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Source and sink solution, finite time blow-up, diffusion, creation and annihilation processes in the Davey–Stewartson equation

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Abstract

Making use of the theory of symmetry transformations in PDE we construct new solutions of the Davey–Stewartson (DS) equation. First, among its reductions one can find a time independent like-DS equation. Starting with the expressions of the well-known coherent structures of the DS equation, we can obtain solutions of this time independent equation. From these solutions, families of solutions of DS equation depending on three arbitrary functions on t are obtained. Besides, new solutions can also be constructed by applying some elements of the symmetry group to known solutions of the model.

Among the solutions constructed using both approaches, one can find source and sink solutions, solutions describing the creation, the diffusion or annihilation of a dromion (or in general, a set of localized structures), finite time blow-up processes, instantaneous source solutions, and coherent structures moving at arbitrary velocities along arbitrary curves.
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1. Introduction

One of the best known $2+1$ dimensional integrable systems that have been found to exhibit solutions describing processes of interaction of exponentially localized structures is the Davey–Stewartson equation. An integro-differential equation that can be written as a system of differential equations in the form:

$$\begin{aligned} &\mathrm{i}q_t + q_{xx} + q_{yy} + 2q(U_{xx} + U_{yy}) = 0, \\ &|q|^2 = 4U_{xy}. \end{aligned} \tag{1}$$

Among its applications, it is known as a model for water waves [4], ferromagnetism [10] or internal gravity waves [7], to cite a few.

An important property of (1) is that the evolution associated to this equation conserves the mass for localized solutions. Indeed, if we define the mass as

$$M = \frac{1}{2}\int_{\mathbb{R}^2} |q|^2\,\mathrm{d}x\,\mathrm{d}y,$$

it is easy to see that

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$$\frac{dM}{dt} = \frac{i}{2}\int_{\mathbb{R}^2} \vec{\nabla}(q^*\vec{\nabla}q - q\vec{\nabla}q^*)\,dx\,dy.$$

Consequently, if q is a localized solution of (1) it satisfies that $dM/dt = 0$.

The basic solution of this equation is the dromion [2]. It is a localized structure that moves at a constant velocity in the plane. Its analytic expression is given by

$$q(x,y,t) = \frac{\sigma(x,y,t)}{\tau(x,y,t)}, \quad U(x,y,t) = \ln\tau(x,y,t), \tag{2}$$

where

$$\begin{aligned} \tau(x,y,t) &= 1 + d_1 e^{p_{1R}(x+p_{1I}t)} + d_2 e^{p_{2R}(y-p_{2I}t)} + d_3 e^{p_{1R}(x+p_{1I}t)+p_{2R}(y-p_{2I}t)},\\ \sigma(x,y,t) &= a p_2 \exp\left[\frac{1}{2}p_1^*(x+p_{1I}t) + \frac{1}{2}p_2(y-p_{2I}t) + \frac{i}{4}(|p_1|^2 + |p_2|^2)t\right], \end{aligned} \tag{3}$$

with p_1, p_2, a, being arbitrary complex parameters, d_1, d_2 arbitrary real constants and

$$p_{iR} = \mathrm{Re}[p_i], \quad p_{iI} = \mathrm{Im}[p_i], \ i = 1,2 \quad \text{and} \quad d_3 = d_1 d_2 + \frac{|p_2|^2}{4p_{1R}p_{2R}}|a|^2.$$

We plot this solution in Fig. 1, for the choice of parameters $p_1 = \frac{1}{2} - i$, $p_2 = \frac{1}{2} + \frac{3}{2}i$, $d_1 = d_2 = 1$, $a = 1 + 2i$. The main property of the dromion, is that in solutions describing interacting processes, dromions emerge from the interaction changing their form [6,9,12]. For example, solutions describing processes of fusion and fission of dromions have been presented in [8]. Other basic solutions of (1), also coherent structures, can be found as degenerated cases of (2) and (3). In this sense we have the kink, which is a localized on a ray, and corresponds to the choice of the parameters $d_i = 0$, $d_j, d_k \neq 0$, with i, j, k different, and the one dimensional soliton which corresponds to $d_i = d_j = 0$, $d_k \neq 0$. A kink solution can be seen in Fig. 2, for the same parameters than the previous dromion but $d_2 = 0$.

On the other hand, most of the 2 + 1 dimensional integrable systems that have been studied from the point of view of the theory of symmetry transformations in PDE have been found to admit infinite dimensional groups of symmetries

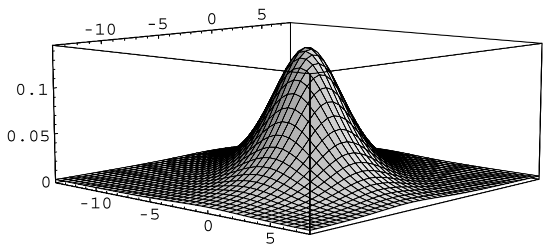

Fig. 1. Dromion.

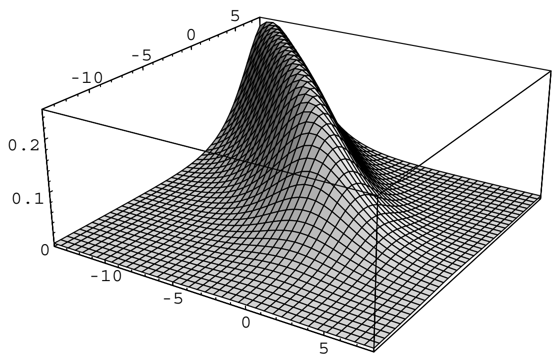

Fig. 2. Kink.

