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New solutions of the 2 + 1 dimensional BKP equation through symmetry analysis: source and sink solutions, creation and diffusion of breathers. . . ☆

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Abstract

Making use of the theory of symmetry transformations in PDEs we construct new solutions of a 2 + 1 dimensional integrable model in the BKP hierarchy.

First, we analyze its reductions and we obtain a BKP equation independent on time. Starting with a solution of this equation we find a family of solutions of the 2 + 1 dimensional BKP equation. These solutions depend on three arbitrary functions on *t*.

On the other hand, new solutions can also be constructed by applying some elements of the symmetry group to known solutions of the model.

We observed that the solutions found by using both approaches describe interesting processes. Among these solutions we present source and sink solutions, solutions describing the creation or the diffusion (or both) of a breather, finite time blow-up processes, finite time source solutions, line solitons and coherent structures moving at arbitrary velocities.

1. Introduction

Among the 2 + 1 dimensional integrable systems that have been found to exhibit solutions describing processes of interaction of exponentially localized structures, one can find the 2 + 1 BKP system, i.e.

$$
\begin{aligned}
&q_t = q_{xxx} + q_{yyy} + 6(qu_1)_x + 6(qu_2)_y, \\
&u_{1y} = q_x, \quad u_{2x} = q_y.
\end{aligned}
\tag{1}
$$

This system [3,8] is one of the first members of a hierarchy of integrable systems emerging from a bilinear identity related to a Clifford algebra which is generated by two neutral fermion fields.

Note that (1) can be written as an integro-differential evolution equation, by taking into account that from the two last equations of (1) one has that

$$
u_1(x, y, t) = \int^y q_x(x, \eta, t)\,\mathrm{d}\eta + \tilde{u}_1(x, t),
$$

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$$u_2(x,y,t) = \int^x q_y(\xi,y,t)\,\mathrm{d}\xi + \tilde{u}_2(y,t).$$

In this way $\tilde{u}_1$ and $\tilde{u}_2$ play the role of potentials for the dependent variable of the evolution equation: $q(x,y,t)$. In [9] it is proved that (1) admits exponentially localized solutions travelling at constant velocity and presenting an internal oscillation, these solutions are referred to as breathers.

The analytic expression of this solution (see [9]) is given by

$$\begin{aligned} q(x,y,t) &= \partial_x\partial_y \log\alpha(x,y,t),\\ u_1(x,y,t) &= \partial_x^2 \log\alpha(x,y,t), \quad u_2(x,y,t) = \partial_y^2\log\alpha(x,y,t),\\ \alpha(x,y,t) &= 1 + a_1\exp[2k_R^1(x-v_1t)] + a_2\exp[2k_R^2(y-v_2t)] + b\exp[2k_R^1(x-v_1t)+2k_R^2(y-v_2t)]\\ &\quad + \exp[k_R^1(x-v_1t)+k_R^2(y-v_2t)]\mathrm{Re}[\mathrm{i}c\exp[\mathrm{i}(k_I^1x - k_I^2y - \omega t)], \end{aligned} \tag{2}$$

where

$$k_R^j = \mathrm{Re}(k^j), \quad k_I^j = \mathrm{Im}(k^j), \quad j=1,2, \quad b = a_1a_2 + \frac{k_I^1k_I^2}{4k_R^1k_R^2}|c|^2,$$

$$v_j = -\frac{[(k^j)^3]_R}{k_R^j}, \quad j=1,2, \quad \omega = -\mathrm{Im}[(k^1)^3-(k^2)^3],$$

a_1, a_2 are arbitrary real parameters and k^j, $j=1,2$ and c are arbitrary complex parameters. We plot this solution in Figs. 1–3 for the choice of the parameters $k^1 = 1+2\mathrm{i}$, $k^2 = \frac{1}{2} - \frac{3}{2}\mathrm{i}$, $a_1 = 10$, $a_2 = 20$, $c = 5-4\mathrm{i}$, and for $t = 0,1,2$ respectively.

Moreover, it is also proved (see [9]) that in solutions describing interaction processes of breathers, they manifest dynamical properties similar to the dromion solutions of the Davey–Stewartson equation [5–7], for instance they change their form under interaction.

It is also worth noting that the evolution associated to (1) conserved the mass, defined by

$$M = \int_{\mathbb{R}^2} q(x,y,t)\,\mathrm{d}x\,\mathrm{d}y$$

for localized solutions. It is a consequence of the fact that the right-hand side of (1) is a divergence.

