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AN APPROACH TO LOCATION MODELS INVOLVING SETS AS EXISTING FACILITIES

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In this paper, we deal with single facility location problems in a general normed space in which the existing facilities are represented by convex sets of points. The criterion to be satisfied by the service facility is the minimization of an increasing, convex function of the distances from the service facility to the closest point of each demand set. We obtain a geometrical characterization of the set of optimal solutions for this problem. Two remarkable cases—the classical Weber problem and the minimax problem with demand sets—are studied as particular instances of our problem. Finally, for the planar polyhedral case, we give an algorithm to find the solution set of the considered problems.

1. Introduction. The classical single facility location problem deals with the location of a point in a real normed space X in order to minimize some function depending on the distances to a finite number of given points (existing facilities or demand points).

The following question arises: Why do we have to consider points as existing facilities? A natural extension is to represent existing facilities as sets of points. This means that we can no longer use the natural distance induced by the norm in X. Therefore, a new decision has to be made before dealing with the problem itself: Which kind of distance measure should be used? Two different alternatives can be considered. The first takes the average behavior into account, so that any point in the set is visited according to a given probability distribution. This approach leads us to the minimization of expected distances as discussed, for instance, in Drezner and Wesolowsky (1980) or Carrizosa et al. (1995). The second alternative measures the distances to the closest points in the sets. Here, the goal is not to serve all points of the set but just to reach the set. Therefore, rather than expected distances, we have to consider the concept of infimal distance to sets. This approach is quite general and includes as particular examples previous approaches in the literature, since infimal distances reduce to regular distances when points are considered instead of sets (see Boffey and Mesa 1996 for a good review on the location of extensive facilities on networks and Brimberg and Wesolowsky 2000, 2002, and Muriel and Carrizosa 1995 for different approaches to locating facilities relative to closest distances).

By allowing sets as clients and using the infimal distance to these sets, different real world situations can be modeled better than in the classical approaches. This concept appears quite naturally in two-level distribution models: Logistics companies usually distribute their products from a central warehouse to medium-sized warehouses in each of the cities of their distribution area (using large trailers). Then, these warehouses deliver the products to final retailers or end customers in the respective city using their own vehicle fleets (small size trucks or vans which can circulate through the city). In this model, the plant is the facility to be located, and the closest points to the plant in each of the cities are the optimal locations of the first-level problem for the local warehouses.

The simultaneous location of a hub together with airports for a given set of cities is another example. The hub would be the facility to be located, and the airports for each city

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should be positioned at the closest point to the hub. A similar argument applies in the case of the location of a recycling plant with respect to local garbage collection plants. Obviously, the cities locate their garbage plants as far away as possible from the city center (to avoid pollution and risks), while staying in their territory (county), and as close as possible to the recycle plant (to minimize transportation costs).

All the above applications are an example of a multilevel logistics system in which a locational decision takes place on a higher level, and the transition points to the lower level can still be chosen accordingly. In traffic planning, this concept applies to the location of a service facility for several cities which should not be accessed by individual transportation means. Therefore, designated park and ride areas for the cities are established at the closest points to this service facility.

Finally, the location model with infimal distances is also directly applicable to the location of a dam and distribution substations of any liquid. Again the locational decision has to be made on the higher level (where the dam should be built) and the distribution substations will be built at the boundary of cities as close as possible to the dam.

The common elements in all of these models are:

(1) A facility must be located.

(2) Existing facilities occupy some nonnegligible area.

(3) The closest points from the existing facilities to the new facility are important (to minimize transportation cost or exposure to risk).

It is worth noting that there are also economic reasons to consider points in the boundaries of the existing facilities: First of all, real estate is cheaper which results in a lower building cost, and secondly, it might be difficult to get licenses to deliver inside the area of the existing facilities without having a representative there.

The aim of this paper is to present a geometrical characterization of the set of optimal solutions of this single-facility location problem with infimal distances. To this end, we will use mainly convex analysis tools. We also address the important cases of the Weber and minimax problem, which are studied in detail. For the very particular case of $\mathbb{R}^2$ with polyhedral norms, a constructive approach is developed. This type of analysis is not new in location analysis. Similar types of optimization problems have deserved the study of researchers, although when facilities are identified with points in their respective spaces. The reader is referred to Durier and Michelot (1985), Durier (1992, 1995), Carrizosa and Puerto (1995), and Puerto and Fernández (2000) for further details.

The rest of the paper is organized as follows. First, we introduce some basic tools and definitions which will be used throughout the paper. In §2, the theory for dealing with set facility location problems is developed. Section 3 studies the existence of optimal solutions and develops optimality conditions based on geometric properties of the problem. In §4, the relationship to some classical location problems is discussed. Section 5 is devoted to the particularities in the planar polyhedral case for which we also give efficient solution algorithms. The paper ends with some conclusions and extensions.

2. Basic tools and definitions. As mentioned in the introduction, everything takes place in a general vector space X equipped with several norms. Let us denote by X^* the topological dual of X equipped with the norm γ and by γ° its dual norm. The unit ball in X with the norm γ (respectively X^*) is denoted by B (respectively B°). The pairing between X and X^* will be indicated by $\langle \cdot, \cdot \rangle$. Nevertheless, for ease of understanding, the reader may replace the space X by $\mathbb{R}^n$. In this case, the topological dual X^* can be identified with X and the pairing is the usual scalar product.

First, we restate some definitions which are needed throughout the paper. Let $B_i \subset X$ be a symmetric, closed, bounded convex set containing the origin in its interior, for $i \in \mathcal{M} := \{1, 2, \dots, M\}$. The norm with respect to B_i is defined as

$$\gamma_i\colon X \to \mathbb{R}, \qquad \gamma_i(x) := \inf\{r > 0\colon x \in rB_i\}. \tag{1}$$

The polar set B_i° of B_i is given by

$$B_i^\circ := \{p \in X^*: \langle p, x\rangle \le 1, \ \forall x \in B_i\}, \tag{2}$$

and the normal cone to B_i at $x \in X$ is given by

$$N_{B_i}(x) := \{p \in X^*: \langle p, y - x\rangle \le 0, \ \forall y \in B_i\}. \tag{3}$$

The case in which each γ_i with $i \in \mathcal{M}$ is a polyhedral norm in a finite dimensional space, which means B_i is a convex polytope with extreme points $\text{ext}(B_i) := \{e_1^i, \dots, e_{G_i}^i\}$, is studied in §5. In this case, we define fundamental directions $\delta_1^i, \dots, \delta_{G_i}^i$ as the directions defined by 0 and $e_1^i, \dots, e_{G_i}^i$.

Let $f: X \to \mathbb{R} \cup \{+\infty\}$ be a convex function. A vector $p \in X$ is said to be a subgradient of f at a point $x \in X$ if

$$f(y) \ge f(x) + \langle p, y - x\rangle$$

for each $y \in X$. The set of all subgradients of f at x is called the subdifferential of f at x and is denoted by $\partial f(x)$.

Given a closed set $A_i \subset X$, we denote by $I_{A_i}(\cdot)$ its indicator function, that is,

$$I_{A_i}(x) = \begin{cases} 0 & \text{if } x \in A_i, \\ +\infty & \text{otherwise}, \end{cases}$$

and we denote by $\sigma_{A_i}(\cdot)$ the support function of the set A_i; i.e.,

$$\sigma_{A_i}(p) = \sup_{x \in A_i} \langle p, x\rangle \qquad \text{for any } p \in X^*.$$

Now, using Hiriart-Urruty and Lemarechal (1993), we know that

$$\partial \gamma_i(x) = \begin{cases} B_i^\circ & \text{if } x = 0, \\ \{p_i \in B_i^\circ: \langle p_i, x\rangle = \gamma_i(x)\} & \text{if } x \ne 0, \end{cases} \tag{4}$$

$$\partial I_{A_i}(x) = N_{A_i}(x) \quad \forall x \in A_i, \tag{5}$$

$$\partial \sigma_{A_i}(u) = \left\{a_i \in A_i: \langle u, a_i\rangle = \sup_{z \in A_i} \langle u, z\rangle\right\}. \tag{6}$$

Let f_1 and f_2 be two functions from X to $\mathbb{R} \cup \{+\infty\}$. Their infimal convolution is a function from X to $\mathbb{R} \cup \{+\infty\}$ defined by

$$\begin{aligned}(f_1 * f_2)(x) &:= \inf\{f_1(x_1) + f_2(x_2): x_1 + x_2 = x\} \\ &= \inf_{y \in X}\{f_1(y) + f_2(x - y)\}.\end{aligned}$$

Another important concept that we need to recall is that of the conjugate functions. Let f be a function from X to $\mathbb{R} \cup \{+\infty\}$ not identically equal to $+\infty$ and minorized by some affine function. The conjugate f^* of f is the function defined by

$$f^*(p) = \sup\{\langle p, x\rangle - f(x): x \in \text{dom} f\} \qquad \text{for any } p \in X^*,$$

where $\text{dom} f$ stands for the effective domain of the function f. It is a well-known result from convex analysis that

$$I_{A_i}^*(p) = \sigma_{A_i}(p) \qquad \text{for any } p \in X^*. \tag{7}$$

Finally, we will denote by $\mathrm{ri}(A)$ the relative interior of the set $A \subset X$, by $\mathrm{bd}(A)$ the boundary of A, by $\mathrm{cl}(A)$ the closure of A, by $\mathrm{co}(A)$ the convex hull of A, and by $\mathrm{cone}(A)$ the convex cone generated by the elements of the set A.

In the next section, we will discuss in more detail some properties of distances from a point to a set.

2.1. Distance to a convex body. Let us consider a convex set $A_i \subset X$ and an arbitrary norm γ_i. The distance from a point $x \in X$ to the set A_i with the norm γ_i is defined as

$$d_i(x, A_i) = \inf\{\gamma_i(x - a_i): a_i \in A_i\},$$

and the set of points $\mathrm{proj}_{A_i}(x) := \{a_i \in A_i: d_i(x, A_i) = \gamma_i(x - a_i)\}$ is called the projection of x onto A_i with the norm γ_i. Note that this set is not necessarily a singleton, and can even be empty if A_i is not closed or not compact. Therefore, ensuring the nonemptyness of the projection set, we will require the sets A_i to be compact. The reader may notice that this is a sufficient condition and that all the results may also be valid under different conditions.

