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THE CALOGERO–BOGOYAVLENSKII–SCHIFF EQUATION IN 2+1 DIMENSIONS

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We use the classical and nonclassical methods to obtain symmetry reductions and exact solutions of the (2+1)-dimensional integrable Calogero–Bogoyavlenskii–Schiff equation. Although this (2+1)-dimensional equation arises in a nonlocal form, it can be written as a system of differential equations and, in potential form, as a fourth-order partial differential equation. The classical and nonclassical methods yield some exact solutions of the (2+1)-dimensional equation that involve several arbitrary functions and hence exhibit a rich variety of qualitative behavior.

Keywords: partial differential equations, symmetries

1. Introduction

The study of multidimensional integrable systems is one of the main themes in integrable systems. Several integrable models have been recently developed in the context of (2+1)-dimensional equations.

In this paper, we discuss the (2+1)-dimensional integrable Calogero–Bogoyavlenskii–Schiff (CBS) equation

$$u_t + uu_z + \frac{1}{2}u_x\partial_x^{-1}u_z + \frac{1}{4}u_{xxz} = 0, \tag{1}$$

where $\partial_x^{-1}f = \int f\,dx$.

This equation was constructed by Bogoyavlenskii and Schiff in different ways. Namely, Bogoyavlenskii used the modified Lax formalism [1]–[3], whereas Schiff obtained the same equation by reducing the self-dual Yang–Mills equation [4]. In [4] and [5], it was shown that Eq. (1) is transformed into a trilinear form. In [1], it was shown that the (2+1)-dimensional equation written in the potential form

$$u_{tx} - 4u_xu_{xz} - 2u_{xx}u_z + u_{xxxz} = 0 \tag{2}$$

admits a Lax representation and is integrable by the one-dimensional inverse scattering transform. In [2], Bogoyavlenskii proved that an equation equivalent to Eq. (2) has an overturning soliton. In [3], several periodic and breaking solutions were constructed for (2) as well as for a modified equation related to Eq. (2) via the Miura transformation $v^2 \pm v_x = u_x$.

In [6], Toda and Yu constructed some new 2+1 integrable models using the Calogero method. In this method, the (2+1)-dimensional equations are derived considering the Lax pair L and T and modifying the T operator to include another spatial dimension z. They also thus derived the (2+1)-dimensional CBS equation from the Korteweg–de Vries equation.

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Although the (2+1)-dimensional CBS equation arises in a nonlocal form, it can be written as the system of differential equations

$$\begin{aligned} &v_x - u_z = 0, \\ &2u_x v + u_{xxz} + 4u_t + 4uu_z = 0, \end{aligned} \tag{3}$$

as well as in the potential form

$$4h_{tx} + 4h_x h_{xz} + 2h_{xx} h_z + h_{xxxz} = 0. \tag{4}$$

In this paper, we apply the classical Lie group method of infinitesimal transformations to system (3), as well as to the related potential equation (4). We point out that the classical Lie symmetries and similarity reduction for system (3) were derived in [7]. Nevertheless, by constructing invariant solutions from the optimal system, we bring out some similarity reductions that do not appear explicitly in [7]. We also obtain some new reduced systems of 1+1 partial differential equations (PDEs) and new systems of ordinary differential equations (ODEs).

By using these new reduced ODEs, we also obtain exact solutions for CBS equation (1). The most interesting solutions are the soliton solutions, localized on a curve and decaying exponentially away from the curve. Some of these solutions were derived by Bogoyavlenskii in [3]. As far as we know, these soliton solutions have not been derived before using symmetry methods.