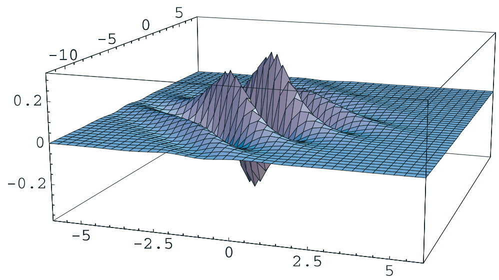

Fig. 1. Breather (2) for $t = 0$ and the above parameters.

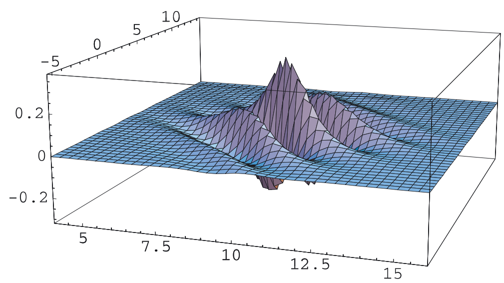

Fig. 2. Breather (2) for $t = 1$ and the above parameters.

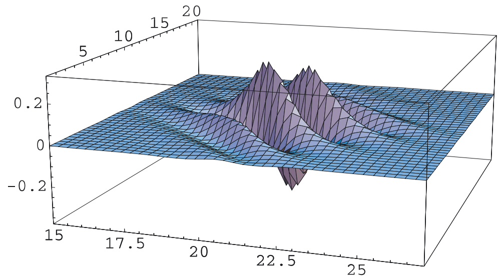

Fig. 3. Breather (2) for $t = 2$ and the above parameters.

On the other hand, most of the $2+1$ dimensional integrable systems that have been studied from the point of view of the the theory of symmetry transformations in PDEs have been found to admit infinite dimensional groups of symmetries (see for example [4,14] for the KP equation or [2] for the Davey–Stewartson equation). Making use of these groups, new solutions with interesting properties have been constructed [10,11].

In this work, we apply these techniques to (1) in order to find new solutions of this system. More specifically, we get the symmetry group admitted by (1) and the corresponding reductions to two independent variables PDE. Making use of both, symmetry group and reductions, together with some known solutions of (1) we can construct new solutions. Among these solutions we find sink and source solutions, solutions describing diffusion processes, finite time blow-up solutions, coherent structures travelling to arbitrary velocities, along arbitrary curves, etc.

2. Reductions and equations of the group

In order to find the Lie algebra associated to the Lie group of symmetry transformations of the $(2+1)$-dimensional BKP equation, we look for vectorial fields of the form:

$$\begin{aligned} V = {} & \xi_1(x,y,t,q,u_1,u_2)\frac{\partial}{\partial x} + \xi_2(x,y,t,q,u_1,u_2)\frac{\partial}{\partial y} + \xi_3(x,y,t,q,u_1,u_2)\frac{\partial}{\partial t} + \phi_1(x,y,t,q,u_1,u_2)\frac{\partial}{\partial q} \\ & + \phi_2(x,y,t,q,u_1,u_2)\frac{\partial}{\partial u_1} + \phi_3(x,y,t,q,u_1,u_2)\frac{\partial}{\partial u_2}, \end{aligned}$$

which leave invariant the third prolongation of the system (1). Using the algorithmic techniques (see for example [1,12,13]) we find that the general element of the Lie algebra has the form

$$V_1(f) + V_2(g) + V_3(h),$$

where

$$\begin{aligned} V_1(f) &= \frac{1}{3}f'(t)x\frac{\partial}{\partial x} + \frac{1}{3}f'(t)y\frac{\partial}{\partial y} + f(t)\frac{\partial}{\partial t} - \frac{2}{3}f'(t)q\frac{\partial}{\partial q} \\ &\quad + \left(-\frac{2}{3}f'(t)u_1 - \frac{1}{18}f''(t)x\right)\frac{\partial}{\partial u_1} + \left(-\frac{2}{3}f'(t)u_2 - \frac{1}{18}f''(t)y\right)\frac{\partial}{\partial u_2} \\ V_2(g) &= g(t)\frac{\partial}{\partial x} - \frac{1}{6}g'(t)\frac{\partial}{\partial u_1}, \\ V_3(h) &= h(t)\frac{\partial}{\partial y} - \frac{1}{6}h'(t)\frac{\partial}{\partial u_2}, \end{aligned} \tag{3}$$

with f, g and h being arbitrary functions on the time variable t.

This general element of the Lie algebra has been obtained by making use of the program YaLie, for MATHEMATICA, by Díaz. This program can be found in the web site: http://library.wolfram.com/infocenter/MathSource/4231/

2.1. Reductions to equations in two independent variables

Solutions of (1) invariant under the action of the symmetry group can be found as solutions of PDEs in two independent variables, the reduced equations. Next, we look for these reduced equations.