First of all, we have that

$$d_i(x, A_i) = \inf_{a_i \in A_i} \gamma_i(x - a_i) = \inf_{y \in X}\{I_{A_i}(y) + \gamma_i(x - y)\} = (I_{A_i} * \gamma_i)(x).$$

It follows that $d_i(\cdot, A_i)$ is a convex function since it is an infimal convolution of two convex functions. Besides, by Corollary VI.4.5.5 in Hiriart-Urruty and Lemarechal (1993), we obtain the following representation of the subdifferential of $d_i(\cdot, A_i)$:

$$\partial d_i(x, A_i) = \partial I_{A_i}(a_i) \cap \partial \gamma_i(x - a_i) \qquad \text{for any } a_i \in \mathrm{proj}_{A_i}(x).$$

Observe that when $x \in A_i$, $\mathrm{proj}_{A_i}(x) = \{x\}$, and since $\partial\gamma_i(0) = B_i^\circ$, we have $\partial d_i(x, A_i) = N_{A_i}(x) \cap B_i^\circ$ if $x \in A_i$, while in general using (4) and (5) we obtain that

$$(8) \quad \partial d_i(x, A_i) = N_{A_i}(a_i) \cap \{p_i \in B_i^\circ: \langle p_i, x - a_i\rangle = \gamma_i(x - a_i)\} \qquad \text{for any } a_i \in \mathrm{proj}_{A_i}(x).$$

REMARK 2.1. It is also possible to obtain the subdifferential set $\partial d_i(\cdot, A_i)$ in a different way using the concept of level sets. The level set $L_i^{\leq}(r)$ of the function $d_i(\cdot, A_i)$ with value $r > 0$ is

$$L_i^{\leq}(r) = \{x \in X: d_i(x, A_i) \leq r\}.$$

Note that we can write $L_i^{\leq}(r) = A_i + rB_i$. Then, for any $x = a_i + rz$ with $a_i \in A_i$ and $z \in B_i$, by Proposition III.5.3.1 in Hiriart-Urruty and Lemarechal (1993), it follows that

$$N_{L_i^{\leq}(r)}(x) = N_{A_i}(a_i) \cap N_{B_i}(z).$$

Since by Theorem VI.1.3.5 in Hiriart-Urruty and Lemarechal (1993), the relation $N_{L_i^{\leq}(r)}(x) = \mathrm{cone}(\partial d_i(x, A_i))$ holds and ∂d_i must be a subset of B_i°, we obtain that

$$\partial d_i(x, A_i) = N_{A_i}(a_i) \cap \{p \in B_i^\circ: \langle p, z\rangle = \gamma_i(z)\}.$$

Now using that $x = a_i + rz$, we get

$$\partial d_i(x, A_i) = N_{A_i}(a_i) \cap \{p \in B_i^\circ: \langle p, x - a_i\rangle = \gamma_i(x - a_i)\}.$$

In the following, we give a description of the subdifferential set $\partial d_i^*(p_i)$ based on the representation of the distance to the set A_i as infimal convolution as described above. Since we have seen that $d_i(x, A_i) = (I_{A_i} * \gamma_i)(x)$ then by Theorem 1, §3.4 in Ioffe and Tihomirov (1979), $d_i^* = I_{A_i}^* + \gamma_i^*$. Now by (7), $I_{A_i}^*$ is the support function of A_i, i.e., $I_{A_i}^* = \sigma_{A_i}$, and the conjugate of the norm γ_i is the indicator function of its unit dual ball, i.e., $\gamma_i^* = I_{B_i^\circ}$.

Hence, since the qualification assumption of Moreau holds (recall that it requires one of the functions to be continuous at one point of the effective domain of the other function; see e.g., Ioffe and Tihomirov 1979), we have

$$(9) \qquad \partial d_i^*(p_i) = \partial(I_{A_i}^* + \gamma_i^*)(p_i) = \partial I_{A_i}^*(p_i) + \partial\gamma_i^*(p_i) = \partial\sigma_{A_i}(p_i) + N_{B_i^\circ}(p_i) =: C_i(p_i).$$

An interesting property of this family of sets $(C_i(p_i))$ is that the function $d_i(\cdot, A_i)$ is linear within them. This result is proved in the next lemma.

LEMMA 2.1. *For each $p_i \in B_i^\circ$, $d_i(\cdot, A_i)$ is an affine function within $\partial d_i^*(p_i)$.*

PROOF. By Fenchel's identity, we have that

$$x \in \partial d_i^*(p_i) \quad \text{iff } p_i \in \partial d_i(x, A_i).$$

Thus, applying (8), for any $x \in \partial d_i^*(p_i)$, we get

$$d_i(x, A_i) = \langle p_i, x - a_i\rangle = \langle p_i, x\rangle - \langle p_i, a_i\rangle \qquad \text{for any } a_i \in \operatorname{proj}_{A_i}(x).$$

Moreover, since $p_i \in \partial d_i(x, A_i)$ we have that $p_i \in N_{A_i}(a_i) = \partial\sigma_{A_i}^*(a_i)$; and this is equivalent to $a_i \in \partial\sigma_{A_i}(p_i)$. Thus, $\langle p_i, a_i\rangle = \sigma_{A_i}(p_i)$ for any $a_i \in \partial\sigma_{A_i}(p_i)$; that is, $\langle p_i, a_i\rangle$ is constant for any $a_i \in \partial\sigma_{A_i}(p_i)$. Besides, since $p_i \in N_{A_i}(a_i)$ for any $a_i \in \operatorname{proj}_{A_i}(x)$, we have that $\langle p_i, a\rangle \le \langle p_i, a_i\rangle$ for all $a \in A_i$; that is, $\operatorname{proj}_{A_i}(x) \subseteq \partial\sigma_{A_i}(p_i)$. Therefore, for any $x \in \partial d_i^*(p_i)$, we get

$$d_i(x, A_i) = \langle p_i, x\rangle - \sigma_{A_i}(p_i) \qquad \text{for any } a_i \in \operatorname{proj}_{A_i}(x),$$

and the result follows. □

It is also possible to give an alternative characterization of $\partial d_i^*(p_i)$ in finite-dimensional spaces. This expression will be used in §5 to develop an algorithm for the facility location problem with infimal distances in $\mathbb{R}^2$. Let us denote by $\mathcal{Y}_i$ the set of all the faces of any dimension of the set A_i with $i \in \mathcal{M}$. That is to say, $\mathcal{Y}_i$ contains faces of any positive dimension and extreme points. Recall that Y_i is an exposed face of A_i if $Y_i = H_i \cap A_i$ for some supporting hyperplane H_i of A_i.

For any $p_i \in B_i^\circ \cap N_{A_i}(y_i)$ and $y_i \in Y_i$ an exposed face of A_i, we introduce

$$(10) \qquad C(Y_i, p_i) := \{x\colon \operatorname{proj}_{A_i}(x) \subseteq Y_i; \text{ and there exists } a_i \in \operatorname{proj}_{A_i}(x);\ \langle p_i, x - a_i\rangle = d_i(x, A_i)\}.$$

REMARK 2.2. In the definition of the set $C(Y_i, p_i)$, we use the existence of a particular point $a_i \in \operatorname{proj}_{A_i}(x)$. Nevertheless, the definition does not depend on this a_i because by the convexity of A_i, if $\hat{y}_i \in \operatorname{ri}(Y_i)$ and $p_i \in N_{A_i}(\hat{y}_i)$, then $p_i \in N_{A_i}(y_i)$ for any $y_i \in \operatorname{ri}(Y_i)$ (notice that $N_{A_i}(y_i)$ is constant in $\operatorname{ri}(Y_i)$). Therefore, we have $\langle p_i, a - a_i\rangle \le 0$ $\forall a \in A_i$. In particular, for all $a \in \operatorname{proj}_{A_i}(x)$, we obtain that $\langle p_i, x - a\rangle \ge \langle p_i, x - a_i\rangle$ meaning that $d_i(x, A_i) = \langle p_i, x - a\rangle$ for all $a \in \operatorname{proj}_{A_i}(x)$.

The following theorem shows that the set $C(Y_i, p_i)$ coincides with $\partial d_i^*(p_i)$ in finite-dimensional spaces.

THEOREM 2.1. *Let X be finite dimensional and $A_i \subset X$ be a compact convex set, let $\mathcal{Y}_i$ denote the set of all its faces, and let $\gamma_i(\cdot)$ be a norm with unit ball B_i.*

(i) *For any $p_i \in B_i^\circ$, there exists $Y_i \in \mathcal{Y}_i$ such that $p_i \in N_{A_i}(y_i)$ for any $y_i \in Y_i$ and $N_{B_i^\circ}(p_i) + \partial\sigma_{A_i}(p_i) = C(Y_i, p_i)$.*

(ii) *Conversely, for any $Y_i \in \mathcal{Y}_i$ such that $p_i \in B_i^\circ \cap N_{A_i}(y_i)$ for any $y_i \in Y_i$, then $C(Y_i, p_i) = N_{B_i^\circ}(p_i) + \partial\sigma_{A_i}(p_i) = \partial d_i^*(p_i)$.*

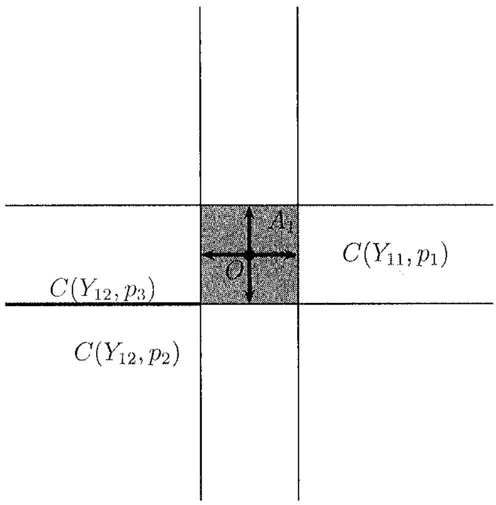

FIGURE 1. Illustration of Example 2.1.

PROOF. Let $x \in N_{B_i^\circ}(p_i) + \partial\sigma_{A_i}(p_i)$. Then there exists $q \in N_{B_i^\circ}(p_i)$ and $a(x) \in \partial\sigma_{A_i}(p_i)$ such that $x = a(x) + q$. Since $q \in N_{B_i^\circ}(p_i)$, $\langle v, q\rangle \le \langle p_i, q\rangle$ $\forall v \in B_i^\circ$. Therefore, $\gamma_i(q) = \gamma_i(x - a(x)) = \langle p_i, x - a(x)\rangle$. Since $\partial\sigma_{A_i}(p_i) = \{y_i \in A_i\colon \langle p_i, y_i\rangle = \sup_{z\in A_i}\langle p_i, z\rangle\}$, it follows that $\langle p_i, a(x)\rangle = \sup_{a_i\in A_i}\langle p_i, a_i\rangle$. Thus,

$$\gamma_i(x - a_i) = \sup_{v\in B_i^\circ}\langle v, x - a_i\rangle \ge \langle p_i, x - a_i\rangle \ge \langle p_i, x - a(x)\rangle = \gamma_i(x - a(x)) \quad \forall a_i \in A_i.$$

As a result, $d_i(x, A_i) = \gamma_i(x - a(x))$. Now, it suffices to consider $Y_i = \{a_i \in A_i\colon \langle p_i, a\rangle = \sigma_{A_i}(p_i)\}$ and we have $N_{B_i^\circ}(p_i) + \partial\sigma_{A_i}(p_i) \subseteq C(Y_i, p_i)$.