2. Lie symmetries for the system

In this section, we perform Lie symmetry analysis for (2+1)-dimensional system (3). We consider a one-parameter Lie group of infinitesimal transformations in (x, z, t, u, v) given by

$$\begin{aligned} x^* &= x + \varepsilon X(x, z, t, u, v) + \mathcal{O}(\varepsilon^2), \\ z^* &= z + \varepsilon Z(x, z, t, u, v) + \mathcal{O}(\varepsilon^2), \\ t^* &= t + \varepsilon T(x, z, t, u, v) + \mathcal{O}(\varepsilon^2), \\ u^* &= u + \varepsilon U(x, z, t, u, v) + \mathcal{O}(\varepsilon^2), \\ v^* &= v + \varepsilon V(x, z, t, u, v) + \mathcal{O}(\varepsilon^2), \end{aligned} \tag{5}$$

where ε is the group parameter. We then require that this transformation leave the set of solutions of system (3) invariant. This leads to an overdetermined linear system of equations for the infinitesimals $X(x, z, t, u, v)$, $Z(x, z, t, u, v)$, $T(x, z, t, u, v)$, $U(x, z, t, u, v)$, and $V(x, z, t, u, v)$. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$\mathbf{v} = X\frac{\partial}{\partial x} + Z\frac{\partial}{\partial z} + T\frac{\partial}{\partial t} + U\frac{\partial}{\partial u} + V\frac{\partial}{\partial v}. \tag{6}$$

After the infinitesimals are determined, the symmetry variables are found by solving the invariant-surface conditions

$$\Phi_1 \equiv X\frac{\partial u}{\partial x} + Z\frac{\partial u}{\partial z} + T\frac{\partial u}{\partial t} - U = 0, \qquad \Phi_2 \equiv X\frac{\partial v}{\partial x} + Z\frac{\partial u}{\partial z} + T\frac{\partial v}{\partial t} - V = 0. \tag{7}$$

Applying the classical method to system (3) yields a system of equations that leads to a six-parameter Lie group. Associated with this Lie group, we have a Lie algebra that can be represented by the generators

$$\begin{aligned} &\mathbf{v}_1 = \frac{\partial}{\partial t}, \qquad \mathbf{v}_2 = \frac{\partial}{\partial z}, \qquad \mathbf{v}_3 = t\frac{\partial}{\partial z} + \frac{\partial}{\partial u}, \\ &\mathbf{v}_4 = t\frac{\partial}{\partial t} + z\frac{\partial}{\partial z} - v\frac{\partial}{\partial v}, \\ &\mathbf{v}_5 = tx\frac{\partial}{\partial x} + 2t^2\frac{\partial}{\partial t} + 2tz\frac{\partial}{\partial z} + (2z - 2tu)\frac{\partial}{\partial u} + (2x - 3tv)\frac{\partial}{\partial v}, \\ &\mathbf{v}_6 = x\frac{\partial}{\partial x} - 2z\frac{\partial}{\partial z} - 2u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} \end{aligned}$$

and the infinite-dimensional

$$\mathbf{v}_\alpha = \alpha(t)\frac{\partial}{\partial x} + 2\alpha'(t)\frac{\partial}{\partial v}.$$

3. Optimal systems and reductions

To construct the one-dimensional optimal system following Olver, we construct the commutator table (Table 1) and the adjoint table (Table 2), which shows the separate adjoint actions of each element in $\mathbf{v}_i$, $i = 1, \ldots, 6$, as it acts on all other elements. This construction is easily done by summing the Lie series.

Table 1

	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$	$\mathbf{v}_6$
$\mathbf{v}_1$	0	0	$\mathbf{v}_2$	$\mathbf{v}_1$	$\mathbf{v}_6 + 4\mathbf{v}_4$	0
$\mathbf{v}_2$	0	0	0	$\mathbf{v}_2$	$2\mathbf{v}_3$	$-2\mathbf{v}_2$
$\mathbf{v}_3$	$-\mathbf{v}_2$	0	0	0	0	$-2\mathbf{v}_3$
$\mathbf{v}_4$	$-\mathbf{v}_1$	$-\mathbf{v}_2$	0	0	$\mathbf{v}_5$	0
$\mathbf{v}_5$	$-(\mathbf{v}_6 + 4\mathbf{v}_4)$	$-2\mathbf{v}_3$	0	$-\mathbf{v}_5$	0	0
$\mathbf{v}_6$	0	$2\mathbf{v}_2$	$2\mathbf{v}_3$	0	0	0

Commutator table for the Lie algebra.