Reduction 1. If $f \not\equiv 0$ the similarity independent variables are given by

$$r = f(t)^{-(1/3)}x - g_1(t), \quad s = f(t)^{-(1/3)}x - h_1(t) \tag{4}$$

where

$$g_1'(t) = \frac{g(t)}{f(t)^{4/3}}, \quad h_1'(t) = \frac{h(t)}{f(t)^{4/3}}.$$

The dependent variables of (1) q, u_1, u_2 are given in terms of the similarity dependent variables q_1, v_1 and v_2 by

$$\begin{aligned} q(x,y,t) &= f(t)^{-(2/3)}q_1(r,s), \\ u_1(x,y,t) &= f(t)^{-(2/3)}v_1(r,s) - \frac{1}{18}\frac{f'(t)}{f(t)^{1/3}}x - \frac{1}{6}f(t)g_1'(t), \\ u_2(x,y,t) &= f(t)^{-(2/3)}v_2(r,s) - \frac{1}{18}\frac{f'(t)}{f(t)^{1/3}}y - \frac{1}{6}f(t)h_1'(t). \end{aligned} \tag{5}$$

Now, introducing (4) and (5) into (1) we obtain the reduced system

$$\begin{aligned} &q_{1rrr} + q_{1sss} + 6(q_1v_1)_r + 6(q_1v_2)_s = 0, \\ &v_{1s} = q_{1r}, \quad v_{2r} = q_{1s}. \end{aligned} \tag{6}$$

Reduction 2. If $f \equiv 0$ and $g \not\equiv 0$ the similarity independent variables are

$$r = \frac{h(t)}{g(t)}x - y \quad \text{and} \quad t,$$

while the dependent variables can be written as

$$\begin{aligned} &q(x,y,t) = q_1(r,t), \\ &u_1(x,y,t) = v_1(r,t) - \frac{g'(t)}{6g(t)}x, \quad u_2(x,y,t) = v_2(r,t) - \frac{h'(t)}{6g(t)}y. \end{aligned}$$

In terms of these variables (1) takes the form

$$\begin{aligned} &q_{1t} + (1 - m(t)^3)q_{1rrr} - 6m(t)(v_1q_1)_r + 6(v_2q_1)_r + n(t)q_1 = 0, \\ &v_{1r} + m(t)q_{1r} = 0, \\ &m(t)v_{2r} + q_{1r} - \frac{1}{6}(m'(t) + n(t)m(t)) = 0, \end{aligned} \tag{7}$$

where

$$m(t) = \frac{h(t)}{g(t)}, \quad n(t) = \frac{g'(t)}{g(t)}.$$

Clearly (7) can be further simplified. In order to do that we need to distinguish the cases $m \not\equiv 0$ and $m \equiv 0$.

Case 2.1. If $m \not\equiv 0$, from the two last equations in (7) we obtain

$$\begin{aligned} &v_1(r,t) = -m(t)q_1(r,t) + m_1(t), \\ &v_2(r,t) = -\frac{1}{m(t)}q_1(r,t) + \frac{1}{6}\frac{m(t)n(t) + m'(t)}{m(t)}r + n_1(t), \end{aligned}$$

with m_1 and n_1 arbitrary functions on t. Substituting these expressions in the first equation in (7) we have that

$$q_{1t} + \left(2n(t) + \frac{m'(t)}{m(t)}\right)q_1 + 6(n_1(t) - m(t)m_1(t))q_{1r} + \left(n(t) + \frac{m'(t)}{m(t)}\right)rq_{1r} + 12\left(m(t)^2 - \frac{1}{m(t)}\right)q_1q_{1r} + (1 - m(t)^3)q_{1rrr} = 0. \tag{8}$$

Case 2.2. If $m \equiv 0$, (7) can be trivially solved and we get

$$q_1(r,t) = p(t), \quad v_1(r,t) = w_1(t), \quad v_2(r,t) = -\frac{1}{6}\left(n(t) + \frac{p'(t)}{p(t)}\right)r + w_2(t),$$

where p, w_1, w_2 are arbitrary functions on t.

Reduction 3. If $f \equiv 0$, $g \equiv 0$, $h \not\equiv 0$ the independent similarity variables are x and t while the dependent variables can be written as

$$q(x,y,t) = q_1(x,t), \quad u_1(x,y,t) = v_1(x,t), \quad u_2(x,y,t) = v_2(x,t) - \frac{h'(t)}{6h(t)}y.$$

Introducing this change of variables into (1) one obtains

$$q_{1t} = q_{1xxx} + 6(q_1v_1)_x - \frac{h'(t)}{h(t)}q_1, \quad q_{1x} = 0, \; v_{2x} = 0$$

and consequently

$$q_1(r,t) = p(t), \quad v_1(r,t) = \frac{1}{6}\left(\frac{p'(t)}{p(t)} + \frac{h'(t)}{h(t)}\right)x + w_1(t), \quad v_2(r,t) = w_2(t),$$

where, again, p, w_1 and w_2 are arbitrary functions on t.

