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Continuous Optimization

Relationships between vector variational-like inequality and optimization problems

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Abstract

In this paper we will establish the relationships between vector variational-like inequality and optimization problems. We will be able to identify the vector critical points, the weakly efficient points and the solutions of the weak vector variational-like inequality problem, under conditions of pseudo invexity. These conditions are more general those existing in the literature.

Keywords: Mathematical programming; Pseudo invex functions; Vector variational-like inequality problems

1. The scalar case

Variational inequalities arise in models for a wide class of mathematical, physical, regional, economic, engineering, optimization and control, transportation, elasticity and applied sciences, etc. (see, for example, [7,10,14] and the references therein).

In the scalar case, as can be seen in [19], if $F : S \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is the gradient of a convex real-valued differentiable function $\theta : S \subseteq \mathbb{R}^n \to \mathbb{R}$ and S is an open convex, then the variational inequality problem (VIP) is to find a vector $\bar{x} \in S$, such that

$$(y - \bar{x})^t F(\bar{x}) \geqslant 0, \quad \forall y \in S.$$

The problem (VIP) is equivalent to the optimization problem

$$\text{(MP)} \quad \begin{array}{l} \min \theta(x) \\ \text{subject to } x \in S. \end{array}$$

The above connection between the variational inequality problem (VIP) and an optimization problem breaks down if F is not a gradient function.

An extension of variational inequality problems (VIP) is the classical scalar variational-like inequality problem (VLIP):

Let S be a subset in $\mathbb{R}^n$ and two continuous maps $F : S \to \mathbb{R}^n$ and $\eta : S \times S \to \mathbb{R}^n$. The variational-like inequality problem (VLIP), is to find a point $\bar{x} \in S$, such that

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$\eta(y,\bar{x})^{t}F(\bar{x}) \geqslant 0, \quad \forall y \in S.$

Parida et al. [21] studied the existence of solutions to variational-like inequalities in $\mathbb{R}^n$, and showed a relationship between the variational-like inequality problem (VLIP) and convex programming as well as with the complementarity problem.

Ruiz et al. [24], proved that the solutions of the variational-like inequality problem (VLIP) are equivalent to the minima of a mathematical programming problem in invex environments. This paper generalizes the previous results in the vectorial case.

These invex environments are characterized by invex sets, defined in [20], as an extension of convex sets, and the vector pseudo invex functions, defined in [22,23], as an extension of invex functions.

Scalar invex functions were introduced by Hanson [9], as an extension of differentiable convex functions. These functions are more general than the convex and pseudo convex ones. This type of invex function is equivalent to the type of function whose stationary points are global minima. Therefore, if θ has no stationary points, then θ is invex. In the scalar case, the concept of invexity and pseudo invexity coincide (see [23]).

The outline of the paper is as follows. In Section 2 we will define vector variational-like inequality problems which generalize the variational inequality problems presented in Section 1. In Section 3 we will recall the vector optimization problems. Section 4 is concerned with establishing various relationships between vector variational-like and optimization problems, in both weak and strong forms. Our final objective is to identify the solutions of the weak vector variational-like inequality problem (WVVLIP) with the vector critical points (VCP), defined by Osuna et al. [23], and the solutions of the weak vector optimization problem (WVOP).

2. Vector variational-like inequality problems

The concept of vector variational inequality problem was introduced by Giannessi [8] in 1980, in the finite dimensional Euclidean space with further applications.

Vector variational inequality has been shown to be a useful tool in vector optimization. Chen and Yang [6] have proved the equivalence and the existence of the vector complementarity problem, the vector variational inequality problem (VVIP), the extremum problem, the minimal element problem and the unilateral minimization problem in setting of Banach spaces.

Yang [28] has also discussed the duality of the vector variational inequality problem, which is related to the dual problem of vector optimization problems.

The following convention for equalities and inequalities will be used throughout this paper.

If $x, y \in \mathbb{R}^n$ then:

(a) $x \leqslant y \Longleftrightarrow x_i \leqslant y_i, \ i = 1,\dots,n,$ with strict inequality holding for at least one i;
(b) $x \leqq y \Longleftrightarrow x_i \leqslant y_i, \quad i = 1,\dots,n;$
(c) $x = y \Longleftrightarrow x_i = y_i, \quad i = 1,\dots,n;$
(d) $x < y \Longleftrightarrow x_i < y_i, \quad i = 1,\dots,n.$

The vector variational-like inequality problem is a generalized form of the vector variational inequality problem, which was introduced and studied by Siddiqi et al. [26] and Yang [27].