Table 2

	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$	$\mathbf{v}_6$
$\mathbf{v}_1$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3 - \epsilon\mathbf{v}_2$	$\mathbf{v}_4 - \epsilon\mathbf{v}_1$	$\mathbf{v}_5 - \epsilon(\mathbf{v}_6 + 4\mathbf{v}_4)$	$\mathbf{v}_6$
$\mathbf{v}_2$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4 - \epsilon\mathbf{v}_2$	$\mathbf{v}_5 - 2\epsilon\mathbf{v}_3$	$\mathbf{v}_6 + 2\epsilon\mathbf{v}_2$
$\mathbf{v}_3$	$\mathbf{v}_1 + \epsilon\mathbf{v}_2$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$	$\mathbf{v}_6 + 2\epsilon\mathbf{v}_3$
$\mathbf{v}_4$	$e^{\epsilon}\mathbf{v}_1$	$e^{\epsilon}\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$	$e^{-\epsilon}\mathbf{v}_5$	$\mathbf{v}_6$
$\mathbf{v}_5$	$\mathbf{v}_1 + \epsilon(\mathbf{v}_6 + 4\mathbf{v}_4)$	$\mathbf{v}_2 + 2\epsilon\mathbf{v}_3$	$\mathbf{v}_3$	$\mathbf{v}_4 + \epsilon\mathbf{v}_5$	$\mathbf{v}_5$	$\mathbf{v}_6$
$\mathbf{v}_6$	$\mathbf{v}_1$	$e^{-2\epsilon}\mathbf{v}_2$	$e^{-2\epsilon}\mathbf{v}_3$	$\mathbf{v}_4$	$\mathbf{v}_5$	$\mathbf{v}_6$

Adjoint table.

The corresponding generators of the optimal system of subalgebras are

$$\langle a\mathbf{v}_1 + \mathbf{v}_6\rangle, \qquad \langle \mathbf{v}_6\rangle, \qquad \langle \mathbf{v}_4 + b\mathbf{v}_6\rangle,$$

$$\langle a\mathbf{v}_1 + \mathbf{v}_3\rangle, \qquad \langle \mathbf{v}_3\rangle, \qquad \langle \mathbf{v}_1 + b\mathbf{v}_2\rangle,$$

where $a \in \mathbb{R}$, $a \neq 0$, and $b \in \mathbb{R}$ is arbitrary.

In the following, we list the corresponding similarity variables and similarity solutions as well as the systems of PDEs obtained when system (3) is reduced via the generators obtained by adding the infinite-dimensional generator $\mathbf{v}_\alpha$ to the generators of the optimal system.

Reduction 1. Using the generator $a\mathbf{v}_1 + \mathbf{v}_6 + \mathbf{v}_\alpha$, we obtain the similarity variables and similarity solutions

$$w_1 = e^{-t/a}x - \frac{1}{a}\int f(t)e^{-t/a}\,dt, \qquad w_2 = e^{2t/a}z,$$

$$u = e^{-2t/a}h_1(w_1, w_2), \qquad v = e^{t/a}\left(h_2(w_1, w_2) + \frac{2}{a}\int f'(t)e^{-t/a}\,dt\right)$$

and the systems of PDEs $\mathbf{S}_1$

$$h_{2w_1} - h_{1w_2} = 0,$$

$$8w_2h_{1w_2} - 4w_1h_{1w_1} + 2ah_2h_{1w_1} + 4ah_1h_{1w_2} + ah_{1w_1w_1w_2} - 8h_1 = 0.$$

Reduction 2. Using the generator $\mathbf{v}_6 + \mathbf{v}_\alpha$, we obtain the similarity variables and similarity solutions

$$w_1 = t, \qquad w_2 = z(x + \alpha(t))^2,$$

$$u = zh_1(w_1, w_2), \qquad v = z^{-1/2}h_2(w_1, w_2) - 2\alpha'(t)$$

and the systems of PDEs $\mathbf{S}_2$

$$w_2h_{1w_2} - 2\sqrt{w_2}h_{2w_2} + h_1 = 0,$$

$$2h_{1w_2w_2w_2}w_2^3 + 7h_{1w_2w_2}w_2^2 + 2h_1h_{1w_2}w_2^2 + \\ + 2h_{1w_2}h_2w_2^{3/2} + 2h_{1w_2}w_2 + 2h_{1w_1}w_2 + 2h_1^2w_2 = 0$$

with $x + \alpha(t) > 0$.