2.2. Integration of the group

From the expression of the arbitrary element of the Lie algebra we can obtain the equations of the symmetry transformation group, by solving a system of ordinary differential equations with respect to the parameter of the group. In this way, starting with a known solution of (1) and applying elements of the symmetry group, new solutions can be constructed. Due to the composition operation in the group, it is enough by computing the expression of the new solutions in terms of the starting solutions, for elements with f arbitrary, $g \equiv 0$, $h \equiv 0$ and elements with g, h arbitrary functions and $f \equiv 0$. In order to get the equations of the transformation associated to $V_1(f)$ we have to solve the system

$$\frac{dX}{ds} = \frac{1}{3}Xf'(T), \quad X(0) = x,$$
$$\frac{dY}{ds} = \frac{1}{3}Yf'(T), \quad Y(0) = y,$$
$$\frac{dT}{ds} = f(T), \quad T(0) = t,$$
$$\frac{dQ}{ds} = -\frac{2}{3}Qf'(T), \quad Q(0) = q,$$
$$\frac{dU_1}{ds} = -\frac{2}{3}U_1f'(T) - \frac{1}{18}Xf''(T), \quad U_1(0) = u_1,$$
$$\frac{dU_2}{ds} = -\frac{2}{3}U_2f'(T) - \frac{1}{18}Yf''(T), \quad U_2(0) = u_2.$$

From the solution of this system we obtain that if $q(x,y,t)$, $u_1(x,y,t)$, $u_2(x,y,t)$ is a solution of (1) a new family of solutions is given by

$$
\begin{aligned}
Q(X,Y,T) &= \left(\frac{f(T)}{f(t)}\right)^{-(2/3)} q(x,y,t),\\
U_1(X,Y,T) &= \left(\frac{f(T)}{f(t)}\right)^{-(2/3)} u_1(x,y,t) - \frac{x}{18}\frac{f'(T)-f'(t)}{f(T)^{2/3}f(t)^{1/3}},\\
U_2(X,Y,T) &= \left(\frac{f(T)}{f(t)}\right)^{-(2/3)} u_2(x,y,t) - \frac{y}{18}\frac{f'(T)-f'(t)}{f(T)^{2/3}f(t)^{1/3}}
\end{aligned} \tag{9}
$$

where

$$
x = \left(\frac{f(T)}{f(t)}\right)^{-(1/3)} \cdot X, \quad y = \left(\frac{f(T)}{f(t)}\right)^{-(1/3)} \cdot Y, \quad t = \Phi^{-1}(\Phi(T)-s) \tag{10}
$$

and

$$
\Phi(T) = \int^T \frac{1}{f(\xi)}\,\mathrm{d}\xi.
$$

Proceeding in the same way with the group element associated to the Lie algebra element $V_2(g)+V_3(h)$, we obtain that the new family of solutions is now given by

$$
\begin{aligned}
Q(X,Y,T) &= q(X-sg(T), Y-sh(T), T),\\
U_1(X,Y,T) &= u_1(X-sg(T), Y-sh(T), T) - \frac{s}{6}g'(T),\\
U_2(X,Y,T) &= u_2(X-sg(T), Y-sh(T), T) - \frac{s}{6}h'(T).
\end{aligned} \tag{11}
$$

3. New solutions

In order to construct new solutions we can proceed in two ways. We can start with the reduced equations and look for solutions of these equations, or we can apply the elements of the symmetry group of (1) to known solutions of this system.

3.1. Solutions associated to the reductions

Let us start with the reduced equation (6). It is clear that this equation is satisfied by any solution of (1) independent on t. Then, if we start with a solution of (1) that does not depend on t, using (4) and (5) we obtain a new family of solutions of (1) which depend of three arbitrary functions on t.