(see for example [5,17] for the KP equation or [3] for the Davey–Stewartson equation). Making use of these groups, new solutions with interesting properties have been constructed [13,14].

In this work, we apply the techniques of this theory (they are thoroughly described, for example, in [1,15,16]) to (1) in order to find new solutions of this system. Making use of them together with some known solutions of (1) we can construct new solutions. Among these, we find source solutions, solutions describing diffusion processes, finite time blow-up, dromions and other coherent structures travelling to arbitrary velocities along arbitrary curves, etc.

The work is organized as follows: in Section 2 we recall the results of [3] for the symmetry group of transformations admitted by (1) and obtain the reductions to PDEs in two independent variables. In Section 3 we use both, the reductions and the elements of the symmetry group in order to construct new solutions of (1) with interesting properties. Finally, we summarize in Section 4 the main results in the work.

2. The symmetry group

In this section we recall some of the results in [3] for (1) and compute the reductions to PDEs in two independent variables. We must point out that in [3] the reductions are obtained for the elements in an optimal system. Then, solutions related to other reductions can be obtained by applying the elements of the group. However, as we see below, the expressions of the solutions found from the reductions are more manageable and easier to interpret than those obtained by applying the elements of the group. For this reason we prefer look for the reduction related to the general element of the symmetry group.

The general element of the Lie algebra of the symmetry group of transformations of (1) [3] is given by

$$V_1(f) + V_2(g) + V_3(h) + V_4(m) + V_5(\mathbf{n})$$

with

$$\begin{aligned}
V_1(f) &= \frac{1}{2}f'(t)x\frac{\partial}{\partial x} + \frac{1}{2}f'(t)y\frac{\partial}{\partial y} + f(t)\frac{\partial}{\partial t} + \frac{1}{192}f'''(t)(x^4+y^4)\frac{\partial}{\partial U} \\
&\quad + \left(-\frac{1}{2}f'(t)v - \frac{1}{8}f''(t)(x^2+y^2)w\right)\frac{\partial}{\partial v} + \left(\frac{1}{8}f''(t)(x^2+y^2)v - \frac{1}{2}f'(t)w\right)\frac{\partial}{\partial w}, \\
V_2(g) &= g(t)\frac{\partial}{\partial x} + \frac{1}{24}x^3 g''(t)\frac{\partial}{\partial U} - \frac{1}{2}xg'(t)w\frac{\partial}{\partial v} + \frac{1}{2}xg'(t)v\frac{\partial}{\partial w}, \\
V_3(h) &= h(t)\frac{\partial}{\partial y} + \frac{1}{24}y^3 h''(t)\frac{\partial}{\partial U} - \frac{1}{2}yh'(t)w\frac{\partial}{\partial v} + \frac{1}{2}yh'(t)v\frac{\partial}{\partial w}, \\
V_4(m) &= \frac{1}{4}m'(t)x^2\frac{\partial}{\partial U} - m(t)w\frac{\partial}{\partial v} + m(t)v\frac{\partial}{\partial w}, \\
V_5(\mathbf{n}) &= \left(\frac{1}{32}(x^2-y^2)n_1(t) + xn_2(t) + yn_3(t) + n_4(t)\right)\frac{\partial}{\partial U},
\end{aligned} \tag{4}$$

where we have put $q = v + \mathrm{i}w$, with v and w real functions of the independent variables. It is clear that the group element generated by $V_5(\mathbf{n})$ transforms a given solution of (1), $q(x,y,t)$, $U(x,y,t)$, into the solution

$$q(x,y,t), \quad U(x,y,t) + \frac{1}{32}(x^2-y^2)n_1(t) + xn_2(t) + yn_3(t) + n_4(t).$$

From the point of view of the applications, one is usually interested in $|q(x,y,t)|^2$. Thus, we can consider that the transformation generated by $V_5(\mathbf{n})$ is trivial, and we take as the general element of the Lie algebra:

$$V_1(f) + V_2(g) + V_3(h) + V_4(m) \tag{5}$$

with $V_1(f)$, $V_2(g)$, $V_3(h)$ and $V_4(m)$ given by (4).

The symmetry group can be integrated from (5) and (4) [3], and in this way, given a solution of (1), a new solution can be obtained, by applying to the first one an element of the group. Due to the involved expression of the general element of the group (see [3]) and due also to the fact that we can use the composition operation in the group, we just provide here the expression of the new solution $Q(X,Y,T)$ $\widetilde{U}(X,Y,T)$ in terms of the known solution $q(x,y,t)$, $U(x,y,t)$, in the cases $g \equiv h \equiv m \equiv 0$ and $f \equiv 0$, respectively. In the first case we have

$$Q(X,Y,T) = \left(\frac{f(T)}{f(t)}\right)^{-(1/2)} \exp\left[\frac{\mathrm{i}}{8}\frac{f'(T)-f'(t)}{f(T)}(X^2+Y^2)\right] q(x,y,t),$$
$$\widetilde{U}(X,Y,T) = \frac{1}{192}\left[\frac{2f(T)f''(T)-f'(T)^2}{2f(T)^2} - \frac{2f(t)f''(t)-f'(t)^2}{2f(T)^2}\right](X^4+Y^4) + U(x,y,t), \tag{6}$$

where

$$x = \left(\frac{f(T)}{f(t)}\right)^{-(1/2)} \cdot X, \quad y = \left(\frac{f(T)}{f(t)}\right)^{-(1/2)} \cdot Y, \quad t = \Phi^{-1}(\Phi(T)-s) \tag{7}$$