Conversely, $x \in C(Y_i, p_i)$ if and only if there exists $a(x) \in Y_i$ such that $d_i(x, A_i) = \gamma_i(x - a(x)) = \langle p_i, x - a(x)\rangle$. However, $\gamma_i(x - a(x)) = \sup_{v\in B_i^\circ}\langle v, x - a(x)\rangle$. Therefore, $\langle v - p_i, x - a(x)\rangle \le 0$ $\forall v \in B_i^\circ$. That is to say, $q := x - a(x) \in N_{B_i^\circ}(p_i)$. Hence, $x = a(x) + q$ with $a(x) \in Y_i$ and $q \in N_{B_i^\circ}(p_i)$. In addition, $p_i \in N_{A_i}(y_i)$ for any $y_i \in Y_i$ and so $\langle p_i, a(x)\rangle \ge \langle p_i, a_i\rangle$ $\forall a_i \in A_i$; that is, $\langle p_i, a(x)\rangle = \sup_{a_i\in A_i}\langle p_i, a_i\rangle$. That means that $a(x) \in \partial\sigma_{A_i}(p_i)$ and also implies that $Y_i = \{a_i \in A_i\colon \langle p_i, a_i\rangle = \sigma_{A_i}(p_i)\}$ which concludes the proof. □

EXAMPLE 2.1. (See Figure 1) Consider $\mathbb{R}^2$ with the l_1-norm and a set $A_1 := \mathrm{co}\{(1, 1), (1, -1), (-1, -1), (-1, 1)\}$. Let $Y_{11} := \mathrm{co}\{(1, 1), (1, -1)\}$ and $p_1 = (1, 0)$, then

$$C(Y_{11}, p_1) = \{x \in \mathbb{R}^2\colon x_1 \ge 1, 1 \le x_2 \le -1\}.$$

For $Y_{12} = \{(-1, -1)\}$, $p_2 = (-1, -1)$, we have

$$C(Y_{12}, p_2) = \{x \in \mathbb{R}^2\colon x_1 \le -1, x_2 \le -1\}.$$

Finally, for $Y_{12} = \{(-1, -1)\}$, $p_3 = (-1, 0)$,

$$C(Y_{12}, p_3) = \{(x, -1)\colon x \le -1\}.$$

3. Set facility location models. Let $\mathscr{A} = \{A_1, \dots, A_M\}$ be a family of sets in X, where each A_i, $i \in \mathscr{M}$ is a compact convex set. Let $\Phi(\cdot)$ be a monotone norm in $\mathbb{R}^M$. Recall that a norm Φ is said to be monotone on $\mathbb{R}^M$ if $\Phi(u) \le \Phi(v)$ for every u, v verifying $|u_i| \le |v_i|$

for each $i = 1, \dots, M$ (see Bauer et al. 1961). We consider the following minimization problem:

$$(P_\Phi(\mathcal{A})) \qquad \inf_{x \in X} F(x) := \Phi(d(x)),$$

where $d(x) = (d_1(x, A_1), \dots, d_M(x, A_M))$. A similar type of objective function has already been considered in standard location analysis; that is, when the facilities are assumed to be points in the framework space (see, e.g., Durier 1992, 1995; Carrizosa and Puerto 1995). Here the novelty comes from considering sets as existing facilities. (The reader may also note that for particular choices of the family $\mathcal{A}$, the former approaches reduce to the one presented in this paper.) We may assume without loss of generality that $\bigcap_{i=1}^M A_i = \varnothing$. ($d(x) \neq 0$ for all $x \in X$.) Indeed, if $\bigcap_{i=1}^M A_i \neq \varnothing$, then the solution set would be $\bigcap_{i=1}^M A_i$ (nonvoid) with objective value of zero.

3.1. Existence of optimal solutions. First of all, the reader can see that the function $F = \Phi \circ d$ is convex on $\mathbb{R}^M$ provided that Φ is monotone (see Proposition IV.2.1.8 in Hiriart-Urruty and Lemarechal 1993). Our first result states a sufficient condition ensuring that the set of optimal solutions of Problem $(P_\Phi(\mathcal{A}))$ is not empty. Thus, it is possible to replace the inf symbol by min.

To this end, we embed the optimization problem $(P_\Phi(\mathcal{A}))$ in a larger space in order to study existence properties of its optimal solution. (See, e.g., Durier 1994, Puerto and Fernández 2000.) Let us consider the normed space $(Y, |||\cdot|||)$, where $Y = X^M$ and for any $y \in Y$, $|||y||| = \Phi(\gamma_1(y_1), \dots, \gamma_M(y_M))$. We define the function

$$\overline{F} : Y \longrightarrow \mathbb{R}$$
$$y \longrightarrow \overline{F}(y) := \Phi(d_1(y_1, A_1), \dots, d_M(y_M, A_M)).$$

Note that $y = (y_1, \dots, y_1)$ implies $\overline{F}(y) = F(y_1)$ for any $y_1 \in X$.

LEMMA 3.1. *Assume that X is reflexive; then the optimal solution set of Problem $(P_\Phi(\mathcal{A}))$ is not empty.*

PROOF. Since the sets A_i are compact for all $i \in \mathcal{M}$, it follows that $m_0 = \overline{F}(0) < +\infty$. Let us define the set $M_0 = \{y \in Y\colon \overline{F}(y) \leq m_0\}$. The set M_0 is convex and closed since F is a continuous, convex function. Moreover, M_0 is a bounded set. Indeed, assume that there exists $\{y^n\}_{n\in\mathbb{N}} \subset M_0$ such that $|||y^n||| \to \infty$. Since $|||y^n||| = \Phi(\gamma_1(y_1^n), \dots, \gamma_M(y_M^n))$ and Φ is a monotone norm in $\mathbb{R}^M$, there exists at least one i and a subsequence $\{n_k\}$ such that $\gamma_i(y_i^{n_k}) \to \infty$.

On the other hand, for any $a \in A_i$, we have that $\gamma_i(y_i^{n_k} - a) \geq \gamma_i(y_i^{n_k}) - \gamma_i(a) \geq \gamma_i(y_i^{n_k}) - \max_{a\in A_i} \gamma_i(a) \to_{n_k\to\infty} \infty$. Hence, since Φ is a monotone norm in $\mathbb{R}^M$, $\overline{F}(y^{n_k}) = \Phi(\gamma_1(y_1^{n_k} - a_1), \dots, \gamma_M(y_M^{n_k} - a_M)) \to \infty$ which contradicts the definition of M_0. Thus, M_0 is bounded and there exists $K > 0$ such that $|||y||| \leq K$ for any $y \in M_0$. Therefore, the problem to be solved is

$$\inf\{\overline{F}(y)\colon y \in M_0 \cap D\},$$

with $D = \{y \in Y\colon y_1 = y_2 = \cdots = y_M\}$. Since D is closed, $M_0 \cap D$ is a nonempty, bounded, closed, convex set. Now, by Proposition 38.12 in Zeidler (1985), the problem has an optimal solution and, hence, the infimum is reached. □

REMARK 3.1. Similar sufficient conditions which ensure that there exist optimal solutions are, for instance, that X has finite dimension or that X is a dual space. It is worth noting that no additional assumptions on Φ nor the shape of the demand sets are needed to ensure existence of optimal solutions. In the remainder of the paper, we will assume that an optimal solution exists, which is, for example, the case if the assumptions of Lemma 3.1 are fulfilled.

3.2. Geometrical characterization of the subdifferential. Recall that for an unconstrained minimization problem with a convex objective function f, x is an optimal solution if and only if $0 \in \partial f(x)$.

Our main objective in this section will be to characterize the set of optimal solutions of $(P_\Phi(\mathcal{A}))$. In order to do that we will study the subdifferential of the objective function F. Our next result characterizes the subdifferential of the objective function of $(P_\Phi(\mathcal{A}))$.

LEMMA 3.2. *Let $x \in X$. $x^* \in \partial F(x)$ iff there exist $a_i \in \operatorname{proj}_{A_i}(x)$, $p_i \in N_{A_i}(a_i) \cap B_i^\circ$ $\forall i \in \mathcal{M}$ and $\lambda = (\lambda_1, \ldots, \lambda_M) \geq 0$ such that*

(1) $x \in \bigcap_{i=1}^M (a_i + N_{B_i^\circ}(p_i))$.
(2) $\Phi^\circ(\lambda) = 1$ *and* $\sum_{i=1}^M \lambda_i d_i(x, A_i) = F(x)$.
(3) $x^* = \sum_{i=1}^M \lambda_i p_i$.

PROOF. First, we consider $t, s \in \mathbb{R}_+^M$ such that $t - s \in \mathbb{R}_+^M$ and $\lambda \in \partial\Phi(t)$. By the monotonicity of Φ and the subgradient inequality, we have that

$$0 \leq \Phi(t) - \Phi(s) \leq \langle \lambda, t - s \rangle.$$

Since this inequality holds for all $s \in \mathbb{R}_+^M$ such that $t - s \in \mathbb{R}_+^M$, this implies that $\lambda \geq 0$ (see Puerto and Fernández 1995, 2000).

Hence, defining the function $\Phi^+(t) := \Phi(t^+)$, where $t^+ = (t_1^+, \ldots, t_M^+)$ with $t_i^+ = \max\{0, t_i\}$ for $i = 1, \ldots, M$, and knowing that Φ is a norm, we have that whenever $t \neq 0$,

$$\partial\Phi^+(t) = \left\{(\lambda_1, \ldots, \lambda_M) \in \mathbb{R}_+^M \colon \Phi^\circ(\lambda) = 1;\ \sum_{i=1}^M \lambda_i t_i^+ = \Phi(t^+)\right\}.$$

On the other hand, since $d(x) > 0$ for any $x \in X$ and $d(x) = d^+(x)$, it follows by Theorem VI.4.3.1 in Hiriart-Urruty and Lemarechal (1993) and Theorem 2 of §8 in Ioffe and Levin (1972) that the subdifferential of the composition of nondecreasing convex functions with convex ones, is given by

$$\partial F(x) = \partial\Phi^+(d(x)) = \left\{\sum_{i=1}^M \lambda_i p_i \colon (\lambda_1, \ldots, \lambda_M) \in \partial\Phi^+(d(x)),\ \ p_i \in \partial d_i(x, A_i)\right\},$$

where $d(x) = (d_1(x, A_1), \ldots, d_M(x, A_M))$. Therefore, we have that λ and p verify

(1) $\lambda = (\lambda_1, \ldots, \lambda_M)$, $\lambda_i \geq 0$, $\Phi^\circ(\lambda) = 1$, $\sum_{i=1}^M \lambda_i d_i(x, A_i) = F(x)$.
(2) $p_i \in N_{A_i}(a_i) \cap \{q \in B_i^\circ \colon \langle q, x - a_i \rangle = \gamma_i(x - a_i)\}$ where $a_i \in \operatorname{proj}_{A_i}(x)$, $\forall i \in \mathcal{M}$.

Finally, using the well-known equivalence between

$$\hat{q} \in \{q \in B^\circ \colon \langle q, x - a \rangle = \gamma(x - a)\} \quad \text{iff } x \in a + N_{B^\circ}(\hat{q}),$$

where B° is the polar set of B and B is the unit ball of γ, the result follows. □

3.3. Generalized elementary convex sets. In order to obtain a characterization of the set of optimal solutions of the Problem $(P_\Phi(\mathcal{A}))$, we need to introduce some additional concepts.

DEFINITION 3.1. Given $p = (p_1, \ldots, p_M) \in (X^*)^M$ with $p_i \in B_i^\circ$ and $I \subseteq \mathcal{M}$, let

$$C_I(p) := \bigcap_{i \in I} \partial d_i^*(p_i),$$

where d_i^* is the conjugate function of $d_i(x, A_i)$, and for any $\lambda = (\lambda_1, \ldots, \lambda_M) \geq 0$, let

$$D_I(\lambda) := \left\{x \colon \sum_{i \in I} \lambda_i d_i(x, A_i) = F(x)\right\}.$$

It is useful to observe that $C_I(p)$ is nonvoid only for some choices of I and p. The sets $C_I(p)$ were previously used in Durier and Michelot (1985) for characterizing optimal solution sets of optimization problems with objective functions given by the sum of convex functions. These sets are called elementary convex sets when the convex functions are norms. For this reason, and since we consider distances to sets rather than norms to points, we will call the sets $C_I(p)$ generalized elementary convex sets (g.e.c.s.).