It is easy to see that in order to find a time independent solution, we can start with the breather solution (2) and choose the complex parameters k^j, $j=1,2$ such that $(k^j)^3$, $j=1,2$ are imaginary numbers and $(k^1)^3=(k^2)^3$, i.e.

$$
k^1 = a\left(\frac{\sqrt{3}}{2}+\frac{\mathrm{i}}{2}\right), \quad k^2 = a\left(-\frac{\sqrt{3}}{2}+\frac{\mathrm{i}}{2}\right) \tag{12}
$$

with a being an arbitrary real parameter. Thus, we obtain the solution of (6) given by:

$$
\begin{aligned}
q_1(r,s) &= \partial_r\partial_s \log\alpha(r,s), \quad v_1(r,s) = \partial_r^2\log\alpha(r,s), \quad v_2(r,s) = \partial_s^2\log\alpha(r,s)\\
\alpha(r,s) &= 1 + a_1\exp[\sqrt{3}ar] + a_2\exp[-\sqrt{3}as] + b\exp[\sqrt{3}a(r-s)]\\
&\quad + \exp\left[\frac{\sqrt{3}}{2}a(r-s)\right]\mathrm{Re}\left[\mathrm{i}c\exp\left[\frac{\mathrm{i}}{2}a(r-s)\right]\right],
\end{aligned} \tag{13}
$$

with $b = a_1a_2 - \frac{|c|^2}{12}$, $a_1, a_2 \in \mathbb{R}$, $c \in \mathbb{C}$ arbitrary constants. Now, from (13) and using (4) and (5) we find the family of solutions of (1)

$$
q(x,y,t) = f(t)^{-(2/3)} q_1\left(f(t)^{-(1/3)}x - g_1(t), f(t)^{-(1/3)}y - h_1(t)\right) \tag{14}
$$

with q_1 given by (13) and f, g_1, h_1 arbitrary functions on t. In order to analyze this family of solutions we start by considering two particular cases:

The case $g_1 \equiv 0$, $h_1 \equiv 0$. In this case (14) takes the form

$$q(x,y,t) = f(t)^{-(2/3)} q_1\left(f(t)^{-(1/3)}x, f(t)^{-(1/3)}y\right). \tag{15}$$

It is easy to see that these solutions have the properties

- If $\lim_{t\to t_1} f(t) = 0$ then

  $$q(x,y,t) \to C\delta(x,y) \quad \text{as } t \to t_1,$$

  where C is a fixed constant. Indeed, that is a consequence of

  $$\int_{\mathbb{R}^2} q(x,y,t)\varphi(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^2} q_1(r,s)\varphi(f(t)^{1/3}r, f(t)^{1/3}s)\,\mathrm{d}r\,\mathrm{d}s.$$

  Note that the same property holds if we replace t_1 by $\pm\infty$.
- If $\lim_{t\to t_2} f(t) = 1$ then

  $$q(x,y,t) \to q_1(x,y) \quad \text{as } t \to t_2$$

  (and the same property holds if we replace t_2 by $\pm\infty$).
- If $\lim_{t\to t_3} f(t) = \infty$ then the amplitude of our solution tends to zero as $t \to t_3$, in fact, the solution is diffusing into the plane as $t \to t_3$. As in the previous limits, the same property holds if we replace t_3 by $\pm\infty$.

Taking into account the previous properties, it is clear that a great variety of solutions can be exhibited by choosing in appropriate way the arbitrary function $f(t)$. Some examples are:

- If we choose $f(t) = \mathrm{e}^{-t}$, the solution (15) behaves as a nonlocalized solution of amplitude tending to zero as $t \to -\infty$ (it can be interpreted as the radiation), as t increases the solution becomes exponentially localized, in particular for $t = 0$ it coincides with the static soliton (13) and as $t \to \infty$, $q(x,y,t) \to C\delta(x,y)$. Thus, this solution can be interpreted as a *sink solution*. It is clear that if we take $f(t) = e^t$ the solution is a *source solution* in which $q(x,y,t) \to C\delta(x,y)$ as $t \to -\infty$ and describes a *diffusion process* as $t \to \infty$. These facts can be appreciated in Figs. 4–7 where we plot solution (13)–(15) with $f(t) = \mathrm{e}^{-t}$, $a = 1$, $a_1 = 10$, $a_2 = 20$, $c = 5 - 4\mathrm{i}$, and we have chosen $t = -2$, $t = 0$, $t = 3$ and $t = 8$ respectively.
- If we choose $f(t) = 1 - t$ and consider the solution for $t \in [0,1)$ the solution describes a *finite time blow up* process. In fact, the solution corresponds initially to (13) and $q(x,y,t) \to C\delta(x,y)$ as $t \to 1^-$. For $t = 1$, $q(x,y,t)$ stops being a solution of (1). We plot this solution in Figs. 8–11, for the same parameters that in the previous case, and for $t = 0.2$, $t = 0.5$, $t = 0.9$ and $t = 0.999$, respectively.