Therefore, given S a subset of $\mathbb{R}^n$, $\eta : S \times S \to \mathbb{R}^n$ a function and $F : S \to \mathbb{R}^{p\times n}$ a matrix-valued function, let us offer the following definition:

A vector variational-like inequality problem (VVLIP), is to find a point $\bar{x} \in S$, such that there exists no $y \in S$, such that
$F(\bar{x})\eta(y,\bar{x}) \leqslant 0.$

A weak vector variational-like inequality problem (WVVLIP), is to find a point $\bar{x} \in S$, such that there exists no $y \in S$, such that

$F(\bar{x})\eta(y,\bar{x}) < 0.$

It is obvious that

$\boxed{\text{(VVLIP)}} \Rightarrow \boxed{\text{(WVVLIP)}}.$

What's more the vector variational inequality problem (VVIP) defined by Yang and Goh [30] is a particular case of the vector variational-like inequality problem (VVLIP) here defined, taking $\eta(y,\bar{x}) = y - \bar{x}$. The same relationship exists between the weak vector variational inequality problem (WVVIP) and the weak vector variational-like inequality problem (WVVLIP).

In recent years, the theorems of existence for vector variational-like problems have been extensively studied, for example, in [1,2,5,11,13,15,16,25]. The aim of the present paper is to show that vector variational-like inequality problems can be a useful tool for studying vector optimization problems.

3. Vector optimization problems

In multi-objective optimization problems, multiple objectives are usually non-commensurable and cannot be combined into a single objective. Moreover, often the objectives conflict with each other. Consequently, the concept of optimality for single-objective optimization problems cannot be applied directly to vector optimization. In this sense we must understand the concept of efficient solutions and that they are the object of study for problems with multiple objectives.

Definition 3.1. Given S an open subset of $\mathbb{R}^n$ and a function $f : \mathbb{R}^n \to \mathbb{R}^p$, a point $\bar{x} \in S$ is said to be efficient (Pareto), if there exists no $y \in S$ such that $f(y) \leqslant f(\bar{x})$. The set of efficient points is denoted by $E(f,S)$.

Given S an open subset of $\mathbb{R}^n$ and a function $f : R^n \to \mathbb{R}^p$, the vector optimization problem (VOP) is to find the points $E(f,S)$ for

$$\text{(VOP)}\quad V - \min f(x) \quad \text{subject to}\, x \in S.$$

It is not always possible to find efficient points, so sometimes it is in our interests to introduce a more general concept of efficiency:

Definition 3.2. Given S an open subset of $\mathbb{R}^n$ and a function $f : R^n \to \mathbb{R}^p$, a point $\bar{x} \in S$ is said to be weakly efficient, if there exists no $y \in S$ such that $f(y) < f(\bar{x})$. The set of weak efficient points is denoted by $\mathrm{WE}(f,S)$.

Given S an open subset of $\mathbb{R}^n$ and a function $f : R^n \to \mathbb{R}^p$, the weak vector optimization problem (WVOP) is to find the points $\mathrm{WE}(f,S)$ for

$$\text{(WVOP)}\quad W - \min f(x) \quad \text{subject to}\, x \in S.$$

Clearly,

$$\boxed{\bar{x} \in E(f,S)} \Rightarrow \boxed{\bar{x} \in \mathrm{WE}(f,S)}.$$

In the strictly convex case it is verified that

$$\boxed{\bar{x} \in E(f,S)} \Longleftrightarrow \boxed{\bar{x} \in \mathrm{WE}(f,S)}.$$

In [30], it is proved that under conditions of convexity of f and if it is verified that $F = \nabla f$ and if $\bar{x}$ solves the vector variational inequality problem (VVIP), then $\bar{x}$ is an efficient solution to the vector optimization problem (VOP). It is also proved that solving the weak vector optimization problem (WVOP) is equivalent to solving the weak vector variational inequality problem (WVVIP). Similar results are found in [18].

The following example shows that an efficient solution to the vector optimization problem (VOP) may not be a solution to the vector variational inequality problem (VVIP).

Example 3.1. Consider the problem

$$\text{(VOP)}\quad V - \min f(x) \quad \text{subject to}\, x \in [-1,0]$$

where $f(x) = (x, x^2)^{\mathrm{t}}$.