Reduction 3. Using the generator $\mathbf{v}_4 + b\mathbf{v}_6 + \mathbf{v}_\alpha$, we obtain the similarity variables and similarity solutions

$$w_1 = \frac{x}{t^b} - \int \frac{f(t)}{t^{b+1}}\,dt, \qquad w_2 = t^{2b-1}z,$$

$$u = \frac{1}{t^{2b}}h_1(w_1, w_2), \qquad v = \left(h_2(w_1, w_2) + 2\int \frac{\alpha'(t)}{t^b}\,dt\right)t^{b-1}$$

and the systems of PDEs $\mathbf{S}_3$

$$h_{2w_1} - h_{1w_2} = 0,$$

$$8bw_2h_{1w_2} - 4w_2h_{1w_2} - 4bw_1h_{1w_1} + 2h_2h_{1w_1} + 4h_1h_{1w_2} + h_{1w_1w_1w_2} - 8bh_1 = 0.$$

Reduction 4. Using the generator $a\mathbf{v}_1 + \mathbf{v}_3 + \mathbf{v}_\alpha$, we obtain the similarity variables and similarity solutions

$$w_1 = x - \frac{1}{a}\int \alpha(t)\,dt, \qquad w_2 = z - \frac{t^2}{2b},$$

$$u = \frac{t}{a} + h_1(w_1, w_2), \qquad v = \frac{2f(t)}{b} + h_2(w_1, w_2)$$

and the systems of PDEs $\mathbf{S}_4$

$$h_{2w_1} - h_{1w_2} = 0,$$

$$2ah_2h_{1w_1} + 4ah_1h_{1w_2} + ah_{1w_1w_1w_2} + 4 = 0.$$

Reduction 5. Using the generator $\mathbf{v}_3+\mathbf{v}_\alpha$, we obtain the similarity variables and similarity solutions

$$w_1 = t, \qquad w_2 = z - \frac{xt}{\alpha(t)},$$
$$u = \frac{z}{t} + h_1(w_1, w_2), \qquad v = \frac{2\alpha'(t)x}{\alpha(t)} + h_2(w_1, w_2)$$

and the systems of PDEs $\mathbf{S}_5$

$$-w_1^2 h_{2w_2} - \alpha h_{1w_2} w_1 + 2\alpha_{w_1} w_1 - \alpha = 0,$$
$$4\alpha^2 h_{1w_2} w_2 + h_{1w_2w_2w_2} w_1^3 - 2\alpha h_{1w_2} h_2 w_1^2 + 4\alpha^2 h_1 h_{1w_2} w_1 + 4\alpha^2 h_1 = 0.$$

Reduction 6. Using the generator $\mathbf{v}_\alpha$, we obtain the similarity variables and similarity solutions

$$w_1 = t, \qquad w_2 = z, \qquad u = h_1(w_1, w_2), \qquad v = 2x\frac{\alpha'(t)}{\alpha(t)} + h_2(w_1, w_2)$$

and the system of 1+1 PDEs $\mathbf{S}_6$

$$2\frac{\alpha_{w_1}}{\alpha} - h_{1w_2} = 0,$$
$$4h_1 h_{1w_2} + h_{1w_1} = 0.$$

Reduction 7. Using the generator $\mathbf{v}_1 + a\mathbf{v}_2 + \mathbf{v}_\alpha$, we obtain the similarity variables and similarity solutions

$$w_1 = x - \int \alpha(t)\,dt, \qquad w_2 = z - at, \qquad u = h_1(w_1, w_2), \qquad v = 2\alpha(t) + h_2(w_1, w_2)$$

and the system of PDEs $\mathbf{S}_7$

$$h_{2w_1} - h_{1w_2} = 0,$$
$$2h_2 h_{1w_1} + 4h_1 h_{1w_2} + h_{1w_1w_1w_2} - 4a h_{1w_2} = 0.$$

Although the classical symmetries for system (3) were derived in [7], some of the reductions do not appear explicitly in that work. Among these systems are $\mathbf{S}_2$, $\mathbf{S}_5$, and $\mathbf{S}_6$.