Different generalizations of elementary convex sets can be found in the literature, see for instance Puerto and Fernández (1995, 2000), and Muriel and Carrizosa (1995).

First of all, it is straightforward to see that the g.e.c.s. are convex because they are defined by a finite intersection of convex sets (recall that subdifferential sets are convex).

First of all, we would like to address an interesting remark that extends a well-known property of location problems. Let us assume that each convex body is the convex hull of its extreme points. Note that this holds in particular when X is locally convex because of the Krein-Milman Theorem.

A first consequence of Lemma 2.1 and the compactness of the solution set (see Plastria 1984) is that there always exists an optimal solution of the infimal distance Weber problem in the set of extreme points of the g.e.c.s. Note that in this result we assume that these convex sets are given by the convex hull of their extreme points. This property extends the intersection point result obtained in $\mathbb{R}^2$ by Wendell and Hurter (1973) for the l_1-norm, by Thisse et al. (1984) for the polyhedral norm case, and by Durier and Michelot (1985) for the Fermat-Weber problem with linear cost. (Notice that the hypothesis on the convex bodies only applies to this remark and is not used in the rest of the paper.)

Furthermore, we may give an alternative geometrical description of g.e.c.s. in finite-dimensional spaces based on Theorem 2.1. This description will be used in §5 to develop an algorithm for solving $(P_\Phi(\mathcal{A}))$ in $\mathbb{R}^2$. Following the notation introduced in §2.1, let $\mathcal{Y}_i$ be the set of all the faces of any dimension of the set A_i with $i \in \mathcal{M}$.

DEFINITION 3.2. Given a family of sets $\mathcal{Y} = \{Y_1, Y_2, \dots, Y_M\}$ where each $Y_i \in \mathcal{Y}_i$, $p = (p_1, \dots, p_M)$ with $p_i \in B_i^\circ \cap N_{A_i}(y_i)$ for any $y_i \in Y_i$, and $I \subseteq \mathcal{M}$, let

$$C_I(\mathcal{Y}, p) := \bigcap_{i \in I} C(Y_i, p_i), \tag{11}$$

with $C(Y_i, p_i)$ as defined in (10).

It should be noted that if the unit balls are polytopes, then the g.e.c.s. can be obtained as the intersection of cones generated by fundamental directions of these balls pointed on the faces or vertices of each demand set (see §5 for details on the construction of these sets).

3.4. Optimality conditions. Let $M_\Phi(\mathcal{A})$ be the set of optimal solutions of $(P_\Phi(\mathcal{A}))$. We call (I, λ, p) a suitable triplet if

(1) $I \neq \varnothing$, $I \subseteq \mathcal{M}$.

(2) $\lambda = (\lambda_1, \dots, \lambda_M)$ with $\lambda_i > 0$ $(i \in I)$, and $\lambda_i = 0$ $(i \notin I)$ satisfying $\Phi^\circ(\lambda) = 1$.

(3) $p = (p_1, \dots, p_M)$ where $p_i \in B_i^\circ \cap N_{A_i}(y_i)$ for any $y_i \in \partial\sigma_{A_i}(p_i)$ $(i \in I)$, with $\sum_{i \in I} \lambda_i p_i = 0$.

LEMMA 3.3. *$x \in M_\Phi(\mathcal{A})$ iff there exists a suitable triplet (I, λ, p) satisfying*

$$x \in C_I(p) \cap D_I(\lambda).$$

PROOF. Observe that $x \in M_\Phi(\mathcal{A})$ iff $0 \in \partial F(x)$. Therefore, applying Lemma 3.2 and the definitions of $C_I(p)$ and $D_I(\lambda)$, the result follows immediately. □

It should be noted that Lemma 3.3 implies

$$C_I(p) \cap D_I(\lambda) \subseteq M_\Phi(\mathcal{A})$$

for any suitable triplet. Therefore, in order to give a complete characterization of $M_\Phi(\mathcal{A})$, we have to prove that a particular triplet exists such that the inclusion becomes an identity.

THEOREM 3.1.
(1) *If* $M_\Phi(\mathcal{A}) \neq \varnothing$, *then there exists a suitable triplet* (I, λ, p) *such that* $M_\Phi(\mathcal{A}) = C_I(p) \cap D_I(\lambda)$.
(2) *For any suitable triplet* (I, λ, p) *such that* $C_I(p) \cap D_I(\lambda) \neq \varnothing$, *one has that* $M_\Phi(\mathcal{A}) = C_I(p) \cap D_I(\lambda)$.

PROOF. Let (I, λ, p) be a suitable triplet with

$$\varnothing \neq C_I(p) \cap D_I(\lambda) \subseteq M_\Phi(\mathcal{A}),$$

existence of which is guaranteed by Lemma 3.3 as soon as $M_\Phi(\mathcal{A}) \neq \varnothing$.

Hence, in order to complete the proof, we have to prove that any $\bar{x} \in M_\Phi(\mathcal{A})$ verifies that $\bar{x} \in C_I(p) \cap D_I(\lambda)$.

Let x^* be such that $x^* \in C_I(p) \cap D_I(\lambda)$; then there exists $a_i(x^*) \in \operatorname{proj}_{A_i}(x^*)$ such that

$$F^* := F(x^*) = \sum_{i=1}^{M} \lambda_i \langle p_i, x^* - a_i(x^*)\rangle = -\sum_{i=1}^{M} \lambda_i \langle p_i, a_i(x^*)\rangle.$$

On the other hand, since F^* is minimal, we get $\langle p_i, a_i(x^*)\rangle = \sup_{a_i \in A_i} \langle p_i, a_i\rangle$; that is, $a_i(x^*) \in \partial\sigma_{A_i}(p_i)$ for any $i \in I$.

For any $x \in X$, we have

$$F^* = -\sum_{i=1}^{M} \lambda_i \langle p_i, a_i(x^*)\rangle \leq -\sum_{i=1}^{M} \lambda_i \langle p_i, a_i(x)\rangle \quad \forall a_i(x) \in \operatorname{proj}_{A_i}(x),$$

$$= \sum_{i=1}^{M} \lambda_i \langle p_i, x - a_i(x)\rangle \quad \forall a_i(x) \in \operatorname{proj}_{A_i}(x).$$

Since $d_i(x, A_i) = \sup_{q_i \in B_i^\circ} \langle q_i, x - a_i(x)\rangle = \gamma_i(x - a_i(x))$, using that $\Phi(\cdot)$ is a norm and $\Phi^\circ(\lambda) = 1$, we obtain

$$F^* \leq \sum_{i=1}^{M} \lambda_i \langle p_i, x - a_i(x)\rangle \leq \sum_{i=1}^{M} \lambda_i d_i(x, A_i) \leq F(x). \tag{12}$$

Hence, if we consider $x = \bar{x} \in M_\Phi(\mathcal{A})$, all these inequalities are equalities; that is,

$$\sum_{i=1}^{M} \lambda_i \langle p_i, \bar{x} - a_i(\bar{x})\rangle = \sum_{i=1}^{M} \lambda_i d_i(\bar{x}, A_i) \quad \forall a_i(\bar{x}) \in \operatorname{proj}_{A_i}(\bar{x})$$

and

$$\sum_{i=1}^{M} \lambda_i \langle p_i, a_i(x^*)\rangle = \sum_{i=1}^{M} \lambda_i \langle p_i, a_i(\bar{x})\rangle.$$

This together with the inequalities existing between corresponding terms leads us to deduce that for all $i \in I$ it holds: (i) $\langle p_i, \bar{x} - a_i(\bar{x})\rangle = d_i(\bar{x}, A_i)$, and (ii) $\langle p_i, a_i(\bar{x})\rangle = \langle p_i, a_i(x^*)\rangle$. From Condition (i), we obtain

$$d_i(\bar{x}, A_i) = \gamma_i(\bar{x} - a_i(\bar{x})) = \langle p_i, \bar{x} - a_i(\bar{x})\rangle \qquad \text{for any } i \in I.$$

Therefore, $p_i \in \partial\gamma_i(\bar{x} - a_i(\bar{x}))$ which is equivalent to $\bar{x} - a_i(\bar{x}) \in \partial\gamma_i^*(p_i)$ for any $i \in I$.

From Condition (ii), and since $a_i(x^*) \in \partial\sigma_{A_i}(p_i)$ for any $i \in I$, we deduce that $a_i(\bar{x}) \in \partial\sigma_{A_i}(p_i)$ for any $i \in I$. Hence,

$$\bar{x} \in a_i(\bar{x}) + \partial\gamma_i^*(p_i) \subset \partial\sigma_{A_i}(p_i) + \partial\gamma_i^*(p_i) = C_i(p_i) \qquad \text{for any } i \in I$$

(see (9) for the definition of $C_i(p_i)$); then we get that $\bar{x} \in C_I(p)$.

Moreover, since $\bar{x} \in M_\Phi(\mathscr{A})$ using the last inequality in (12), we have $F(\bar{x}) = \sum_{i\in I} \lambda_i d_i(\bar{x}, A_i)$ and $\bar{x} \in D_I(\lambda)$. Hence, $\bar{x} \in C_I(p) \cap D_I(\lambda)$ and the result follows. □

The reader may note that these conditions extend previous optimality conditions given in Durier (1995) for a similar location problem but only with point-facilities.

EXAMPLE 3.1. Consider in $\mathbb{R}^2$ the following problem: $\mathscr{A} = \{A_1, A_2, A_3\}$, where A_i, $i = 1, 2, 3$, are circles of radius 1 centered at $(-a, 0)$, $(0, 0)$, and $(a, 0)$, respectively. $\Phi(x_1, x_2, x_3) = |x_1| + |x_2| + |x_3|$ and $\gamma_1 = \gamma_2 = \gamma_3 = l_2$, the Euclidean norm in $\mathbb{R}^2$. (See Figure 2.)

Applying the theorem above we can obtain the entire set of optimal solutions. Indeed, take $I = \{1, 3\}$, $\lambda = (1, 1, 1)$, $p_1 = (1, 0)$, and $p_3 = (-1, 0)$. With these choices one has: $C_1(p_1) = \{(x, 0): x \geq -a + 1\}$, $C_3(p_3) = \{(x, 0): x \leq a - 1\}$, and $D_I(1, 1, 1) = A_2$. Hence,

$$M_\Phi(\mathscr{A}) = C_1(p_1) \cap C_3(p_3) \cap D_I(1, 1, 1) = \{(x, 0):\ -1 \leq x \leq 1\}.$$

The last part of this section is devoted to some properties of the optimal solution set $M_\Phi(\mathscr{A})$ of $(P_\Phi(\mathscr{A}))$. The first property states the relationship between $(P_\Phi(\mathscr{A}))$ and a particular Weber problem.

Let us denote by $F_W^*(A)$ and $M_W(A)$ the optimal value and the set of optimal solutions of the following Weber problem, respectively,

$$(P_W(A)) \qquad F_W^*(A) = \min_{x\in X} \sum_{i=1}^{M} w_i \gamma_i(x - a_i),$$

where $A = \{a_1, \ldots, a_M\}$ and $W = \{w_1, \ldots, w_M\}$. Finally, let F^* denote the optimal value of $(P_\Phi(\mathscr{A}))$.