and

$$\Phi(T) = \int^{T} \frac{1}{f(\xi)}\,\mathrm{d}\xi.$$

On the other hand, if $f \not\equiv 0$ and g, h, m are arbitrary functions, one finds

$$Q(X,Y,T) = \exp\left[\mathrm{i}\left(\frac{1}{4}\frac{g'(T)}{g(T)}X^2 + \frac{1}{4}\frac{h'(T)}{h(T)}Y^2 + m(T)s\right)\right] q(X-g(T)s, Y-h(T)s, T),$$
$$\widetilde{U}(X,Y,T) = \frac{X^3 m'(t)}{12g(T)} - \frac{(X-sg(T))^3 m'(T)}{12g(T)} + \frac{1}{96}\left(\frac{X^4 g''(T)}{g(T)} + \frac{Y^4 h''(T)}{h(T)}\right)$$
$$- \frac{1}{96}\left[\frac{g''(T)}{g(T)}(X-sg(T))^4 + \frac{h''(T)}{h(T)}(Y-sh(T))^4\right] + U(X-sg(T), Y-sh(T), T). \tag{8}$$

2.1. Reductions to equations in two independent variables

Solutions of (1) invariant under the action of the symmetry group can be found as solutions of PDEs in two independent variables, the reduced equations. Next, we look for these reduced equations.

Reduction 1. If $f \not\equiv 0$ the similarity independent variables are given by

$$r(x,y,t) = f(t)^{-1/2}x - g_1(t),$$
$$s(x,y,t) = f(t)^{-1/2}y - h_1(t), \tag{9}$$

where

$$g_1'(t) = \frac{g(t)}{f(t)^{3/2}}, \quad h_1'(t) = \frac{h(t)}{f(t)^{3/2}}.$$

The dependent variables of (1) q, U are given in terms of the similarity dependent variables q_1 and U_1 by

$$q(x,y,t) = \frac{1}{\sqrt{f(t)}} \exp\left[\mathrm{i}\left[\phi_0(t) + \frac{1}{8}\left(f(t)^{-1/2}x - g_1(t)\right)^2 f'(t) + \frac{1}{8}\left(f(t)^{-1/2}y - h_1(t)\right)^2 f'(t)\right.\right.$$
$$\left.\left. + \left(f(t)^{-1/2}x - g_1(t)\right)\left(\frac{g_1(t)f'(t)}{4} + \frac{f(t)g_1'(t)}{2}\right) + \left(f(t)^{-1/2}y - h_1(t)\right)\left(\frac{h_1(t)f'(t)}{4} + \frac{f(t)h_1'(t)}{2}\right)\right]\right]$$
$$\times q_1(r(x,y,t), s(x,y,t)) \tag{10}$$

with

$$\phi_0(t) = \int\left(\frac{3f'(t)(g_1(t)g_1'(t) + h_1(t)h_1'(t))}{4} + \frac{(g_1(t)^2 + h_1(t)^2)f''(t)}{8} + \frac{f(t)(g_1(t)g_1''(t) + h_1(t)h_1''(t))}{2} + \frac{m(t)}{f(t)}\right)\mathrm{d}t \tag{11}$$

and

$$U(x,y,t) = U_1(r(x,y,t), s(x,y,t)) + x^2\left(\frac{m(t)}{4f(t)} - \frac{f(t)g_1'(t)^2}{16}\right) - \frac{y^2 f(t)h_1'(t)^2}{16} + \frac{x^3\left(f'(t)g_1'(t) + f(t)g_1''(t)\right)}{24\sqrt{f(t)}}$$
$$+ \frac{y^3\left(f'(t)h_1'(t) + f(t)h_1''(t)\right)}{24\sqrt{f(t)}} + \frac{2f''(t)f(t) - f'(t)^2}{384f(t)^2}(x^4+y^4) + \alpha_1(t)x + \alpha_2(t)y + \alpha_3(t), \tag{12}$$

where $\alpha_1(t)$, $\alpha_2(t)$ and $\alpha_3(t)$ are certain functions of t with complicated expressions in terms of f, g_1 and h_1. From (1), it is clear that we do not need these expressions. Now, introducing (9)–(12) into (1) we obtain the reduced system:

$$\begin{aligned} &q_{1rr} + q_{1ss} + 2q_1(U_{1rr} + U_{1ss}) = 0,\\ &|q_1|^2 = 4U_{rs}. \end{aligned} \tag{13}$$

Reduction 2. If $f \equiv 0$, $g, h \not\equiv 0$, the similarity independent variables are given by

$$r(x,y,t) = h(t)x - g(t)y \quad \text{and} \quad t,$$

while dependent variables of (1) q, U are given in terms of the similarity dependent variables q_1 and U_1 by

$$q(x,y,t) = \exp\left[\mathrm{i}\left(\frac{1}{4}\left(\frac{g'(t)}{g(t)}x^2 + \frac{h'(t)}{h(t)}y^2\right) + \frac{m(t)}{g(t)}x\right)\right]q_1(r(x,y,t),t),$$

$$U(x,y,t) = \frac{1}{96}\left(\frac{g''(t)}{g(t)}x^4 + \frac{h''(t)}{h(t)}y^4\right) + \frac{1}{12}\frac{m'(t)}{g(t)}x^3 + U_1(r(x,y,t),t).$$