4. Further symmetries and exact solutions

In several cases, the reduced systems of 1+1 PDEs admit symmetries that lead to further reductions to systems of ODEs. We again use the techniques of Lie group theory. We consider the symmetries of systems $\mathbf{S}_2$ and $\mathbf{S}_7$ here.

System $\mathbf{S}_2$ admits the symmetries

$$\mathbf{v}_{21} = \frac{\partial}{\partial w_1},$$
$$\mathbf{v}_{22} = z_1^2\frac{\partial}{\partial z_1} + 2z_1 z_2\frac{\partial}{\partial z_2} + (1 - 2z_1 h_1)\frac{\partial}{\partial h_1} + (z_2^{1/2} - z_1 h_2)\frac{\partial}{\partial h_2},$$
$$\mathbf{v}_{23} = 2z_1\frac{\partial}{\partial z_1} + 2z_2\frac{\partial}{\partial z_2} - 2h_1\frac{\partial}{\partial h_1} - h_2\frac{\partial}{\partial h_2}.$$

Using $\mathbf{v}_{23}$, we obtain the similarity variable and similarity solutions

$$w = \frac{z_1}{z_2}, \qquad h_1 = \frac{1}{z_1} g(w), \qquad h_2 = \frac{1}{z_1^2} k(w)$$

and the system of ODEs

$$\begin{aligned} &wg' - g - 2w^{3/2}k' = 0, \\ &2w^4 g''' - 5w^3 g'' - 2w^{3/2} g' k - (2g - 2)wg' + 2g^2 - 2g = 0. \end{aligned}$$

We also consider system $\mathbf{S}_7$, which admits the symmetries

$$\begin{aligned} \mathbf{v}_{71} &= \frac{\partial}{\partial w_1}, \\ \mathbf{v}_{72} &= z_1 \frac{\partial}{\partial z_1} - 2(h_1 - a)\frac{\partial}{\partial h_1} - h_2 \frac{\partial}{\partial h_2}, \\ \mathbf{v}_\beta &= \beta(w_2)\frac{\partial}{\partial z_2} - \beta'(z_2) h_2 \frac{\partial}{\partial h_2}. \end{aligned}$$

Using $c\mathbf{v}_{72} + \mathbf{v}_\beta$, we obtain the similarity variable and similarity solutions

$$\begin{aligned} w &= w_1 \exp\left(-\int \frac{c}{\beta(w_2)}\, dw_2\right), \\ h_1 &= a + g(w)\exp\left(-2\int \frac{c}{\beta(w_2)}\, dw_2\right), \\ h_2 &= k(w)\frac{1}{\beta(w_2)} \exp\left(-\int \frac{c}{\beta(w_2)}\, dw_2\right) \end{aligned}$$

and the system of PDEs

$$\begin{aligned} &cg'w + k' + 2gc = 0, \\ &2kg' - cg'''w - 4cgg'w - 4cg'' - 8cg^2 = 0. \end{aligned}$$

Setting $k = 0$, we obtain the explicit solution

$$g(w) = k_1 e^{-w^2}, \qquad k(w) = 0.$$

The corresponding solution of the (2+1)-dimensional CBS equation is

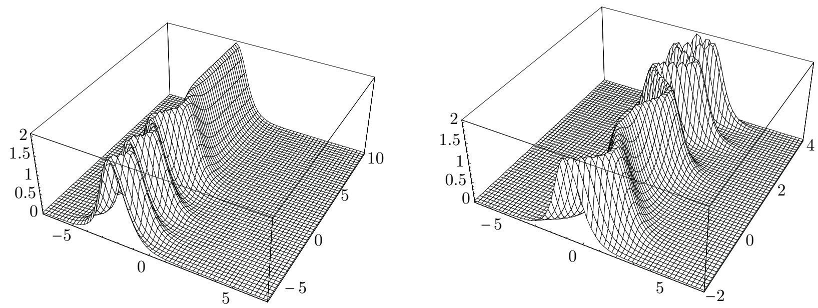
$$u = k_1 \exp\left(-2\delta(z - at) - \left(x - \int \alpha(t)\, dt\right)^2 e^{-2\delta(z-at)}\right),$$