It is clear that every $x \in [-1,0]$ is an efficient solution.

Let $x = 0$, then $\exists y = -1$, such that

$$\nabla f(x)(y-x) = \big(f_1'(x)(y-x), f_2'(x)(y-x)\big) = (-1,0)^{\mathrm{t}} \leqslant (0,0)^{\mathrm{t}}.$$

Thus $x = 0$ is not a solution to (VVIP).

We shall extend the results given by Lee and Kum [18] and Yang and Goh [30], of convex functions to pseudo invex functions.

In [3,12,29,31], the weakly efficient points are identified with the solutions of the vector variational inequality problem under conditions of pre-invexity. We shall extend the previous results under the condition of pseudo invexity. This condition is more general than the pre-invexity.

As in the scalar case, the concept of invex function plays an important role. In [23], for the vectorial case, we have:

Definition 3.3. Let $S \subset \mathbb{R}^n$ be an open set, $f : S \subset \mathbb{R}^n \to \mathbb{R}^p$ be differentiable and $\nabla f(x)$ be the Jacobian which is a $p \times n$ matrix. The function f is said to be

(a) invex (IX) if and only if there exists a function $\eta : S \times S \to \mathbb{R}^n$, such that

$$f(y) \geqq f(x) + \nabla f(x)\eta(y,x), \quad \forall x, y \in S;$$

(b) strictly invex (SIX) if and only if there exists a function $\eta : S \times S \to \mathbb{R}^n$, such that

$$f(y) > f(x) + \nabla f(x)\eta(y,x), \quad \forall x, y \in S, \ x \neq y;$$

(c) pseudo invex (PIX) if and only if there exists a function $\eta : S \times S \to \mathbb{R}^n$, such that

$$f(y) - f(x) < 0 \Rightarrow \nabla f(x)\eta(y,x) < 0, \quad \forall x, y \in S.$$

It is clear that

$$\boxed{(\text{SIX})} \Rightarrow \boxed{(\text{IX})} \Rightarrow \boxed{(\text{PIX})}.$$

The following example, given by Arana et al. [4], shows that in the vectorial case, the concepts of invexity and pseudo invexity do not coincide.

Example 3.2. Consider the function $f : S \subset \mathbb{R} \to \mathbb{R}^2$ defined as $f(x) = (f_1(x), f_2(x)) = (x^2, -x^2)$. Observe that $\forall y, x \in \mathbb{R}$ verifies that $y^2 - x^2 < 0 \Longleftrightarrow -y^2 + x^2 > 0$, which is to say,

$$f_1(y) - f_1(x) < 0 \Longleftrightarrow f_2(y) - f_2(x) > 0.$$

Therefore, $\nexists y, x \in \mathbb{R}$ such that $f_i(y) - f_i(x) < 0$, $i = 1, 2$, so it is never verified that $f(y) - f(x) < 0$, and as a consequence f is pseudo invex with respect to any η.

On the other hand, if we take $y = 1$, $x = 0$, we have

$$\nabla f_2(x) = 0 \quad \text{and} \quad f_2(y) - f_2(x) = -1$$

with which

$$f_2(y) - f_2(x) \ngeqq \eta(y,x)\nabla f_2(x);$$

therefore f_2 is not invex and so neither is f.

In the next section, we will relate the solutions to vector variational-like inequality and optimization problems, in both weak and strong forms.

4. Relationships between vector variational-like inequality and optimization problems

In this section, using the vector pseudo invex functions, we shall extend the results given by Yang and Goh [30] for convex functions and the results given by Ansari and Siddiqi [3], Kazmi [12] and Yang [31], for pre-invex functions. We will set out by endeavouring to calculate the efficient points starting with the solutions of the vector variational-like inequality problem (VVLIP).

Using a similar reasoning to that in [17], we will demonstrate the following theorem:

Theorem 4.1. *Let $f : S \subset \mathbb{R}^n \to \mathbb{R}^p$ be a differentiable function on the open set S. If $F = \nabla f$, f is invex* (IX) *with respect to η and $\bar{x}$ solves the vector variational-like inequality problem* (VVLIP) *with respect to the same η, then $\bar{x}$ is an efficient point to the vector optimization problem* (VOP).