Introducing these expressions into (1) we obtain the reduced system

$$\mathrm{i}q_{1t} + (g(t)^2 + h(t)^2)(q_{1rr} + 2q_1U_{1rr}) + \mathrm{i}\left[r\left(\frac{g'(t)}{g(t)} + \frac{h'(t)}{h(t)}\right) + 2\frac{h(t)m(t)}{g(t)}\right]q_{1r} + \left[\frac{\mathrm{i}}{2}\left(\frac{g'(t)}{g(t)} + \frac{h'(t)}{h(t)}\right) - \frac{m(t)^2}{g(t)^2}\right]q_1 = 0,$$

$$|q_1|^2 = -4g(t)h(t)U_{1rr},$$

that can be easily transformed into a single equation of the form

$$\begin{aligned} &\mathrm{i}q_{1t} + (g(t)^2 + h(t)^2)q_{1rr} - \frac{g(t)^2 + h(t)^2}{2g(t)h(t)}q_1|q_1|^2 + \mathrm{i}\left[r\left(\frac{g'(t)}{g(t)} + \frac{h'(t)}{h(t)}\right) + 2\frac{h(t)m(t)}{g(t)}\right]q_{1r}\\ &\quad + \left[\frac{\mathrm{i}}{2}\left(\frac{g'(t)}{g(t)} + \frac{h'(t)}{h(t)}\right) - \frac{m(t)^2}{g(t)^2}\right]q_1 = 0. \end{aligned} \tag{14}$$

Note that in the particular case that g and h are constant and $m \equiv 0$, Eq. (14) becomes the nonlinear Schrödinger equation in $1+1$ dimensions.

Reduction 3. If $f \equiv h \equiv 0$, $g \not\equiv 0$, the similarity independent variables are y and t, while the dependent variables of (1) q, U are given in terms of the similarity dependent variables q_1 and U_1 by

$$\begin{aligned} &U(x,y,t) = \frac{1}{96}\frac{g''(t)}{g(t)}x^4 + \frac{1}{12}\frac{m'(t)}{g(t)}x^3 + U_1(y,t),\\ &q(x,y,t) = \exp\left[\mathrm{i}\left(\frac{1}{4}\frac{g'(t)}{g(t)}x^2 + \frac{m(t)}{g(t)}x\right)\right]q_1(y,t). \end{aligned} \tag{15}$$

Introducing these expressions into (1) it is found the trivial solution $q \equiv 0$ and U given by (15) with U_1 an arbitrary function of y and t.

Reduction 4. If $f \equiv g \equiv 0$, $h \not\equiv 0$, the similarity independent variables are x and t, while the dependent variables of (1) q, U are given in terms of the similarity dependent variables q_1 and U_1 by

$$\begin{aligned} &U(x,y,t) = \frac{1}{96}\frac{h''(t)}{h(t)}y^4 + \frac{1}{12}\frac{m'(t)}{h(t)}x^2y + U_1(x,t),\\ &q(x,y,t) = \exp\left[\mathrm{i}\left(\frac{1}{4}\frac{h'(t)}{h(t)}y^2 + \frac{m(t)}{h(t)}y\right)\right]q_1(x,t). \end{aligned} \tag{16}$$

Introducing (16) into (1) we find the solution

$$q(x,y,t) = \exp\left[\mathrm{i}\left(\frac{1}{4}\frac{h'(t)}{h(t)}y^2 + \frac{m(t)}{h(t)}y + \phi_1(y,t)\right)\right]\sqrt{2\frac{m'(t)}{h(t)}x}$$

and U given by (16), with ϕ_1, U_1 satisfy the equation

$$\phi_{1t} - \frac{\mathrm{i}}{x}\phi_{1x} + \phi_{1x}^2 - \mathrm{i}\phi_{1xx} + 2U_{1xx} + \frac{1}{x^2} + \frac{m(t)^2}{h(t)^2} - \frac{\mathrm{i}}{2}\frac{m''(t)}{m'(t)} = 0,$$

if $m' \not\equiv 0$. In the case $m' \equiv 0$ the reduction leads us to the trivial solution $q \equiv 0$ and U given by (16) with U_1 an arbitrary function of x and t.

3. New solutions

In order to construct new solutions we can proceed in two ways. We can start with the reduced equations and look for solutions of these equations, or we can apply the elements of the symmetry group of (1) to known solutions of this system.

3.1. Solutions associated to the reductions

Let us start with the reduced equation (13). It is clear that this equation is satisfied for any solution of (1) independent on t. Then, if we start with a solution of (1) that does not depend on t, using (9)–(12) we obtain a new family of solutions of (1) which depends on four arbitrary functions of t.

As we point out in Section 1, a well-known solution of (1) is the dromion. From (2) and (3) it is clear that if we choose the parameters p_1 and p_2 real parameters we obtain

$$q(x,y,t) = \mathrm{e}^{\mathrm{i}\alpha t}\hat{q}(x,y), \quad U(x,y,t) = \widehat{U}(x,y) \quad \text{with } \alpha = \frac{1}{4}(p_1^2 + p_2^2).$$

Then, it is easy to see that $q_1(r,s) = \hat{q}(r,s)$, $U_1(r,s) = \widehat{U}(r,s) - \frac{\alpha}{8}(r^2+s^2)$, is a solution of (13). In particular we have

$$\begin{aligned} q_1(r,s) &= \frac{ap_2\mathrm{e}^{(1/2)(p_1 r + p_2 s)}}{1 + d_1\mathrm{e}^{p_1 r} + d_2\mathrm{e}^{p_2 s} + d_3\mathrm{e}^{p_1 r + p_2 s}}, \\ U_1(r,s) &= \ln[1 + d_1\mathrm{e}^{p_1 r} + d_2\mathrm{e}^{p_2 s} + d_3\mathrm{e}^{p_1 r + p_2 s}] - \frac{1}{32}(p_1^2 + p_2^2)(r^2 + s^2). \end{aligned} \tag{17}$$