where

$$\delta(z_2) = c\int \beta(z_2)^{-1}\, dz_2, \qquad z_2 = z - at.$$

Using $c\mathbf{v}_{71} + \mathbf{v}_\beta$, we obtain the similarity variable and similarity solutions

$$w = z_1 - c\int \frac{1}{\beta(z_2)}\, dz_2, \qquad h_1 = g(w), \qquad h_2 = \frac{1}{\beta(z_2)} k(w)$$



a b

Fig. 1

and the system of ODEs

$$cg' + k' = 0,$$

$$2kg' - cg''' - 4cgg' + 4acg' = 0.$$

This system can be transformed into the second-order ODE

$$k_2 - cg'' - 3cg^2 + 2k_1 g + 4acg = 0$$

and

$$k = -cg + k_1.$$

Setting $k_2 = 0$, we obtain the explicit solution

$$g = k_3 \operatorname{sech}^2\left(\sqrt{\frac{k_3}{2}}w\right).$$

The corresponding solution of (2+1)-dimensional CBS equation (1) is

$$u = k_3 \operatorname{sech}^2\left(\sqrt{\frac{k_3}{2}}(x - \varphi(t) - \delta(z - at))\right), \tag{8}$$

where

$$k_3 = \frac{k_1 + 2ac}{c}, \qquad \varphi(t) = \int \alpha(t)\,dt, \qquad \delta(z_2) = c\int \beta(z_2)^{-1}\,dz_2, \qquad z_2 = z - at.$$

We note that the arbitrary functions $\varphi(t)$ and $\delta(z - at)$ in the soliton solutions given by (8) allows a wide variety of qualitative and physical behaviors for these solutions. In particular, these solutions given by (8) are localized on the curve $x + \varphi(t) + \delta(z - at) = 0$ and decay exponentially away from the curve. Solutions (8) with $\delta(z - at) = \mathrm{Ai}(z)$, where Ai is the Airy function, and with $\delta(z - at) = \sin(z^2)$ are respectively plotted in Figs. 1a and 1b.

Choosing $\varphi(t)$ as a constant and $a = 0$ in (8), we obtain stationary solutions. We can find coherent structures by setting $\delta(z - at) = 0$ and $\varphi(t) = t$.

5. Lie symmetries for the potential CBS equation

To study the invariance properties, we consider the CBS equation written in the potential form

$$4h_{tx} + 4h_x h_{xz} + 2h_{xx}h_z + h_{xxxz} = 0. \tag{9}$$

We consider the one-parameter Lie group of infinitesimal transformations in (x, z, t, h) given by

$$\begin{aligned} x^* &= x + \varepsilon X(x, z, t, h) + \mathcal{O}(\varepsilon^2), \\ z^* &= z + \varepsilon Z(x, z, t, h) + \mathcal{O}(\varepsilon^2), \\ t^* &= t + \varepsilon T(x, z, t, h) + \mathcal{O}(\varepsilon^2), \\ h^* &= h + \varepsilon H(x, z, t, h) + \mathcal{O}(\varepsilon^2), \end{aligned} \tag{10}$$

where ε is the group parameter. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$\mathbf{v} = X\frac{\partial}{\partial x} + Z\frac{\partial}{\partial z} + T\frac{\partial}{\partial t} + H\frac{\partial}{\partial h}. \tag{11}$$

After the infinitesimals are determined, the symmetry variables are found by solving the invariant-surface condition

$$\Phi \equiv X\frac{\partial h}{\partial x} + Z\frac{\partial h}{\partial z} + T\frac{\partial h}{\partial t} - H = 0. \tag{12}$$

Applying the classical method to Eq. (9) yields a system of equations that leads to a six-parameter Lie group. Associated with this Lie group, we have a Lie algebra that can be represented by the generators

$$\begin{aligned} &\mathbf{v}_1 = \frac{\partial}{\partial t}, && \mathbf{v}_2 = \frac{\partial}{\partial z}, \\ &\mathbf{v}_3 = x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} + t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}, && \mathbf{v}_4 = 2tx\frac{\partial}{\partial x} + 2tz\frac{\partial}{\partial t} + t^2\frac{\partial}{\partial z} + (4xz - 2tu)\frac{\partial}{\partial u}, \\ &\mathbf{v}_5 = 2t\frac{\partial}{\partial t} + 4x\frac{\partial}{\partial u}, && \mathbf{v}_6 = -x\frac{\partial}{\partial x} + z\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} \end{aligned}$$

and the infinite-dimensional vector fields of the form

$$\mathbf{v}_\alpha = \alpha(t)\frac{\partial}{\partial x} + 2(\alpha'(t)z)\frac{\partial}{\partial u}, \qquad \mathbf{v}_\beta = \beta(t)\frac{\partial}{\partial u}.$$