Proof. To reduce this to the absurd, suppose that $\bar{x}$ is not an efficient point to (VOP) $\Rightarrow \exists y \in S$ such that

$$f(y) - f(\bar{x}) \leqslant 0.$$

Since f is invex with respect to η, we have ensured that $\exists y \in S$, such that

$$\nabla f(\bar{x})\eta(y, \bar{x}) \leqslant 0;$$

therefore $\bar{x}$ cannot be a solution to the vector variational-like inequality problem (VVLIP). This contradiction leads to the result. □

Hence, under conditions of invexity, the solutions of the vector variational-like inequality problem (VVLIP) are efficient points.

In order to see the converse of the previous theorem, we must impose stronger conditions, as can be observed in the two following theorems.

Theorem 4.2. *Let $f : S \subset \mathbb{R}^n \to \mathbb{R}^p$ be a differentiable function on the open set S. Suppose that $F = \nabla f$ and that $-f$ is strictly invex* (SIX) *with* *respect to η. If $\bar{x}$ solves the weak vector optimization problem* (WVOP) *then $\bar{x}$ also solves the vector variational-like inequality problem* (VVLIP).

Proof. Suppose that $\bar{x}$ solves the (WVOP), but not the (VVLIP). Then, there exists a point $y \in S$ such that $\nabla f(\bar{x})\eta(y,\bar{x}) \leqslant 0$.

By the strict invexity (SIX) of $-f$ respect to η, we have that

$$f(y) - f(\bar{x}) < \nabla f(\bar{x})\eta(y,\bar{x}) \leqslant 0; \tag{1}$$

therefore $\exists y \in S$ such that $f(y) < f(\bar{x})$, which contradicts $\bar{x}$ being a weakly efficient point. □

Corollary 4.1. *Let $f : S \subset \mathbb{R}^n \to \mathbb{R}^p$ be a differentiable function on the open set S. Suppose that $F = \nabla f$ and $-f$ is strictly invex* (SIX) *with respect to η. If $\bar{x}$ solves the vector optimization problem* (VOP) *then $\bar{x}$ solves the vector variational-like inequality problem* (VVLIP).

Proof. Because every efficient point is weakly efficient and because of Theorem 4.2, this corollary is proved. □

Let us now look for the conditions under which we might identify solutions of the weak vector variational-like inequality problem (WVVLIP) with the weakly efficient points. In order to do so, in [20] the invex sets are defined in the following way:

Definition 4.1. Let $x \in S$. Then, the set S is said to be invex at x with respect to η, if, for each $y \in S$, $0 \leqslant t \leqslant 1$, $x + t\eta(y,x) \in S$.

S is said to be an invex set with respect to η, if S is invex at each $x \in S$.

Obviously, the convex sets are a particular case of the invex sets if $\eta(y,x) = y - x$.

Theorem 4.3. *Suppose that S is an open invex set and $F = \nabla f$. If $\bar{x}$ is weakly efficient for the weak vector optimization problem* (WVOP) *then $\bar{x}$ solves the weak vector variational-like inequality problem* (WVVLIP).

If f is a pseudo invex (PIX) *function with respect to η and $\bar{x}$ solves the weak vector variational-like inequality problem* (WVVLIP) *with respect to the same η then $\bar{x}$ is weakly efficient to the weak vector optimization problem* (WVOP).

Proof. ($\Rightarrow$) Let $\bar{x}$ be the solution to (WVOP), since S is an invex set, we have that $\nexists y \in S$, such that $f(\bar{x} + t\eta(y,\bar{x})) - f(\bar{x}) < 0$, $0 < t < 1$.

Dividing the above inequality by t and taking the limit as t tends to 0, we get to $\nexists y \in S$ such that $\nabla f(\bar{x})\eta(y,\bar{x}) < 0$.