On the other hand, other coherent structures, solutions of (1) are exhibited in [11]. These solutions are bound states among the basic coherent structures for the DS equation, the dromion and the kink. By choosing in these solutions, some of the arbitrary complex parameters as real constants, we obtain that q and U are of the form $q(x,y,t) = \mathrm{e}^{\mathrm{i}\alpha t}\hat{q}(x,y)$, $U(x,y,t) = \widehat{U}(x,y)$, consequently, proceeding as in the previous case, we get the solution of (13)

$$q_1(r,s) = \frac{\sigma(r,s)}{\tau(r,s)}, \quad U_1(r,s) = \ln\tau(r,s) - \frac{1}{32}(p_1^2 + p_2^2)(r^2 + s^2), \tag{18}$$

where

$$\begin{aligned} \sigma(r,s) &= a\exp\left[\frac{p_1 r}{2} + \frac{p_2 s}{2}\right]p_2\left(1 + d_4\frac{p_1 - p_3}{p_1 + p_3}\mathrm{e}^{p_3 r}\right) \\ \tau(r,s) &= 1 + d_1\mathrm{e}^{p_1 r} + d_4\mathrm{e}^{p_3 r} + d_2\mathrm{e}^{p_2 s} + d_3\mathrm{e}^{p_1 r + p_2 s} + d_2 d_4\mathrm{e}^{p_3 r + p_2 s} + d_1 d_4\frac{(p_1 - p_3)^2}{(p_1 + p_3)^2}\mathrm{e}^{(p_1 + p_3)r} + d_3 d_4\frac{(p_1 - p_3)^2}{(p_1 + p_3)^2}\mathrm{e}^{(p_1 + p_3)r + p_2 s}, \end{aligned} \tag{19}$$

with $d_3 = d_1 d_2 + a^2\frac{p_2}{4p_1}$ and p_1, p_2, p_3, d_1, d_2, d_4, a real arbitrary constants. Now, from (9) and (10) we have that our new family of solutions satisfies

$$|q(x,y,t)|^2 = f(t)^{-1}|q_1(f(t)^{-(1/2)}x - g_1(t), \quad f(t)^{-(1/2)}y - h_1(t))|^2 \tag{20}$$

with q_1 given by (17) or (18) and (19) and f, g_1, h_1 arbitrary functions on t. In order to analyze this family of solutions we start by considering two particular cases

The case $g_1 \equiv 0$, $h_1 \equiv 0$

In this case (20) takes the form

$$|q(x,y,t)|^2 = f(t)^{-1}|q_1(f(t)^{-(1/2)}x, f(t)^{-(1/2)}y)|^2. \tag{21}$$

It is easy to see that these solutions have the properties:

- If $\lim_{t\to t_1} f(t) = 0$ then

  $$|q(x,y,t)|^2 \to C\delta(x,y) \quad \text{as} \quad t \to t_1,$$

  where C is a fixed constant. Indeed, that is a consequence of

  $$\int_{\mathbb{R}^2} |q(x,y,t)|^2 \varphi(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^2} |q_1(r,s)|^2 \varphi\left(f(t)^{1/2}r, f(t)^{1/2}s\right)\mathrm{d}r\,\mathrm{d}s.$$

  Note that the same property holds if we replace t_1 by $\pm\infty$.
- If $\lim_{t\to t_2} f(t) = 1$ then

  $$|q(x,y,t)|^2 \to |q_1(x,y)|^2 \quad \text{as} \quad t \to t_2$$

  (and the same property holds if we replace t_2 by $\pm\infty$).
- If $\lim_{t\to t_3} f(t) = \infty$ the amplitude of our solution tends to zero as $t \to t_3$, in fact, the solution is diffusing into the plane as $t \to t_3$. As in the previous limits, the same property holds if we replace t_3 by $\pm\infty$.

Taking into account the previous properties, it is clear that a great variety of solutions can be exhibited by choosing the arbitrary function $f(t)$ in an appropriate way. Some examples are:

- If we choose $f(t) = \mathrm{e}^{-t}$, the solution (21) behaves as a nonlocalized solution of amplitude tending to zero as $t \to -\infty$ (it can be interpreted as the radiation), as t increases the solution becomes exponentially localized, in particular for $t = 0$ it coincides with the static solution $q_1(x,y)$ (given by (17) or (18) and (19)), and as $t \to \infty$, $|q(x,y,t)|^2 \to C\delta(x,y)$. Thus, this solution can be interpreted as a *sink solution*. We plot this solution, corresponding to q_1, U_1, (17), in Figs. 3–5 for the choice of the parameters $p_1 = p_2 = \frac{1}{2}$, $d_1 = d_2 = a = 1$ and the values of t, $t = -3$, $t = 0$ and $t = 5$, respectively.