THEOREM 3.2. *For each monotone norm Φ, such that $M_\Phi(\mathscr{A}) \neq \varnothing$, the following results hold*:

(1) *There exists a set of nonnegative weights $W = \{w_1, \ldots, w_M\}$ and a set of points $A = \{a_1, \ldots, a_M\}$ with $a_i \in A_i$, $i \in \mathscr{M}$ such that*

$$M_W(A) \cap M_\Phi(\mathscr{A}) \neq \varnothing \quad \text{and} \quad F^* = F_W^*(A).$$

(2) *If $D_I(W) := \{x \in X\colon \sum_{i=1}^{M} w_i d_i(x, A_i) = F^*\} \neq \varnothing$ for a given $W = \{w_1, \ldots, w_M\}$, then there exists $A = \{a_1, \ldots, a_M\}$ such that $(P_W(A))$ and $(P_\Phi(\mathscr{A}))$ have common optimal solutions.*

PROOF. If $x^* \in M_\Phi(\mathscr{A})$, then there exists a suitable triplet (I, λ, p) such that $x^* \in C_I(p) \cap D_I(\lambda)$. In particular, $x^* \in C_I(p) = \bigcap_{i\in I} C_i(p_i)$. Therefore, for each $i \in I$ there exists

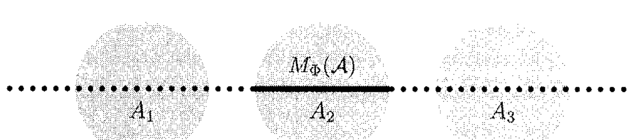

FIGURE 2. Illustration of Example 3.1.

$a_i \in \text{proj}_{A_i}(x^*) \subseteq \partial\sigma_{A_i}(p_i)$ and $p_i \in B_i^\circ \cap N_{A_i}(y_i)$ for any $y_i \in \partial\sigma_{A_i}(p_i)$, $\forall i \in I$, such that

$$x^* \in a_i + N_{B_i^\circ}(p_i) \ \forall i \in I \quad \text{and} \quad \sum_{i\in I} \lambda_i p_i = 0.$$

In addition, since $x^* \in D_I(\lambda)$ we have that

$$F^* = \Phi(d(x^*)) = \sum_{i\in I} \lambda_i d_i(x^*, A_i) = \sum_{i\in I} \lambda_i \gamma_i(x^* - a_i).$$

Therefore, if we take $W = \{w_1, \ldots, w_M\}$ with $w_i = \lambda_i, \forall i \in I$, $w_i = 0$, $i \notin I$ and a_i is given as above for $i \in I$ and otherwise arbitrarily chosen within A_i; it follows that

$$M_W(A) \cap M_\Phi(\mathcal{A}) \neq \varnothing \quad \text{and} \quad F^* = F_W^*(A). \quad \square$$

If γ_i are strict norms, a more precise relation can be shown.

COROLLARY 3.1. *If $\gamma_i(\cdot)$ $\forall i \in \mathcal{M}$ are strict norms and there exists a suitable triplet (I, λ, p) with $|I| \geq 3$ and three demand sets A_j, A_k, A_l, $j, k, l \in I$, which cannot be stabbed by a line, then $W = \{w_1, \ldots, w_M\}$ and $A = \{a_1, \ldots, a_M\}$ exist such that $M_W(A) \subseteq M_\Phi(\mathcal{A})$ and $F^* = F_W^*(A)$.*

PROOF. It is well known that if $\gamma_i(\cdot)$ is a strict norm and the existing facilities are not colinear then for any set of weights, the classic Weber problem has a unique optimal solution (see Pelegrin et al. 1985). Since under the assumptions of this corollary any family of points $A = \{a_i\}_{i\in I}$ with $a_i \in A_i$ cannot be colinear, Theorem 3.2 leads us to the desired result:

$$M_W(A) \subseteq M_\Phi(\mathcal{A}). \quad \square$$

REMARK 3.2. It is important to observe that this corollary is only a sufficient condition and that, in general, inclusion cannot be ensured. The following examples show that (1) the same result can be obtained without the assumptions of Corollary 3.1, (2) there is not a general inclusion relationship between the set of optimal solutions.

Consider once more the problem in Example 3.1 whose configuration is displayed in Figure 2.

The optimal solution set $M_\Phi(\mathcal{A})$ is given by the segment indicated by the thick line in Figure 2, that is, the diameter of A_2 on the line through the three centers. Now consider the Weber problem $(P_W(A))$ with existing facility set A given by any point in the diameter of the central circle and the points in each one of the external circles closest to the central one, and weights $w_1 = w_3 = 1$, $w_2 = 3$. The optimal solution set $M_W(A)$ is the point of the central circle. Obviously, $M_W(A) \subset M_\Phi(\mathcal{A})$ and the objective value of both problems coincide. However, the assumption of Corollary 3.1 does not hold. Moreover, if we had taken weights $w_1 = w_3 = 1$ and $w_2 = 0$, the optimal objective value of both problems would have been the same, but the solution set $M_W(A)$ would be the segment joining the points in the external circles. Note that in this case $M_\Phi(\mathcal{A}) \neq M_W(A)$. $(M_\Phi(\mathcal{A}) \subset M_W(A).)$

4. Relationships with two classical problems: Some important examples. We consider a collection of sets $\mathcal{A} = \{A_1, \ldots, A_M\}$ where each set A_i is compact and convex. Moreover, let $W = \{w_1, \ldots, w_M\}$ denote a set of positive weights, and let $\gamma_i(\cdot)$ with $i \in \mathcal{M}$ be a set of norms in X with unit ball B_i.

4.1. The Weber problem with infimal distances. The Weber problem with infimal distances with respect to $\mathcal{A}$ and W is defined as

$$(P_W(\mathcal{A})) \qquad \min_{x\in X} G(x) := \sum_{i=1}^{M} w_i d_i(x, A_i).$$

Recall that $d_i(x, A_i) = \inf_{a\in A_i} \gamma_i(x-a)$.

Our main goal will be to characterize the set of optimal solutions $M_W(\mathcal{A})$ of $P_W(\mathcal{A})$. The following results are particular cases of Lemma 3.2 and Theorem 3.1 taking $\Phi = l_1$-norm in $\mathbb{R}^M$, and $\gamma_i' = w_i\gamma_i$ for all $i \in \mathcal{M}$. Therefore, the proofs are omitted here.

LEMMA 4.1. *For any $x \in X$, we have that $x^* \in \partial G(x)$ for some $x \in X$ iff there exist $a_i \in \operatorname{proj}_{A_i}(x)$, $i = 1, 2, \ldots, k$, $p_i \in N_{A_i}(a_i) \cap B_i^\circ$, such that*

(1) $x \in \bigcap_{i=1}^{M} (a_i + N_{B_i^\circ}(p_i))$.

(2) $x^* = \sum_{i=1}^{M} w_i p_i$.

THEOREM 4.1.

(1) *If $M_W(\mathcal{A}) \neq \varnothing$, then there exists a suitable triplet (I, λ, p) with $\lambda_i = w_i$ $\forall i \in I$, such that $M_W(\mathcal{A}) = \bigcap_{i\in I} (\partial\sigma_{A_i}(p_i) + N_{B_i^\circ}(p_i))$.*

(2) *If there exists a suitable triplet (I, λ, p) with $\lambda_i = w_i$ $\forall i \in I$ such that*

$$\bigcap_{i\in I} (\partial\sigma_{A_i}(p_i) + N_{B_i^\circ}(p_i)) \neq \varnothing,$$

then $M_W(\mathcal{A}) = \bigcap_{i\in I} (\partial\sigma_{A_i}(p_i) + N_{B_i^\circ}(p_i))$.

EXAMPLE 4.1. Consider three sets in $\mathbb{R}^2$ with $\gamma_i = l_1$ for every i. The demand sets are $A_1 := \operatorname{co}\{(0, 1), (-1, 2), (1, 2)\}$, $A_2 := \operatorname{co}\{(2, -0.5), (2, 0.5), (3, 0.5), (3, -0.5)\}$, and $A_3 := \operatorname{co}\{(-2, -2), (-2, -1), (-3, -1), (-3, -2)\}$ with $w_1 = w_2 = w_3 = 1$ (see Figure 3).

We see that the g.e.c.s. are those sets delimited by the lines drawn in Figure 3 (in §5, the characterization of these sets is described in detail). The optimal solution is given by $p_1 = (0, -1)$, $p_2 = (-1, 0)$, and $p_3 = (1, 1)$. Moreover,

$$C_I\big((p_1, p_2, p_3)\big) = \operatorname{co}\{(0, -0.5), (0, 0.5)\}.$$

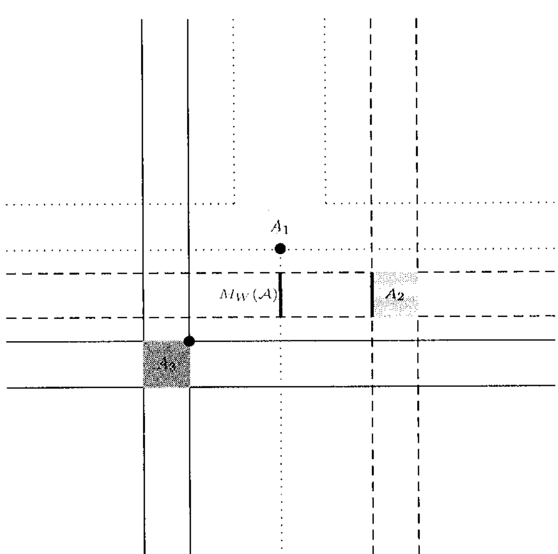

FIGURE 3. Illustration of Example 4.1.

COROLLARY 4.1. *The Weber problem with infimal distances always has an optimal solution in the set of extreme points of the corresponding g.e.c.s.*

To prove the last result in this section, let us assume that $\gamma_i = \gamma$ for all $i = 1, \dots, M$. In addition, let us denote by $d_1(x, y) = \gamma(x - y)$ the common distance generated by the unique norm in the problem. We can derive a majority theorem similar to the one valid for the classical case with points as existing facilities.

THEOREM 4.2. *If γ is a norm in X, $M_W(\mathcal{A}) \neq \varnothing$ and there exists $A_i \in \mathcal{A}$ such that $w_i \geq \sum_{i \neq j} w_j$, then there exists an optimal solution in A_i.*

PROOF. Let $x^* \in M_W(\mathcal{A})$ and assume that $x^* \notin A_i$. If $x \in \mathrm{proj}_{A_i}(x^*)$, then we have

$$G(x^*) = \sum_{i=1}^{M} w_i d_1(x^*, A_i) \leq G(x) = \sum_{j \neq i} w_j d_1(x, A_j).$$

Now by the triangular inequality, we obtain

$$\sum_{j \neq i} w_j d_1(x, A_j) \leq \sum_{j \neq i} w_j (\gamma(x - x^*) + d_1(x^*, A_j)) \leq w_i \gamma(x - x^*) + \sum_{j \neq i} w_j d_1(x^*, A_j) = G(x^*).$$

Hence, x is also an optimal solution. □

4.2. Minimax problem with infimal distances. The minimax problem with infimal distances is defined as

$$\min_{x \in X} H(x) := \max_{1 \leq i \leq M} w_i d_i(x, A_i), \tag{13}$$

where $d_i(x, A_i) = \inf_{a \in A_i} \gamma_i(x - a)$. Denote by $M_W^{l_\infty}(\mathcal{A})$ the set of optimal solutions of (13) and let us define for $I \subseteq \mathcal{M}$ and $\alpha \geq 0$ the following set:

$$AS_I(\alpha) = \{x \in X \colon w_i d_i(x, A_i) = \alpha, \ \forall i \in I; \ w_i d_i(x, A_i) < \alpha, \ \forall i \notin I\}.$$

Then the following theorem gives a characterization of $M_W^{l_\infty}(\mathcal{A})$.