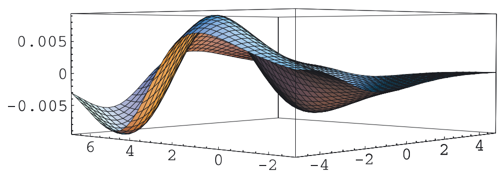

Fig. 4. Solution (5), (13) with $f(t) = \mathrm{e}^{-t}$, $g \equiv h \equiv 0$ for $t = -2$.

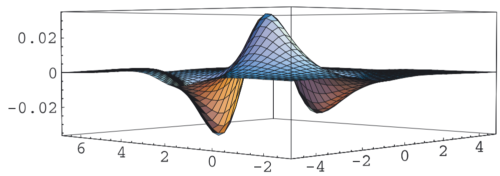

Fig. 5. Solution (5), (13) with $f(t) = \mathrm{e}^{-t}$, $g \equiv h \equiv 0$ for $t = 0$.

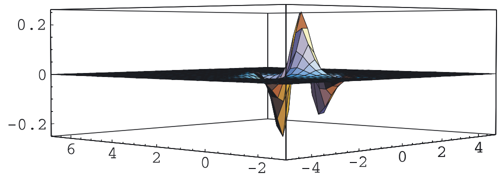

Fig. 6. Solution (5), (13) with $f(t) = e^{-t}$, $g \equiv h \equiv 0$ for $t = 3$.

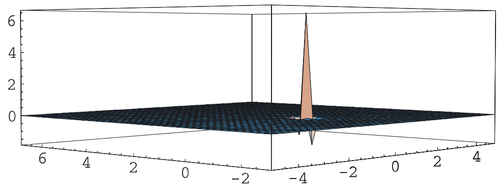

Fig. 7. Solution (5), (13) with $f(t) = e^{-t}$, $g \equiv h \equiv 0$ for $t = 8$.

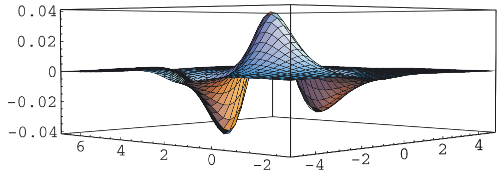

Fig. 8. Solution (5), (13) with $f(t) = 1 - t$, $g \equiv h \equiv 0$ for $t = 0.2$.

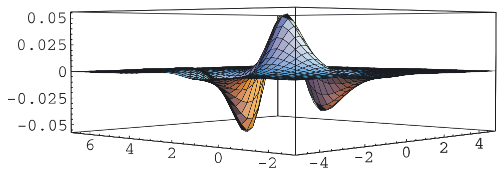

Fig. 9. Solution (5), (13) with $f(t) = 1 - t$, $g \equiv h \equiv 0$ for $t = 0.5$.

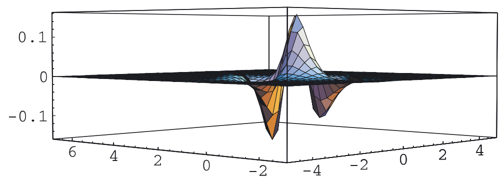

Fig. 10. Solution (5), (13) with $f(t) = 1 - t$, $g \equiv h \equiv 0$ for $t = 0.9$.

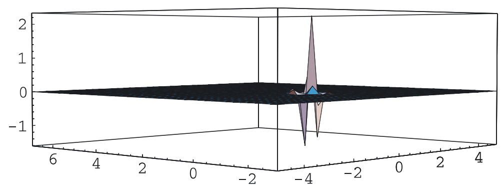

Fig. 11. Solution (5), (13) with $f(t) = 1 - t$, $g \equiv h \equiv 0$ for $t = 0.999$.

- If we choose $f(t) = 1 + t^2$ the solution (15) describes a diffusion process for both limits $t \to \pm\infty$. It can also be seen as the process of first, the *creation* of the coherent structure from the radiation and then, its *diffusion* to radiation again.

The case $f \equiv 1$. In this case (14) takes the form

$$q(x,y,t) = q_1(x - g_1(t), y - g_2(t)), \tag{16}$$

which describes a *coherent structure* corresponding to the statical localized solution (13) moving to an *arbitrary velocity* along an *arbitrary curve* on the plane.

The general case. If f, g_1 and h_1 are arbitrary functions on t, Then, the solution describes an structure that depending on the choice of f describes processes as those previously discussed, and besides, moves along an arbitrary curve to an arbitrary velocity (depending on the choices of g_1 and h_1). We point out that in the case of collapse (blow up at a point), one finds

$$q(x,y,t) \to C\delta(x + g_1(t_1), y + h_1(t_1)) \quad \text{as } t \to t_1,$$

if $\lim_{t \to t_1} f(t) = 0$, i.e. the collapse is in this case at an arbitrary point.