  It is clear that if we take $f(t) = \mathrm{e}^{t}$ the solution is a *source solution* in which $|q(x,y,t)|^2 \to C\delta(x,y)$ as $t \to -\infty$ and describes a *diffusion process* as $t \to \infty$.
- If we choose $f(t) = 1 - t$ and consider $t \in [0,1)$ the solution describes a *finite time blow up* process. In fact, the solution corresponds initially to (17) (or (18) and (19)) and $|q(x,y,t)|^2 \to C\delta(x,y)$ as $t \to 1^-$. For $t = 1$, $q(x,y,t)$ stops being a solution of (1). We plot this solution with q_1 and U_1 given by (18) and (19), and the parameters $p_1 = p_2 = \frac{1}{2}$, $p_3 = 1$, $d_1 = d_2 = 1$, $d_3 = 3$, $a = 1$, in Figs. 6–9. We take $t = 0$, $t = 0.5$, $t = 0.95$ and $t = 0.99$ respectively.
- If we choose $f(t) = 1 + t^2$ the solution (21) describes a diffusion process for both limits $t \to \pm\infty$. It can be seen as the process of first, the *creation* of the localized structure from the radiation and then, its *diffusion* to radiation again.

The case $f \equiv 1$

In this case (20) takes the form

$$|q(x,y,t)|^2 = |q_1(x - g_1(t), y - h_1(t))|^2, \tag{22}$$

which describes a *coherent structure* corresponding to (17) (or (18) and (19)), moving to an *arbitrary velocity* along an *arbitrary curve* on the plane.

The general case

If f, g_1 and h_1 are arbitrary functions on t, the solution describes an structure with an evolution that, depending on the choice of f consists in processes as those previously discussed, and besides, moves along an arbitrary curve to an arbitrary velocity (depending on the choices of g_1 and h_1). We point out that in the case of collapse (blow up at a point), one finds

$$|q(x,y,t)|^2 \to C\delta(x + g_1(t_1), y + h_1(t_1)) \quad \text{as} \quad t \to t_1,$$

if $\lim_{t\to t_1} f(t) = 0$, i.e. the collapse is in this case at an arbitrary point.

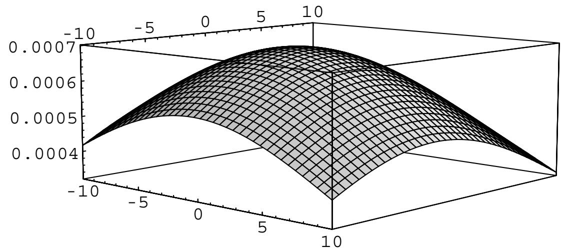

Fig. 3. Solution (21), (17), with $f(t) = e^{-t}$ for $t = -3$.

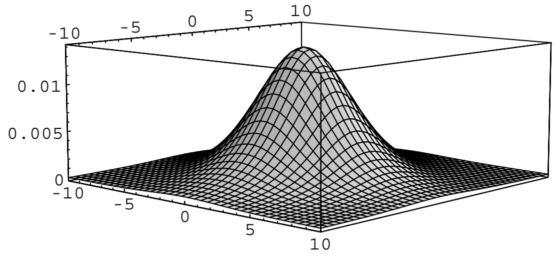

Fig. 4. Solution (21), (17), with $f(t) = e^{-t}$ for $t = 0$.

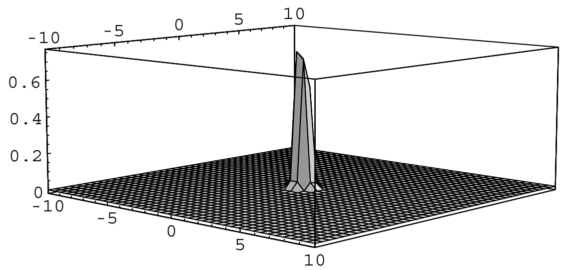

Fig. 5. Solution (21), (17), with $f(t) = e^{-t}$ for $t = 5$.

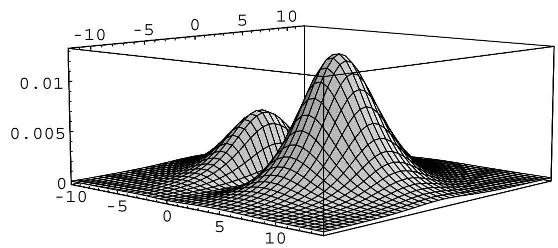

Fig. 6. Solution (21), (18) and (19), with $f(t) = 1 - t$ for $t = 0$.

3.2. Solutions associated to the action of the symmetry group elements

We can also construct new solutions of the Davey–Stewartson equation, by applying the symmetry groups admitted by (1) to its known solutions. Note, in this sense, that a lot of solutions of (1) have been obtained (see for example [2,6,8,9,11,12]).

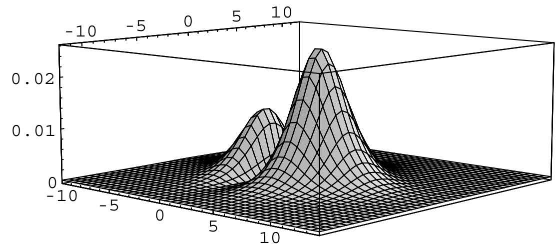

Fig. 7. Solution (21), (18) and (19), with $f(t) = 1 - t$ for $t = 0.5$.

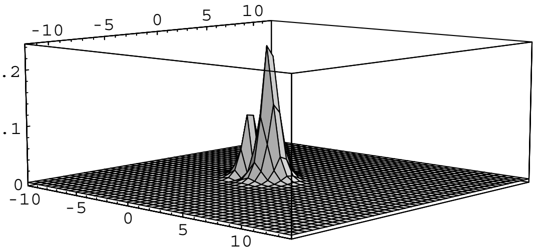

Fig. 8. Solution (21), (18) and (19), with $f(t) = 1 - t$ for $t = 0.95$.