THEOREM 4.3.
(1) *If $M_W^{l_\infty}(\mathcal{A}) \neq \varnothing$, then there exists a suitable triplet (I, λ, p) and $\alpha \geq 0$ such that*

$$M_W^{l_\infty}(\mathcal{A}) = C_I(p) \cap AS_I(\alpha).$$

(2) *For any suitable triplet (I, λ, p) and $\alpha \geq 0$ such that $C_I(p) \cap AS_I(\alpha) \neq \varnothing$, one has that*

$$M_W^{l_\infty}(\mathcal{A}) = C_I(p) \cap AS_I(\alpha).$$

PROOF. The proof consists of applying the general Theorem 3.1 for $\Phi = l_\infty$-norm in $\mathbb{R}^M$ and $\gamma_i' = w_i \gamma_i$ for all $i \in \mathcal{M}$. Since $\Phi = l_\infty$-norm, this implies that the dual norm $\Phi^\circ = l_1$-norm. Therefore, $\gamma^\circ(\lambda) = 1$ if and only if there exists $I \subseteq \mathcal{M}$ such that $\sum_{i \in I} \lambda_i = 1$. Hence, $\alpha = H(x) = \max_{1 \leq i \leq M} w_i d_i(x, A_i) = \sum_{i \in I} \lambda_i w_i d_i(x, A_i)$ with $\sum_{i \in I} \lambda_i = 1$ is equivalent to $w_i d_i(x, A_i) = \alpha$ for any $i \in I$ and $w_i d_i(x, A_i) < \alpha$ for any $i \notin I$. In other words, $x \in D_I(\lambda)$ if and only if $x \in AS_I(\alpha)$ for $\alpha = H(x)$, and the desired result follows. □

REMARK 4.1. The value of α which defines the optimal solution set $AS_I(\alpha)$ is the optimal objective value of Problem (13).

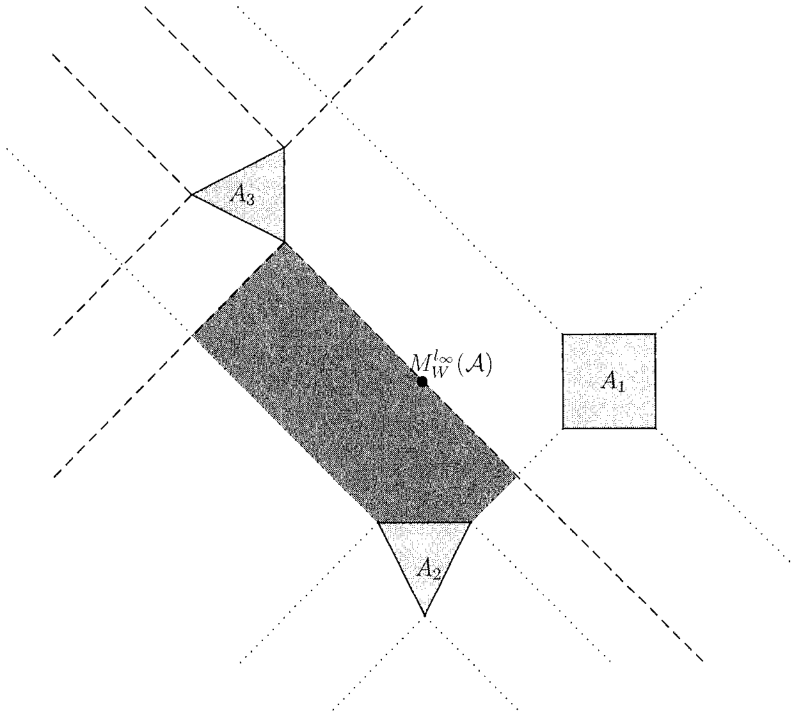

FIGURE 4. Illustration of Example 4.2.

EXAMPLE 4.2. Consider a problem with $\Phi(x_1, x_2, x_3) = \max\{|x_1|, |x_2|, |x_3|\}$ and the following demand sets $\mathscr{A} = \{A_1 := \text{co}\{(5, 1), (5, -1), (3, -1), (3, 1)\}, A_2 := \text{co}\{(1, -3), (0, -5), (-1, -3)\}, A_3 := \text{co}\{(-3, 3), (-3, 5), (-5, 4)\}\}$ with weights $W = \{1, 1, 1\}$ and $\gamma_i = l_\infty$-norm in $\mathbb{R}^2$ for $i = 1, 2, 3$ (see Figure 4).

The problem to be solved is

$$\min_{x \in \mathbb{R}^2} H(x) := \max_{i=1,2,3} d_i(x, A_i).$$

Taking $I = \{2, 3\}$, $p_2 = (0, 1), p_3 = (0, -1)\}$ it follows that

$$C_I(p) = \{x \in \mathbb{R}^2 \colon x_2 \geq -3, x_1 + x_2 \leq 0, x_1 + x_2 \geq -4, x_1 - x_2 \geq -6, x_1 - x_2 \leq 4\}.$$

Now for $\alpha = 3$, we have that $AS_I(3) = \{(0, 0)\}$, which equals $D_I((0, 0.5, 0.5))$. Note that for $\lambda = (0, 0.5, 0.5)$, we have $\Phi^\circ(\lambda) = 1$. In fact, this set is defined by

$$D_I((0, 0.5, 0.5)) = \left\{x\colon \max_{i=1,2,3} d_i(x, A_i) = \frac{1}{2}\langle(0, 1), x - a_2\rangle + \frac{1}{2}\langle(0, -1), x - a_3\rangle\right\} = \{(0, 0)\},$$

where $a_2 = (0 - 3)$ and $a_3 = (-3, 3)$. Hence,

$$M_W^{l_\infty}(\mathscr{A}) = C_I(p) \cap AS_I(\alpha) = \{(0, 0)\}.$$

5. The polyhedral planar case: Interpretations. In order to obtain the solution set of $(P_\Phi(\mathscr{A}))$, it is important to realize that within the sets $C_I(\mathscr{Y}, p)$ (defined in (11)) the infimal distance function is linear.

In this section, we restrict ourselves to $\mathbb{R}^2$ and total polyhedrality. This reduction allows us to describe in an easy way the geometrical characterization given in the previous sections.

In this section, we take advantage of the properties of our characterizations in the planar case to develop a polynomial algorithm to find the g.e.c.s. in $\mathbb{R}^2$ whenever polyhedral norms are used to measure the distances and the demand sets are convex polygons.

We will do that by applying the following scheme. Having already proven that the g.e.c.s. are the sets of points projecting onto faces of the existing facilities, we will characterize the maximal projection domains using the norm associated with each facility. To do that, we first characterize the projection onto lines, then onto segments, and finally onto cones. After that, we can characterize the projection onto convex polygons, since they can be seen as segments plus corners (cones).

Let γ be a polyhedral norm with unit ball B having G extreme points. In the following, we say that a point x projects onto the line r with direction δ if there exists $\bar{x} \in \text{proj}_r(x)$ such that $x = \bar{x} + \lambda\delta$ with $\lambda > 0$.

LEMMA 5.1. *Let r be a line with normal vector $p \in B^\circ$. It holds that*

$$\bar{x} \in \text{proj}_r(x) \quad \textit{iff } x - \bar{x} \in \partial\gamma^*(p).$$

PROOF. By definition, $\bar{x} \in \text{proj}_r(x)$ iff $\bar{x} \in \arg\min_{y\in r} \gamma(x-y)$. Since this is a constrained convex problem, its optimality condition is: $\bar{x}$ is an optimal solution iff $0 \in \partial\gamma(x-\bar{x}) + N_r(\bar{x})$. Finally, this relationship holds iff $x - \bar{x} \in \partial\gamma^*(p)$. □

COROLLARY 5.1. *Let π_1 be an open halfspace determined by a line r. The projection of a point belonging to π_1 onto r can be*:

(1) *Unique. In this case, all points of π_1 project with the same fundamental direction.*

(2) *Not unique. In this case, all points of π_1 project with the cone of directions generated by two consecutive fundamental directions.*

PROOF. Since, $N_{B^\circ}(p)$ is either a halfline or a full dimension cone, the result is a straightforward consequence of Lemma 5.1. □

In the following corollary, we determine the direction projections according to the two cases analyzed in Corollary 5.1.

COROLLARY 5.2. *Let π_1 be an open halfspace determined by a line r and let $\overline{AE}$ be a segment included in r.*

(1) *If $x \in \pi_1$ projects with the fundamental direction δ_1 onto r and $\bar{x} = \text{proj}_r(x)$, then*

$$\exists p \in B^\circ, \qquad d(x,r) = \langle p, x - \bar{x}\rangle, \quad \forall x \in \pi_1,$$

and if $x \in \pi_1$ and $\text{proj}_r(x) \in \overline{AE}$, then

$$x \in \overline{AE} + \mu\delta_1, \qquad \mu \geq 0.$$

(2) *If $x \in \pi_1$ projects with the directions δ_1 and δ_2 onto r, then*

$$\exists q \in B^\circ, \qquad d(x,r) = \langle q, x - \bar{x}\rangle, \quad \forall x \in \pi_1 \textit{ and any } \bar{x} \in \text{proj}_r(x).$$

Moreover, if $x \in \pi_1$ and $\text{proj}_r(x) \cap \overline{AE} \neq \varnothing$, then

$$x \in \overline{AE} + \text{cone}(\delta_1, \delta_2). \quad \square$$

(*See Figure* 5.)

Once we have described the set of points in π_1 whose projections belong to a segment, we proceed studying the case of a cone.

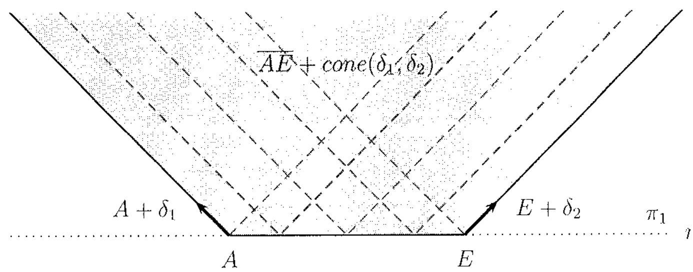

FIGURE 5. Set of points belonging to π_1, whose projection onto r with the l_∞-norm has a nonempty intersection with the line segment $\overline{AE}$.

THEOREM 5.1. *Let h_1 and h_2 be two halflines with the same origin O, contained in the lines r_1 and r_2 respectively. Let π_1, π_2 be the two open halfspaces determined by r_1 and r_2 such that $h_1 \cap \pi_2 = \varnothing$ and $h_2 \cap \pi_1 = \varnothing$. The following two statements hold:*

(1) *If $x \in \pi_1$ and $\bar{x}_1 \in \text{proj}_{r_1}(x) \cap (h_1 \backslash \{O\})$, then there exists $p_1 \in B^\circ$ verifying*

$$d(x, \text{co}(h_1, h_2)) = \langle p_1, x - \bar{x}_1 \rangle.$$

(The analogous result holds for π_2.)