Another possibility for constructing solutions of (1) is looking for solutions of (8). It is worth to noting that for the choice of the arbitrary functions $n \equiv m_1 \equiv n_1 \equiv 0$, $m \equiv c$ (constant), Eq. (8) is the KdV equation

$$q_{1t} + 12\left(c^2 - \frac{1}{c}\right)q_1 q_{1r} + (1 - c^3)q_{1rrr} = 0. \tag{17}$$

Taking into account the similarity variables that lead us to these reduction, we have that if $q_1(r,t)$ is a solution of (17) then

$$q(x,y,t) = q_1(cx - y, t), \quad u_1(x,y,t) = -cq_1(cx - y, t), \quad u_2(x,y,t) = -\frac{1}{c}q_1(cx - y, t)$$

is a solution of (1). Thus, (1) admits *line solitons* and *interaction processes among line solitons* as solutions.

3.2. Solutions associated to the action of the symmetry group elements

We can also construct new solutions of (1) by applying the symmetry groups admitted by this system to its known solutions. In this sense, we note that large families of solutions of (1) are obtained in [9]. These solutions describe processes of interaction of breathers, in which the interacting structures change their forms under the interaction, although they conserve their velocities. The simplest of these structures is the breather (2). On the other hand, equations (9), (10) or (11) allow us to construct new solutions starting with the known ones. In fact, let us present some examples.

Example 1. If we choose $f(t) = e^{-t}$, $g \equiv h \equiv 0$ the new solution is given by

$$Q(X,Y,T) = (1 - se^{-T})^{-2/3} \times q((1 - se^{-T})^{-1/3} \cdot X, (1 - se^{-T})^{-1/3} \cdot Y, T + \ln(1 - se^{-T})), \tag{18}$$

where $q(x,y,t)$ is the solution we have applied the group element and s is the group parameter (we do not provide the expression of the potentials U_1, U_2, as they are involved, easy to find from (9) to (10) and they are not useful for our discussion). Now, from (18) we find:

- If $s > 0$, $T \in (-\infty, \ln s)$, we have that $Q \to 0$ as $T \to -\infty$ for each (X,Y), while $Q(X,Y,T) \to C\delta(X,Y)$ as $T \to (\ln s)^-$. Thus, if we take q as (2) the solution describes the *creation* of the breather and then, its *finite time blow-up* to a point.
- If $s > 0$, $T \in (\ln s, \infty)$, taking into account that $Q(X,Y,T) \sim q(x,y,t)$ as $T \to \infty$, we have the *creation* of a breather from a *finite time source*.
- If $s < 0$ (18) is a solution of (1) for $T \in \mathbb{R}$. We also have that as $T \to -\infty$, $Q \to 0$ for each (X,Y), provided that q is a bounded solution, while as $T \to \infty$

$$Q(X,Y,T) \sim q(x,y,t).$$

Thus, for example if we choose for q the one breather solution, Q describe the *creation of the breather from the radiation*. We can also obtain a solution in which the *radiation evolve in several breathers interacting among them*, if we take for q a solution (see [9]) describing an interacting process among some breathers. We plot solution (18), (2) in

Figs. 12–14 for the choice of parameters $a_1 = 1$, $a_2 = 2$, $c = -\mathrm{i}$, $k_1 = 1 - 2\mathrm{i}$, $k_2 = \frac{1}{2} - \mathrm{i}$, $s = -1$ and for $t = -5$, $t = -1$ and $t = 10$ respectively.

Example 2. Another example can be obtained by taking $f(t) = 1 + t^2$, $g \equiv h \equiv 0$. In this case we have

$$Q(X,Y,T) = \frac{(\sec s)^{4/3}}{(1+(\tan s)T)^{4/3}}\, q\left(\frac{(\sec s)^{2/3}}{(1+(\tan s)T)^{2/3}}X, \frac{(\sec s)^{2/3}}{(1+(\tan s)T)^{2/3}}Y, \frac{T-\tan s}{1+(\tan s)T}\right), \tag{19}$$

- If $T \in (-\infty, -\mathrm{cotan}\, s)$ we have that the amplitude tends to zero as $T \to -\infty$ while

  $Q(X,Y,T) \to C\delta(X,Y) \quad \text{as } T \to (-\mathrm{cotan}\, s)^-.$

  Thus, the solution describes the creation of a localized structure (or a set of localized structures, depending on our choice of q), and later a *finite time blow-up* to a point
- If $T \in (-\mathrm{cotan}\, s, \infty)$ we find

  $$Q(X,Y,T) \to C\delta(X,Y) \quad \text{as } T \to (-\mathrm{cotan}\, s)^+,$$

  where C is a fixed constant and provided that q is a localized solution. We also have that as $T \to \infty$ the amplitude of our solution tends to zero. Thus, if for example we choose for q the one breather solution, (19) describes a processes in which *a breather emerges from a source at a finite time* and afterwards a *diffusion* process takes place. This solution is illustrated in Figs. 15–17 for the same parameters of the breather than in the previous solution, $s = \frac{\pi}{4}$ and $t = -0.9$, 1, 10.