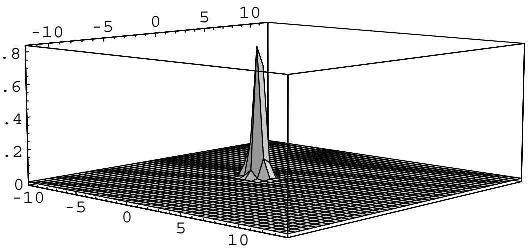

Fig. 9. Solution (21), (18) and (19), with $f(t) = 1 - t$ for $t = 0.99$.

From (8), it is clear that if we choose an element of the symmetry group with $f \equiv 0$ and we start with a solution $q(x, y, t)$, $U(x, y, t)$, we have for the new solution

$$|Q(X, Y, T)|^2 = |q(X - sg(T), Y - sh(T), T)|^2,$$

i.e., the action of the transformation is just a translation at an arbitrary velocity and on an arbitrary curve.

Let us consider some examples with $f \not\equiv 0$, $g \equiv h \equiv m \equiv 0$.

Example 1. $f(t) = \mathrm{e}^{-t}$.

In this case we have

$$Q(X, Y, T) = (1 - s\mathrm{e}^{-T})^{-(1/2)} \exp\left[\frac{\mathrm{i}}{8}\frac{s\mathrm{e}^{-T}}{(1 - s\mathrm{e}^{-T})}(X^2 + Y^2)\right] q\Big((1 - s\mathrm{e}^{-T})^{-(1/2)}X, (1 - s\mathrm{e}^{-T})^{-(1/2)}Y, T + \ln(1 - s\mathrm{e}^{-T})\Big),$$

$$\widetilde{U}(X, Y, T) = \frac{s^2\mathrm{e}^{-2T} - 2s\mathrm{e}^{-T}}{384(1 - s\mathrm{e}^{-T})^2}(X^4 + Y^4) + U((1 - s\mathrm{e}^{-T})^{-(1/2)}X, (1 - s\mathrm{e})^{-T})^{-(1/2)}Y, T + \ln(1 - s\mathrm{e}^{-T})). \tag{23}$$

- If $s > 0$, (23) gives us a solution of (1) for $T \in (\ln s, \infty)$ satisfying

$$|Q(X,Y,T)|^2 \to C\delta(X,Y) \quad \text{as } T \to (\ln s)^+,$$

$$|Q(X,Y,T)|^2 \to |q(X,Y,T)|^2 \quad \text{as } T \to \infty.$$

Thus, (23) is a *instantaneous source* solution, verifying that for $T \to \infty$, the solution we have started with, is recovered. In order to illustrated this solution we start with the solution in [8] that describes a process of fusion between two dromions. This solution [8] corresponds to

$$\tau(x,y,t) = 1 + 2e^{x-2t} + 5e^{x-t} + e^{2(y-2t)} + 4e^{(x-2t)/2}e^{(x-t)/2}\cos\left(\frac{x}{2} - \frac{3t}{4}\right) + e^{(x-2t)/2}e^{((x-t)/2} + 3e^{x-2t}e^{2(y-2t)} + 5e^{x-t}e^{2(y-2t)} + 4e^{(x-2t)/2}e^{(x-t)/2}e^{2(y-2t)}\cos\left(\frac{x}{2} - \frac{3t}{4}\right) + 2e^{(x-2t)/2}e^{(x-t)/2}e^{2(y-2t)}. \tag{24}$$

From the preceding discussion and the previous formula it is clear that the solution (23) with $s > 0$ and $|q|$, U determined by (24) describes a process in which two dromions travelling with velocities $(2,2)$, $(1,2)$ emerge from an instantaneous source, and later, interact experimenting a fusion process. We plot this solution in Figs. 10–15 for $s = e^{-5}$ and $t = -4.999, -4.9, -4.5, -3, -1$ and 5 respectively.

- If $s < 0$, the amplitude of (23) tends to zero as $T \to -\infty$, while $|Q(X,Y,T)| \to |q(X,Y,T)|$ as $T \to \infty$. If for example we start with the dromion solution (2) and (3), then (23) describes the *creation* of a dromion.

Example 2. $f(t) = t^2$.

Now, we have

$$|Q(X,Y,T)|^2 = \frac{1}{(1+sT)^2}\left|q\left(\frac{X}{1+sT}, \frac{Y}{1+sT}, \frac{T}{1+sT}\right)\right|^2. \tag{25}$$

- For $T \in (-\infty, -\frac{1}{s})$, the *creation* of a certain set of structures (depending on the choice of q) and afterwards a *finite time blow-up*.
- For $T \in (-\frac{1}{s}, \infty)$ the solution is an *instantaneous source* solution, with a *diffusion* process as $T \to \infty$.

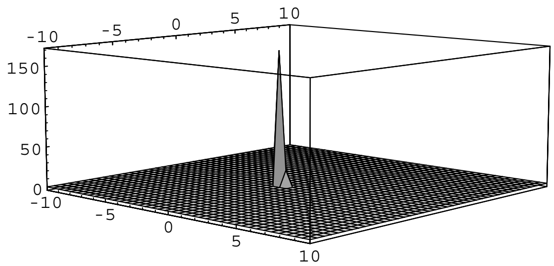

Fig. 10. Solution (23) and (24) with $s = e^{-5}$ for $t = -4.999$.