Our aim is to apply the theory of symmetry reductions to find traveling-wave solutions of the 2+1 CBS equation. For this, we consider the following reduction arising from translations and the infinite-dimensional vector field, i.e., $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_\alpha$.

Reduction. Using the generator $\mu\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_\alpha$, we obtain the similarity variables and similarity solutions

$$z_1 = x - \int \alpha(t)\,dt, \qquad z_2 = z - \mu t, \qquad h = 2\alpha'(t)z + f(z_1, z_2)$$

and the PDE

$$4\mu f_{z_1z_2} - 2f_{z_1z_1}f_{z_2} - 4f_{z_1}f_{z_1z_2} + f^2 f_{z_1z_1z_1z_2} = 0. \tag{13}$$

6. Further reductions to ODEs and exact solutions

The reduced PDE of 1+1 dimensions admits symmetries with arbitrary functions that lead to further reductions to an ODE. The corresponding solutions of the (2+1)-dimensional equation involve up to three arbitrary smooth functions.

Equation (13) admits the symmetries

$$\mathbf{v}_1 = \frac{\partial}{\partial z_1}, \qquad \mathbf{v}_2 = \frac{\partial}{\partial f}, \qquad \mathbf{v}_3 = z_1\frac{\partial}{\partial z_1} + (2\mu z_1 - f)\frac{\partial}{\partial f}, \qquad \mathbf{v}_\beta = \beta(z_2)\frac{\partial}{\partial z_2}.$$

Using $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_\beta$, we obtain the similarity variable and similarity solutions

$$w = z_1 - \int \frac{dz_2}{\beta(z_2)}, \qquad f = z_1 + g$$

and the ODE

$$g^{(\mathrm{IV})} + 6g'g'' + 4(1-\mu)g'' = 0.$$

Integrating once with respect to w and using $y = g'$, we obtain the second-order ODE

$$y'' + 3y^2 + 4(1-\mu)y - c = 0.$$

The solutions can be written in terms of the elliptic functions.

Using $\mathbf{v}_3 + \mathbf{v}_\beta$, we obtain the similarity variable and similarity solutions

$$w = z_1 \exp\biggl(-\int \frac{dz_2}{\beta(z_2)}\biggr), \qquad f = \mu z_1 + \frac{1}{z_1}g$$

and the ODE

$$w^3 g^{(\mathrm{IV})} + 6w^2 g'g'' - 4wgg'' - 4w(g')^2 + 4gg' = 0.$$

Integrating once with respect to w, we obtain the third-order ODE

$$w^2 g''' - wg'' + 3w(g')^2 - 4gg' + kw = 0.$$

A particular solution is $g = \sqrt{k}w$.

7. Nonclassical symmetries of the potential CBS equation

To apply the nonclassical method to (2+1)-dimensional potential equation (9), we consider the one-parameter Lie group of infinitesimal transformations in (x, t, z, h) given by (10) with the associated set of vector fields (11). After the infinitesimals are determined, the symmetry variables are found by solving invariant-surface condition (12).

We then require that this transformation leave the subset of solutions of Eqs. (9) and (12) invariant. This leads to an overdetermined system of nonlinear equations for the infinitesimals $X(x, z, t, h)$, $Z(x, z, t, h)$, $T(x, z, t, h)$, and $H(x, z, t, h)$. Applying the nonclassical method to (9) yields a system of 17 equations that lead to

$$X = \alpha(t)x + \beta(t), \qquad Z = \eta(z, t), \qquad T = 1,$$

$$\phi = \gamma + \beta(4\alpha z + 2\eta) + 2\beta' z + (\eta\eta_z + \eta_t + 2\alpha\eta)x - \alpha u,$$

where $\alpha(t)$, $\beta(t)$, $\gamma(t)$, and $\eta = \eta(z, t)$ are related by the conditions

$$\eta\eta_{zz} + (\eta_z)^2 + \eta_{tz} - 2\alpha' - 4\alpha^2 = 0,$$

$$\eta(\eta_z)^2 + 2\eta_t\eta_z + 4\alpha\eta\eta_z + \eta_{tt} + \eta\eta_{tz} + 4\alpha\eta_t + 2\alpha'\eta + 4\alpha^2\eta = 0.$$