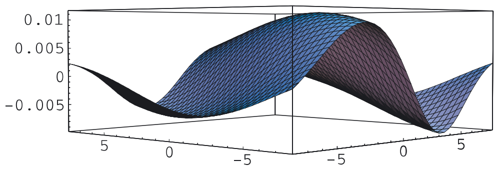

Fig. 12. Solution (18), (2) for $t = -5$.

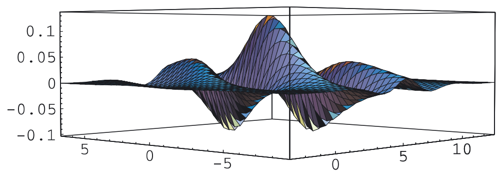

Fig. 13. Solution (18), (2) for $t = -1$.

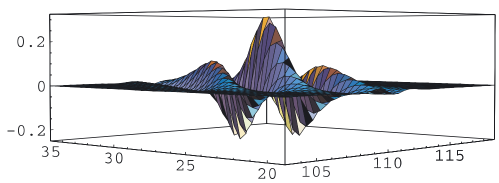

Fig. 14. Solution (18), (2) for $t = 10$.

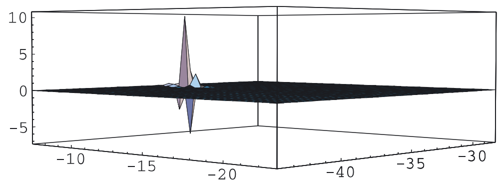

Fig. 15. Solution (19), (2) for $t = -0.9$.

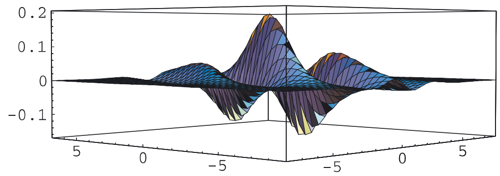

Fig. 16. Solution (19), (2) for $t = 1$.

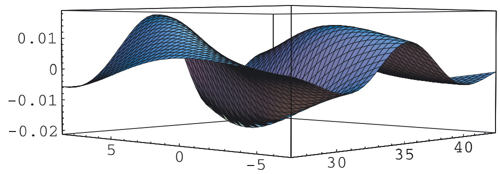

Fig. 17. Solution (19), (2) for $t = 10$.

Example 3. Finally, if we choose $f \equiv 0, g, h$ arbitrary functions, from (11) it is clear that the action of the group element only modifies the two components of the velocity of the possible structures in the solution. The difference with (16) is that we can apply (11) to any solution (interaction processes among breathers, interaction processes among line solitons, other solutions discussed in this work,...) and not only to time independent solutions.

4. Conclusions

In this work we have made use of the theory of symmetry transformations in PDEs in order to construct new solutions of a well known $2+1$ dimensional integrable model, the BKP equation. Using these techniques, we characterize solutions which describe interesting processes. For example, we can find:

- *Sink solutions*. Solutions verifying $q(x,y,t) \to C\delta(x-x_0, y-y_0)$ as $t \to \infty$.
- *Source solutions*. Solutions verifying $q(x,y,t) \to C\delta(x-x_0, y-y_0)$ as $t \to -\infty$.
- Solutions describing blow-up at a point, at finite time, i.e. $q(x,y,t) \to C\delta(x-x_0, y-y_0)$ as $t \to t_0^-$.
- Solutions describing the creation of a breather from a source at finite time, i.e. $q(x,y,t) \to C\delta(x-x_0, y-y_0)$ as $t \to t_0^+$.
- Solutions describing the creation of a breather (or in general a set of interacting breathers) from the radiation, and eventually its diffusion.
- Solutions describing the creation of a set of breathers which interact among them.
- Line solitons and interaction processes among them.
- Coherent structures moving at arbitrary velocities, along arbitrary curves.

Note that these techniques can also be applied to other integrable models which admit infinite dimensional groups of symmetries. For example, we have constructed in this way new solutions of the KP equation [10], of the Davey–Stewartson equation [11].

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