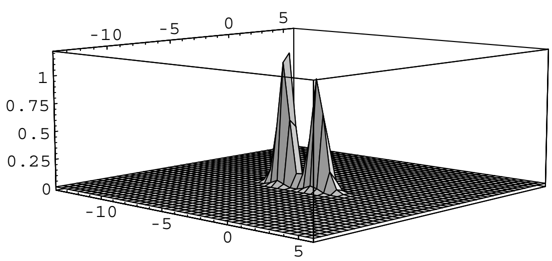

Fig. 11. Solution (23) and (24) with $s = e^{-5}$ for $t = -4.9$.

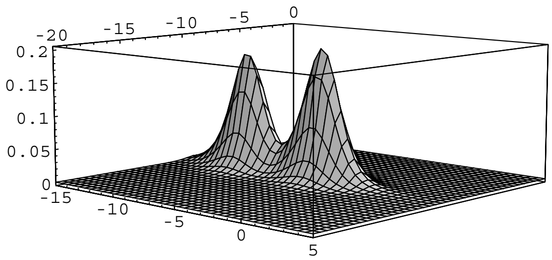

Fig. 12. Solution (23) and (24) with $s = e^{-5}$ for $t = -4.5$.

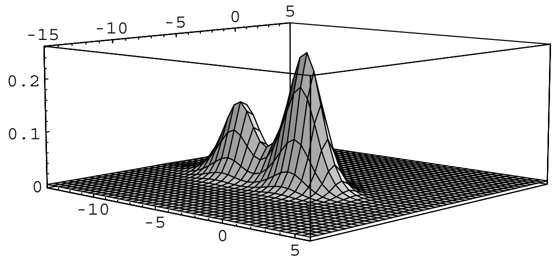

Fig. 13. Solution (23) and (24) with $s = e^{-5}$ for $t = -3$.

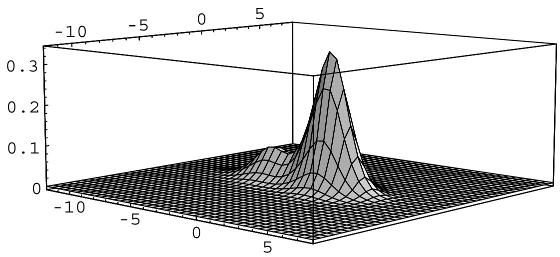

Fig. 14. Solution (23) and (24) with $s = e^{-5}$ for $t = -1$.

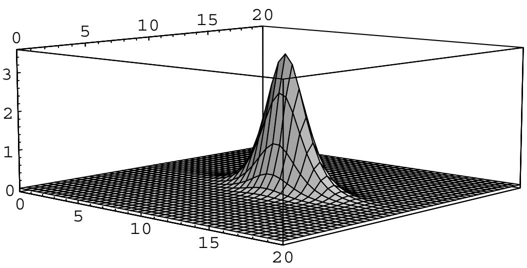

Fig. 15. Solution (23) and (24) with $s = e^{-5}$ for $t = 5$.

Example 3. $f(t) = \frac{1}{t}$.

For this choice of f we find

$$|Q(X,Y,T)|^2 = \frac{1}{(1-2sT^{-2})^{1/2}}\left|q\left(\frac{X}{(1-2sT^{-2})^{1/4}}, \frac{Y}{(1-2sT^{-2})^{1/4}}, T(1-2sT^{-2})^{1/2}\right)\right|^2. \tag{26}$$

In this case we see that $|Q(X, Y, T)| \to |q(X, Y, T)|$ in both asymptotic limits $T \to \pm\infty$. Moreover, depending on the sign of s and the range of T, (26) describes the following processes:

- If $s < 0$, $T \in (-\infty, 0)$ an *annihilation* process (the amplitude of the solution tends to zero as T tends to the finite value $T_0 = 0$).
- If $s < 0$, $T \in (0, \infty)$ the *creation* of the structures given by $q(X, Y, T)$ in the limit $T \to \infty$.
- If $s > 0$, $T \in (-\infty, -\sqrt{2s})$, a *blow-up* at finite time.
- If $s > 0$, $T \in (\sqrt{2s}, \infty)$, a *instantaneous source* solution.

4. Conclusions

In this work we have made use of the theory of symmetry transformations in PDEs in order to construct new solutions of the Davey–Stewartson equation. Using these techniques, we obtain solutions which describe interesting processes. For example, we can find:

- *Sink solutions*. Solutions verifying $|q(x, y, t)|^2 \to C\delta(x - x_0, y - y_0)$ as $t \to \infty$.
- *Source solutions*. Solutions verifying $|q(x, y, t)|^2 \to C\delta(x - x_0, y - y_0)$ as $t \to -\infty$.
- Solutions describing blow-up at a point, at finite time, i.e. $|q(x, y, t)|^2 \to C\delta(x - x_0, y - y_0)$ as $t \to t_0^-$.
- Solutions describing the creation of some localized structures (for example a dromion, or a set of interacting dromions) from an instantaneous source, i.e. $|q(x, y, t)|^2 \to C\delta(x - x_0, y - y_0)$ as $t \to t_0^+$.
- Solutions describing the creation of some localized structures from the radiation, and eventually its diffusion, i.e. $|q(x, y, t)|^2 \to 0$ as $t \to \pm\infty$.
- Solutions describing the creation or annihilation of some localized structures, i.e. $|q(x, y, t)|^2 \to 0$ as $t \to t_0^{\mp}$.
- Coherent structures moving at arbitrary velocities, along arbitrary curves.

Note that these techniques are also be applied to other integrable models which admit infinite dimensional groups of symmetries. For example, solutions with properties similar to the solutions in this work have been constructed for the $2+1$ dimensional BKP equation [14]. Also using these approaches we have constructed new solutions of the KP equation [13].

Acknowledgements

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