(2) *If $x \in \pi_1 \cup \pi_2$ and $\text{proj}_{r_i}(x) \cap (h_i \backslash \{O\}) = \varnothing$ with $i = 1, 2$, then $\text{proj}_{\text{co}(h_1, h_2)}(x) = O$ and there exists $p_x \in B^\circ$ verifying*

$$d(x, \text{co}(h_1, h_2)) = \langle p_x, x - O \rangle,$$

where $\text{co}(h_1, h_2)$ is the convex hull of h_1 and h_2.

PROOF.

(1) This is a straightforward consequence of Corollary 5.2.

(2) Let $x \in \pi_1 \cup \pi_2$ and $\bar{x} \in \text{proj}_{\text{co}(h_1, h_2)}(x)$. Since $x \notin \text{co}(h_1, h_2)$, using the convexity of γ, we have that $\bar{x} \in h_1 \cup h_2$. Now, we have to prove that $\bar{x} = O$. Let us assume that $\bar{x} \in (h_1 \cup h_2) \backslash \{O\}$.

Since $\text{proj}_{r_i}(x) \cap (h_i \backslash \{O\}) = \varnothing$ for $i = 1, 2$, using the convexity of γ, we have that $\gamma(x - O) < \gamma(x - y)\ \forall y \in (h_i \backslash \{O\})$, $i = 1, 2$. This contradicts that $\bar{x} \in (h_1 \cup h_2) \backslash \{O\}$. Thus, we obtain that $\bar{x} = O$.

Therefore, there exists a cone $\text{cone}(D_O)$ (maybe degenerated to a line), which is generated by halflines defined by O and the points whose unique projection onto $\text{co}(h_1, h_2)$ is O. That is, if $x \in O + \text{cone}(D_O)$, then $\text{proj}_{\text{co}(h_1, h_2)}(x) = O$. Thus, for all $x \in O + \text{cone}(D_O)$ there exists $p_x \in \partial\gamma(x - O)$, verifying that

$$d(x, \text{co}(h_1, h_2)) = \gamma(x - O) = \langle p_x, x - O \rangle. \quad \square$$

COROLLARY 5.3. *The function $d(x, \text{co}(h_1, h_2))$ is linear in the following sets (see Figure 6):*

(1) *$h_i + \text{cone}(D_i)$, where D_i is the set of fundamental directions of projection of π_i onto r_i with $i = 1, 2$.*

(2) *$O + \text{cone}(\delta_s, \delta_{s+1})$ being δ_s and δ_{s+1} two consecutive fundamental directions of D_O and where D_O is the set of consecutive fundamental directions verifying that $|D_1 \cap D_O| = |D_2 \cap D_O| = 1$ and that $O + \text{cone}(D_O) \subseteq \text{cl}(\pi_1 \cup \pi_2)$ (where we denote by cl the topological closure).*

REMARK 5.1. It should be noted that D_O may only have one element. In this case, $O + \text{cone}(\delta_s, \delta_{s+1})$ is a cone degenerated to a line.

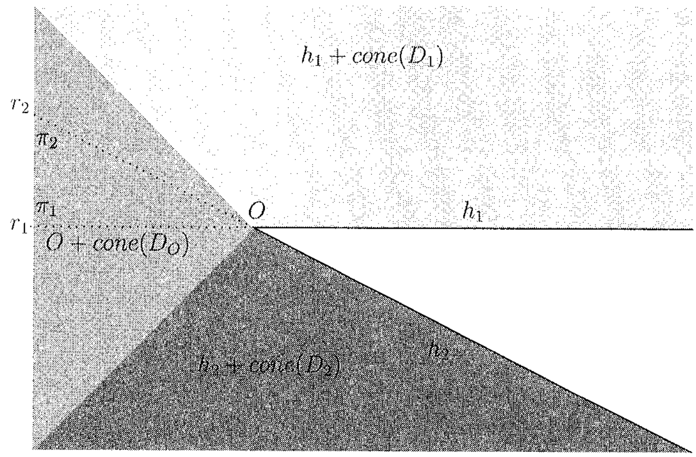

FIGURE 6. Different sets where the distance to $\text{co}(h_1, h_2)$, using the l_∞-norm, is a linear function.

PROOF.
(1) This is a straightforward consequence of Theorem 5.1.
(2) The set of points included in $\pi_1 \cup \pi_2$ whose unique projection onto $\text{co}(h_1, h_2)$ is O, is the set

$$P_O = \{x\colon x \in \text{cl}((\pi_1 \cup \pi_2)\backslash(h_1 + \text{cone}(D_1) \cup h_2 + \text{cone}(D_2)))\}.$$

Therefore, P_O is a pointed cone at O generated by the set of fundamental directions, D_O, enclosed by D_1 and D_2 such that $O + \text{cone}(D_O) \subseteq \text{cl}(\pi_1 \cup \pi_2)$. Thus, if δ_s and δ_{s+1} (two consecutive fundamental directions) belong to D_O we have that there exists $p_s \in \partial\gamma(\delta_s) \cap \partial\gamma(\delta_{s+1})$ such that

$$d(x, O) = \langle p_s, x - O\rangle \quad \forall x \in O + \text{cone}(\delta_s, \delta_{s+1}). \quad \square$$

In the previous result we have characterized the sets where the infimal distance to a cone is linear. Now, in the following corollary, we extend these results to the infimal distance to a polygon.

COROLLARY 5.4. *Let A be a convex polygon, where $F_1, \ldots, F_L$ are its facets and $O_1, \ldots, O_L$ are its vertices (see Figure 7). Let r_j be the line containing the facet F_j, and*

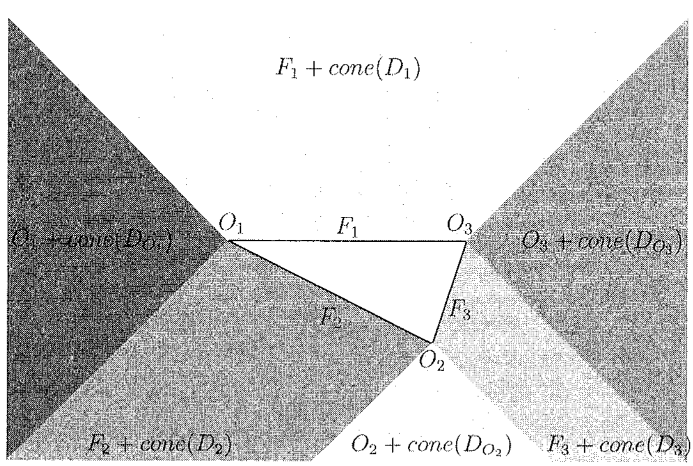

FIGURE 7. Different sets where the distance to this triangle, using the l_∞-norm, is a linear function.

π_j the open halfspace defined by r_j and not containing A, with $j = 1, \ldots, L$. There exist $p(D_j)$, $p(O_j, s) \in B_i^\circ$ for all $j = 1, \ldots, L$, such that

$$d(x, A) = \begin{cases} \langle p(D_j), x - \bar{x}_j \rangle & \forall x \in F_j + \text{cone}(D_j) \text{ and } \bar{x}_j \in \text{proj}_A(x), \\ \langle p(O_j, s), x - O_j \rangle & \forall x \in O_j + \text{cone}(\delta_s, \delta_{s+1}) \text{ with } \delta_s, \delta_{s+1} \in D_{O_j}, \end{cases}$$

where D_j and D_{O_j} with $j = 1, \ldots, L$ are defined as in the previous corollary.

REMARK 5.2. Since $\partial d(x, A_i) \neq \varnothing$, we can choose $p(D_j) \in N_{A_i}(\bar{x})$ for any $\bar{x} \in \text{proj}_{A_i}(x) \cap \text{ri}(F_j)$ such that $\langle p(D_j), x - \bar{x} \rangle = d(x, A_i)$. Therefore, we obtain that $C(F_j, p(D_j)) = F_j + \text{cone}(D_j)$. In the same way, there exists $p(O_j, s) \in N_{A_i}(O_j)$ such that $C(O_j, p(O_j, s)) = O_j + \text{cone}(\delta_s, \delta_{s+1})$. (Recall that the sets $C(Y_i, p_i)$ were defined in (10).)

After these results, we construct the maximal sets $C_l(\mathcal{Y}, p)$ (defined in (11)). In fact, in the Appendix, we develop an algorithm which gives us a methodology to build the maximal domain of linearity of the infimal distance to each set of the family $\mathcal{A}$.

The algorithm in the Appendix performs a loop over the extreme points $O_1, \ldots, O_L$ and the facets $F_1, \ldots, F_L$ of an existing facility, $A \in \mathcal{A}$. During this loop we can compute the sets D_j and D_{O_j} and their corresponding vectors $p(D_j)$ and $p(O_j, s)$, with $j = 1, \ldots, L$, defined in Corollaries 5.3 and 5.4. Finally, we calculate $C(F_j, p(D_j))$ and $C(O_j, p(O_j, s))$ as described in Remark 5.2.

Assuming that the facets are given in a sorted circular list, we can obtain the domains of linearity in $O(L + G)$ time. (Recall that L is the number of facets of the polygon A and G is the number of extreme points of the unit ball of γ.) A detailed description of this algorithm is given in the Appendix.

Once we have described the algorithm to compute the maximal domain of linearity of the infimal distance to any polygon, we can obtain the domain of linearity of any problem where the demand sets are polygons as the intersection of the maximal domain of linearity of the infimal distance to each demand set. These maximal domains of linearity are called cells and they are the natural extension of the elementary convex sets when we consider a problem with demand points.

In order to solve a general problem $P_W(\mathcal{A})$ with polygons as demand sets, we describe an algorithm to compute the optimal solution of this problem. As a straightforward extension of the results in Plastria (1984), one can prove compactness of the optimal solution set. Then by the discussion prior to Definition 3.2 and Corollary 4.1 we only need to look at the extreme points of the g.e.c.s.

ALGORITHM 5.1. (SOLVING THE PROBLEM $(P_W(\mathcal{A}))$ IN $\mathbb{R}^2$).

Step 1. *COMPUTE the planar graph generated by the cells and let V be its set of vertices using the maximal domains of linearity.*

Step 2. *Perform a local search in the vertices of V with the neighbor structure induced by the adjacent vertices.*

The planar graph generated by the cells of the problem can be obtained by employing a sweep line technique applying the algorithm by Bentley and Ottmann (1979) and described in more detail in Weißler (1999) and Nickel et al. (1999). In order to use the sweep line technique, we need to consider a bounded region on the plane which follows from the compactness property mentioned above. Since the objective function F is convex and in the polyhedral case, the number of intersection points is polynomial, the algorithm ends in polynomial time with the optimal solutions given by the convex hull of the intersection points attaining the lowest F value.

The complexity of Algorithm 5.1 is determined by the complexity of computing the planar graph generated by the cells and the time needed to evaluate the objective function for each $v \in V$ $(MG_{\max})$.

