^0 + t_\alpha),\end{aligned}$$

therefore $\mathbb{K}(e_\alpha^0 + t_\alpha) \subset L_{\alpha_1}$.

In a similar way we can verify that $\mathbb{K}(e_\alpha^0 - t_\alpha) \subset L_{\alpha_2}$. Then we have $L_\alpha^0 \oplus T_\alpha = \mathbb{K}e_\alpha^0 + \mathbb{K}t_\alpha = \mathbb{K}(e_\alpha^0 + t_\alpha) + \mathbb{K}(e_\alpha^0 - t_\alpha) \subset L_{\alpha_1} \oplus L_{\alpha_2}$.

If $e_{\alpha_1} + e_{\alpha_2} \in L_{\alpha_1} \oplus L_{\alpha_2}$, we obtain that

$$[h^0, e_{\alpha_1} + e_{\alpha_2}] = \alpha_1(h^0) e_{\alpha_1} + \alpha_2(h^0) e_{\alpha_2} = \alpha(h^0)(e_{\alpha_1} + e_{\alpha_2}),$$

and therefore $e_{\alpha_1} + e_{\alpha_2} \in L_\alpha^0 \oplus T_\alpha$. The proof is complete. □

Theorem 3.1. *Let T be a split Lie triple system respect to H^0 such that $\Lambda^1 = \Lambda_{a,i}^1 \cup \Lambda_{s,i}^1$. Then its standard embedding L is split respect to $H^0 \oplus T_0$.*

Proof. First, if $T_0 = 0$ then H^0 is a MASA of L. We assert that in this case $\Lambda^0 \cap \Lambda^1 = \emptyset$. Indeed, if $\alpha \in \Lambda^0 \cap \Lambda^1$, α must be strongly integrable and by Corollary 3.1-1, $T_0 \neq 0$, a contradiction. Hence $\Lambda^0 \cup \Lambda^1$ is the root system of L respect to H^0, and we have the splitting decomposition

$$L = H^0 \oplus \left(\bigoplus_{\beta \in \Lambda^0} L_\beta^0 \right) \oplus \left(\bigoplus_{\alpha \in \Lambda^1} T_\alpha \right).$$

Second, if $T_0 \neq 0$ then $H^0 \oplus T_0$ is a MASA of L, and: If either $\alpha \notin \Lambda^1$ and $\alpha \in \Lambda^0$, (then $[T_0, L_\alpha^0] = 0$), or α is an abelian integrable root of T, (then $[T_0, T_\alpha] = 0$); we can define the root $\alpha' : H^0 \oplus T_0 \to \mathbb{K}$ as $\alpha'(h_0, t_0) := \alpha(h_0)$ for any $(h_0, t_0) \in H^0 \oplus T_0$, the associated root space being $L_{\alpha'} = L_\alpha^0$ in the first case and $L_{\alpha'} = T_\alpha$ in the second one.

If α is a strongly integrable root of T then, by Proposition 3.8, there exists two nonzero roots of L respect to $H^0 \oplus T_0$, α'_1 and α'_2, such that $L^0_\alpha \oplus T_\alpha = L_{\alpha'_1} \oplus L_{\alpha'_2}$. As by Proposition 3.1, $L = H^0 \oplus (\bigoplus_{\alpha \in \Lambda_0} L^0_\alpha) \oplus T_0 \oplus (\bigoplus_{\beta \in \Lambda_1} T_\beta)$, we conclude that L admits the splitting decomposition respect to $H^0 \oplus T_0$ given by

$$L = (H^0 \oplus T_0) \oplus \left(\bigoplus_{\alpha'} L_{\alpha'} \right).$$

4. LOCALLY FINITE SPLIT LIE TRIPLE SYSTEMS

Definition 4.1. A Lie triple subsystem of a Lie triple system T is a submodule U such that $[U, U, U] \subseteq U$. We say that a Lie triple system T is locally finite if every finite subset of T is contained in a finite dimensional subtriple of T.

We recall that a similar concept is defined for Lie algebras (c.f. Neeb, 2000 or Stumme, 1999).

Proposition 4.1. *The standard embedding $L = L^0 \oplus L^1$ of a locally finite Lie triple system T is a locally finite Lie algebra.*

Proof. Let V^0 be a finite subset of L^0. By the construction of L^0, the subset V^0 can be described from a suitable finite subset V of T. As T is locally finite, there exists a finite dimensional subtriple of T, U, such that V is contained in U. The Lie subalgebra of L generated by U is finite dimensional and V^0 is contained in it. Therefore, L^0 is locally finite. It is easy to conclude that L is also locally finite. □

Let us note that in a locally finite split Lie triple system all abelian (resp. strongly) simple roots of Λ^1 are abelian (resp. strongly) integrable.

Theorem 4.1. *Let T be a locally finite split simple Lie triple system such that $\Lambda^1 = \Lambda^1_{a,i} \cup \Lambda^1_{s,i}$. Then T is isomorphic to one of the followings:*

(1) *$T = \mathscr{L}$ where $\mathscr{L}$ is a locally finite split simple Lie algebra.*
(2) *$T = Skw(\mathscr{L}, \xi)$ with $\mathscr{L}$ as in the previous case and ξ an involutive automorphism of $\mathscr{L}$.*

Proof. By Propositions 2.1 and 2.2, either $T = L$ or $T = Skw(L', \xi)$, where L, L' are simple Lie algebras and ξ is an involutive automorphism of L'. By Proposition 4.1, L and L' are locally finite. By Proposition 3.1, L is split and finally by Theorem 3.1, L' is split. The proof is complete. □

Remark 4.1. If T is of infinite dimension, following Neeb (2001), the algebra $\mathscr{L}$ of Theorem 4.1, must be one of the following $sl(J, \mathbb{K})$, $d(J, \mathbb{K})$ or $sp(J, \mathbb{K})$ where J is an infinite set whose cardinality equals the dimension of the Lie algebra.

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