($\Leftarrow$) We will prove the converse by reduction to the absurd. If $\bar{x}$ is not a weakly efficient point then

$$\exists y \in S \quad \text{such that}\, f(y) < f(\bar{x}).$$

By pseudo invexity (PIX) of f with respect to η we have ensured that

$$\exists y \in S \quad \text{such that}\, \nabla f(\bar{x})\eta(y,\bar{x}) < 0.$$

This contradicts the fact that $\bar{x}$ is a solution to the (WVVLIP). □

Thus, if it is verified that f is a pseudo invex function and if $F = \nabla f$ then we can identify the solutions of the (WVOP) and (WVVLIP) problems. Then the weak vector optimization problem and the weak vector variational-like inequality problem have the same solution set. Obviously, a solution of the (VVLIP) is also a solution of the (WVVLIP), but not the converse, and a solution of the (VOP) is also a solution of the (WVOP), but not the converse. Nevertheless, under a further assumption on f, the converse is true:

Theorem 4.4. *Let $f : S \subset \mathbb{R}^n \to \mathbb{R}^p$ be a differentiable function on the open invex set S. Assume that $F = \nabla f$ and the f is strictly invex* (SIX) *with respect to η. If $\bar{x}$ solves the weak vector optimization problem* (WVOP) *then $\bar{x}$ also solves the vector optimization problem* (VOP).

Proof. Suppose that $\bar{x}$ solves the (WVOP), but not the (VOP). Then, there exists $y \in S$ such that $f(y) \leqslant f(\bar{x})$. By the strict invexity (SIX) of f with respect to η we have that

$$0 \geqslant f(y) - f(\bar{x}) > \nabla f(\bar{x})\eta(y,\bar{x})$$

which is to say, $\exists y \in S$ such that $\nabla f(\bar{x})\eta(y,\bar{x}) < 0$; therefore, $\bar{x}$ does not solve the (WVVLIP) problem.

The contradiction arises from, on the other hand, the earlier Theorem 4.3, we have that if $\bar{x}$ solves the (WVOP) then $\bar{x}$ solves also the (WVVLIP). □

Hence if

$$\boxed{\bar{x} \in E(f,S)} \Rightarrow \boxed{\bar{x} \in \mathrm{WE}(f,S)}.$$

In the strictly invex case (SIX) it holds that

$$\boxed{\bar{x} \in E(f,S)} \Longleftrightarrow \boxed{\bar{x} \in \mathrm{WE}(f,S)}.$$

The following definition is established in [22,23], a concept analogous to the stationary point or critical point for the scalar function.

Definition 4.2. A feasible point $\bar{x} \in S$ is said to be a vector critical point (VCP) for the problem (VOP) if there exists a vector $\lambda \in \mathbb{R}^p$ with $\lambda \geqslant 0$ such that $\lambda^t \nabla f(\bar{x}) = 0$.

Scalar stationary points are those whose vector gradients are zero. For vector problems, the vector critical points are those such that there exists a non-negative linear combination of the gradient vectors of each component objective function, valued at that point, equal to zero.

The following result is proved in [22,23]:

Theorem 4.5. *All vector critical points are weakly efficient solutions if and only if the vectorial function f is pseudo invex on S.*

In the light of Theorems 4.3 and 4.5 we could relate the vector critical points to the solutions of the weak vector variational-like inequality problem (WVVLIP), with the following result:

Corollary 4.2. *Suppose that S is an open invex set and $F = \nabla f$. If the objective function is pseudo invex* (PIX) *with respect to η then the vector critical points, the weakly efficient points and the solutions of the weak vector variational-like inequality problem* (WVVLIP) *are equivalents.*

We summarize the findings so far with the following diagram:

$$\begin{array}{ccccc} \text{VVLIP} & \underset{-f(\text{SIX}),F=\nabla f}{\overset{f(\text{IX}),F=\nabla f}{\rightleftarrows}} & \text{VOP} & & \\ \downarrow & & \Updownarrow f(\text{SIX}) & & \\ \text{WVVLIP} & \underset{F=\nabla f}{\overset{f(\text{PIX}),F=\nabla f}{\rightleftarrows}} & \text{WVOP} & \overset{f(\text{PIX}),F=\nabla f}{\rightleftarrows} & \text{VCP.} \end{array}$$

5. Conclusions

In [24], it is proved that the solutions of the variational-like inequality problem (VLIP) in the scalar case, are equivalent to the minima of the mathematical programming problem in invex environments. In this paper we have generalized the previous results to the vectorial case. Furthermore, we have extended the vectorial results given by Ansari and Siddiqi [3], Kazmi [12], Yang and Goh [30] and Yang [31] from convex and pre-invex functions to pseudo invex functions. Under the condition of pseudo invexity, we have seen the relationship that exists between vector variational-like problems and vector optimization problems, and managed to identify the weakly efficient points, the solutions of the weak vector variational-like inequality problem (WVVLIP) and the vector critical points.

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