Integrating once with respect to z allows the first condition to be written as

$$(-2\alpha' - 4\alpha^2)z + \eta\eta_z + \eta_t + \delta = 0$$

with $\delta = \delta(t)$. For $\alpha = 0$ and $\delta = 0$, these conditions are satisfied if $\eta\eta_z + \eta_t = 0$. Choosing $\beta = 0$ leads to the additional reduction

$$z_1 = x, \qquad z_2 = a(z,t), \qquad h = f(z_1, z_2) + d(z,t), \tag{14}$$

and the (1+1)-dimensional PDE

$$2\psi(z_2) f_{z_1 z_1} + 4\phi(z_2) f_{z_1 z_2} + 2 f_{z_1 z_1} f_{z_2} + 4 f_{z_1} f_{z_1 z_2} + f^2 f_{z_1 z_1 z_1 z_2} = 0$$

with $z_{2t} = \phi(z_2) z_{2z}$ and $d_z = \psi(z_2) z_{2z}$. For $\psi = -2\phi$, this equation is invariant under translations and leads to the ODE

$$g^{(\mathrm{IV})} + 6g'g'' = 0$$

with

$$w = z_1 + z_2, \qquad f = g(w). \tag{15}$$

Integrating once with respect to w and setting $g' = y$ yields the second-order ODE

$$y'' + 3y^2 = k_1.$$

The solutions of this equation can be written in terms of the elliptic functions. A particular solution for $k1 = 3k^2$ is

$$y = 3k \operatorname{sech}^2\left(\sqrt{\frac{3k}{2}} w\right) - k.$$

Considering the symmetry reductions (14) and (15), we obtain the corresponding exact solution of potential equation (9)

$$h = (6k)^{1/2} \tanh\left(\sqrt{\frac{3k}{2}}(x + a(z,t))\right) - k(x + a(z,t)) + d(z,t), \tag{16}$$

where $\phi = \phi(a)$ is any function such that

$$a_t = \phi(a) a_z, \qquad d_z = -2a_t.$$

The breaking solutions derived by Bogoyavlenskii in [3] are

$$h = -c\lambda \tanh\left(\frac{c}{2}(\lambda x - \varphi)\right)$$

with $\lambda = \lambda(z,t)$, $\varphi = \varphi(z,t)$, and $\lambda_t = \alpha\lambda^2\lambda_z$, $\varphi_t = \alpha\lambda^2\varphi_z$, $\alpha = -c^2$.

The corresponding solution of (2+1)-dimensional CBS equation (1) derived from (16) is

$$u = 3k \operatorname{sech}^2\left(\sqrt{\frac{3k}{2}}(x + a(z,t))\right) + k_1,$$

where $a = a(z,t)$ satisfies $a_t = \phi(a) a_z$ and k_1 is an arbitrary constant.

8. Conclusions

We have discussed symmetry reductions and exact solutions of the (2+1)-dimensional integrable CBS equation. This (2+1)-dimensional equation has been written in a local form as a system of PDEs, Eqs. (3), and as a fourth-order PDE, Eq. (9). Although the classical symmetries for (3) were derived in [7], we have derived some new reductions from the optimal system of subalgebras. We have also applied the classical and nonclassical methods to potential equation (9). These reductions yield some exact solutions of the (2+1)-dimensional equation, which have a rich variety of qualitative behaviors because of the freedom in choosing the arbitrary functions $\varphi(t)$ and $\delta(z - at)$. We have obtained soliton solutions. Because these arbitrary functions are included in single-soliton solution (8), the solution is localized on a curve, and the curve can have quite a free form. Some of these solutions were derived in [3] by Bogoyavlenskii.

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