By applying the results of Weißler (1999) and Nickel et al. (1999), the complexity of Algorithm 5.1 is $O(M^2 G_{\max} \log(MG_{\max})) + O(|V|MG_{\max}) = O(M^2 G_{\max} \log(MG_{\max}) + |V|MG_{\max})$. The number of vertices $|V|$ can be bounded by $M^2 G_{\max}$, where $G_{\max}$ is the maximum number of fundamental directions of the norms associated to each demand set, $A \in \mathcal{A}$.

The reader may note that there exist very powerful alternative approaches to solve this problem. For instance using Cohen and Megiddo (1993), one can get subquadratic complexity (in $MG_{\max}$) using an optimal convex algorithm for piecewise convex functions in fixed dimension.

EXAMPLE 5.1. Let A_1, A_2, and A_3 be the demand sets defined as follows: $A_1 = \text{co}\{(4.5, 10), (10.5, 10), (10.5, 13.5), (4.5, 13.5)\}$, $A_2 = \text{co}\{(19.5, 15), (23.5, 17), (24, 15)\}$, and $A_3 = \text{co}\{(18.5, 4), (18.5, 6), (20.5, 6), (18.5, 6)\}$. We consider that $\gamma_1 = l_1$-norm, and $\gamma_2 = \gamma_3 = l_\infty$-norm. The problem to be solved is given by

$$\min_{x \in \mathbb{R}^2} 2d_1(x, A_1) + d_2(x, A_2) + d_3(x, A_3).$$

In order to solve this problem, we compute the generalized elementary convex sets using Algorithm A.1 (see Figure 8). Knowing all elementary convex sets, we use Algorithm 5.1 to obtain as optimal solution the shaded region $M_\Phi(\mathcal{A})$.

6. Concluding remarks. There exists another natural extension that can be addressed: the location of a regional facility with respect to existing facilities that are sets.

Let us consider a fixed set B closed, compact, and convex. The problem consists of determining the translation vector x such that x solves the following problem:

$$\min_{x \in X} \Phi(d_1(x + B, A_1), \ldots, d_M(x + B, A_M)),$$

where $d_i(x + B, A) = \inf_{b \in B} \inf_{a_i \in A_i} \gamma_i(x + b - a_i)$.

Now, it is straightforward to see that

$$\inf_{b \in B} \inf_{a_i \in A_i} \gamma_i(x + b - a) = \inf_{c_i \in B - A_i} \gamma_i(x - c_i).$$

Therefore, we reduce this problem to the first one by considering a new family $\mathcal{A}' = \{B - A_1, \ldots, B - A_M\}$. (Set-to-set expected distance location problems have been already considered in Carrizosa et al. 1995.)

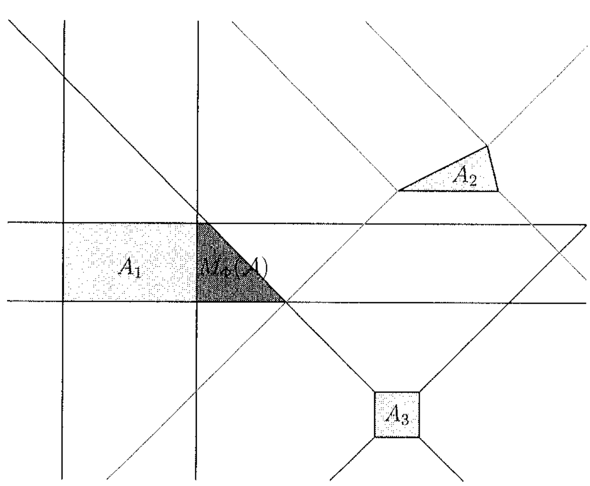

FIGURE 8. Illustration of the generalized elementary convex sets in Example 5.1.

Finally, we would like to mention that similar results to the ones developed in this paper can also be obtained when the norms γ_i associated with each set A_i are replaced by gauges.

Appendix. In this section, we give a detailed description of the algorithm for finding the maximal domains of linearity. Recall that we consider, as in §5, a polyhedral norm γ in $\mathbb{R}^2$ with unit ball B having G extreme points and fundamental directions $\{\delta_1, \dots, \delta_G\}$.

Before starting with the description of the algorithm, we need the following lemma that allows us to identify the projection directions onto a line.

LEMMA A.1. *Let π_1 be an open halfspace determined by a line r. If $((\bar{x}-\delta_1)+B)\cap \mathrm{cl}(\pi_1) \subseteq r$ with $\bar{x} \in r$, then the points of π_1 project onto r at least with δ_1.*

PROOF. We can assume without loss of generality that every fundamental direction δ verifies that $\gamma(\delta) = 1$.

There exists a fundamental direction δ_1, such that $((\bar{x}-\delta_1)+B)\cap \mathrm{cl}(\pi_1) \subseteq r$ with $\bar{x} \in r$. Then, two cases can occur:

(1) $((\bar{x}-\delta_1)+B)\cap \mathrm{cl}(\pi_1) = (\bar{x}-\delta_1)+\delta_1$.

(2) $((\bar{x}-\delta_1)+B)\cap \mathrm{cl}(\pi_1) = \theta((\bar{x}-\delta_1)+\delta_1)+(1-\theta)((\bar{x}-\delta_1)+\delta_2)$ with $\theta \in [0,1]$ and δ_2 a consecutive fundamental direction of δ_1.

Now, consider a fundamental direction δ, such that $\delta \neq \delta_1$ in Case 1. Moreover, $\delta \neq \theta\delta_1 + (1-\theta)\delta_2$, $\forall \theta \in [0,1]$ in Case 2. Then again one of two cases can occur:

(1) $\forall \lambda > 0$ we have that $(\bar{x}-\delta_1)+\lambda\delta \notin r$.

(2) $\exists \lambda > 0$ such that $(\bar{x}-\delta_1)+\lambda\delta \in r$.

The first case implies that any point of π_1 does not project onto r with the direction δ.

In the second case (see Figure 9), let $x = \bar{x}+\delta_1$, and $y = (\bar{x}-\delta_1)+\lambda\delta \in r$. Since $((\bar{x}-\delta_1)+B)\cap \mathrm{cl}(\pi_1) \neq (\bar{x}-\delta_1)+\delta$, it follows that $\lambda > 1$.

We have that $x = \bar{x}+\delta_1$ or equivalently, $x = \bar{x}+\delta_1-\lambda\delta+\lambda\delta$. Moreover, since $\bar{x} \in r$ and $(\bar{x}-\delta_1)+\lambda\delta \in r$, then $\bar{x}-(-\delta_1+\lambda\delta)$ also belongs to r. Thus, x is equal to an element of r, namely $\bar{x}-(-\delta_1+\lambda\delta)$, plus $\lambda\delta$. This means that the distance from r to x with direction δ is λ. We know that $\lambda > 1$ and the distance from r to x with δ_1 is 1. Therefore, x does not project with δ. This implies that x has to project with δ_1. □

Using this lemma and the results in §5, we derive an algorithm that performs a loop over the extreme points $O_1, \dots, O_L$ and facets $F_1, \dots, F_L$ of a convex polygon, A, in order to obtain the maximal domain of linearity of the infimal distance function to A.

ALGORITHM A.1.

Preprocessing:

- *For existing facility $A \in \mathcal{A}$, we denote by $-n_1, \dots, -n_L$ the negative normal vectors of the facets of A. They are sorted in counterclockwise order.*

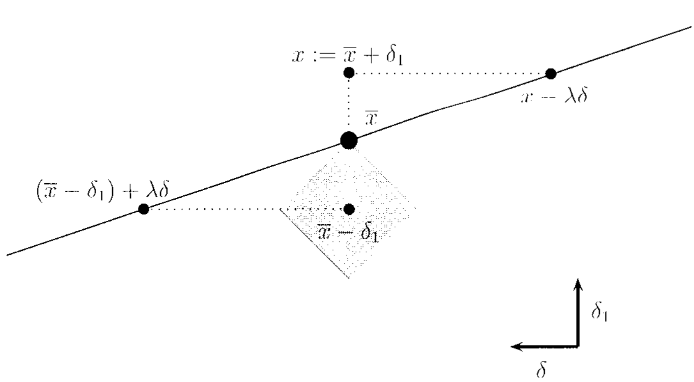

FIGURE 9. Illustration of the proof of Lemma A.1.

• *For each fundamental direction δ_i of the unit ball B, we build $B-\delta_i$ and denote by $C\delta_i$ the cone generated by the two facets δ_i° and δ_i^C which start in the origin (of $B-\delta_i$). Also the δ_i (and therefore also the $C\delta_i$) are assumed to be sorted in counterclockwise order. Moreover, we assume that we have the elements in a circular list, i.e., $G+1=1$.*

A test routine: **bool IsActive**$(C\delta_i, -n_j)$

(1) *IF* $\langle -n_j, \delta_i^{\circ}\rangle \geq 0$ *and* $\langle -n_j, \delta_i^C\rangle \geq 0$
then return TRUE;
(2) *else return FALSE.*

The main algorithm:

(1) $i := 1$;
(2) *WHILE NOT IsActive*$(C\delta_i, -n_1)$ $i := i+1$. (* *Find the active projections for* $-n_1$ *);
(3) *ActiveCones* $:= \{C\delta_i\}$;
(4) *IF* $(i=1)$ *AND IsActive*$(C\delta_G, -n_1)$;
then ActiveCones $:=$ *ActiveCones* $\cup \{C\delta_G\}$.
(5) *IF IsActive*$(C\delta_{i+1}, -n_1)$
then ActiveCones $:=$ *ActiveCones* $\cup \{C\delta_{i+1}\}$, $i := i+1$;
(6) *ActiveDirs*$(-n_1) :=$ *ActiveCones*;
(7) *FOR* $j := 2$ *TO L DO*
  (a) *FOR all cones* $C\delta \in$ *ActiveCones DO*
    (i) *IF NOT IsActive* $(C\delta, -n_j)$
    then ActiveCones $:=$ *ActiveCones*$\setminus\{C\delta\}$.
    (* *Note, that we have maximally* 2 *active cones* *);
  (b) *IF* $|ActiveCones| = 1$ *then*
  IF IsActive$(C\delta_{i+1}, -n_j)$
  then ActiveCones $:=$ *ActiveCones* $\cup \{C\delta_{i+1}\}$;
  (c) *IF ActiveCones* $= \varnothing$ *then*
    (i) *WHILE NOT IsActive*$(C\delta_i, -n_j)$ $i := i+1$;
    (ii) *ActiveCones* $:= \{C\delta_i\}$;
    (iii) *IF IsActive*$(C\delta_{i+1}, -n_1)$
    then ActiveCones $:=$ *ActiveCones* $\cup\{C\delta_{i+1}\}$, $i := i+1$;
  (d) *ActiveDirs*$(-n_j) :=$ *ActiveCones.*
(8) *FOR* $j := 1$ *TO* $L-1$
  (a) *ActiveDirs*$(p_j) :=$ *Cone*(*last*(*ActiveDirs*$(-n_j)$), *first*(*ActiveDirs*$(-n_{j+1})$)).
(9) *ActiveDirs*$(p_L) :=$ *Cone*(*last*(*ActiveDirs*$(-n_L)$), *first*(*ActiveDirs*$(-n_1)$)).
(10) *END*

The running time of the algorithm is $O(L+G)$ and the *ActiveDirs*$(-n_j)$ and *ActiveDirs*(p_j) contain the directions spanning the maximal linearity domains.

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