Math. Ann. 327, 117–134 (2003) **Mathematische Annalen**

DOI: 10.1007/s00208-003-0442-9

Adjoints of linear fractional composition operators on the Dirichlet space

Eva A. Gallardo-Gutiérrez · Alfonso Montes-Rodríguez

Received: 3 April 2002 / Revised version: 8 October 2002 /
Published online: 16 May 2003 –

Abstract. The adjoint of a linear fractional composition operator acting on the classical Dirichlet space is expressed as another linear fractional composition operator plus a two rank operator. The key point is that, in the Dirichlet space modulo constant functions, many linear fractional composition operators are similar to multiplication operators and, thus, normal. As a particular application, we can easily deduce the spectrum of each linear fractional composition operator acting on such spaces. Even the norm of each linear fractional composition operator is computed on the Dirichlet space modulo constant functions. It is also shown that all this work can be carried out in the Hardy space of the upper half plane.

Mathematics Subject Classification (2000): 47B38

1. Introduction

Let $\mathbb{D}$ denote the open unit disk of the complex plane and $A(z)$ the normalized Lebesgue area measure of the unit disk. The Dirichlet space $\mathcal{D}$ is the Hilbert space of functions f analytic on $\mathbb{D}$ for which the norm

$$\|f\|_{\mathcal{D}}^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \, dA(z)$$

is finite. Observe that the integral above is just the area of the image of $\mathbb{D}$ under f, counting multiplicity. The term $|f(0)|^2$ avoids that constant functions have norm zero.

E.A. Gallardo-Gutiérrez
Departamento de Matemáticas, Universidad de Cádiz, Apartado 40 Puerto Real (Cádiz) 11510, SPAIN (e-mail: eva.gallardo@uca.es)

A. Montes-Rodríguez
Departamento de Análisis Matemático, Universidad de Sevilla, aptdo. 1160, Avenida Reina Mercedes, Sevilla 41080, SPAIN (e-mail: amontes@us.es)

This work was partially supported by Plan Nacional I+D Ref. BFM2000-0360 and Junta de Andalucía Ref. FQM-260. The first named author was also supported by Plan Propio de la Universidad de Cádiz.

If φ is an analytic function on $\mathbb{D}$ with $\varphi(\mathbb{D}) \subset \mathbb{D}$, then the composition operator induced by φ is defined by

$$C_\varphi f = f \circ \varphi,$$

for $f \in \mathcal{D}$. A necessary condition for C_φ to act boundedly on $\mathcal{D}$ is that $C_\varphi z = \varphi$ belongs to $\mathcal{D}$. However, it is not difficult to construct an unbounded composition operator C_φ on $\mathcal{D}$ with $\varphi \in \mathcal{D}$. In this work, we deal with linear fractional composition operators on $\mathcal{D}$, that is, composition operators C_φ induced by linear fractional maps of $\mathbb{D}$

$$\varphi(z) = \frac{az + b}{cz + d}$$

such that $ad - bc \neq 0$ and $\varphi(\mathbb{D}) \subset \mathbb{D}$. Since linear fractional maps are univalent functions, they always induce bounded composition operators on the Dirichlet space.

In 1988 Cowen [4] (see also [6, Chap. 9]) proved that the adjoints of linear fractional composition operators on the Hardy space can be expressed as products of Toeplitz and linear fractional composition operators. Cowen used his theorem to compute the norm of C_{sz+t} whenever $|s|+|t| \leq 1$ and $|t| < 1$. Cowen's theorem was extended by Hurst [10] to weighted Bergman spaces. For a weighted Hardy space $\mathcal{H}^2(\beta)$ the adjoint is the product of a composition operator by a Toeplitz operator but acting on a different space (see [10], Thm. 5). Appel, Bourdon and Thrall [2] have studied the norm of composition operators in the Hardy space. In the paper by Vukotić [15], it is shown that the norm of univalently induced composition operators in the Bergman space can be computed using a geometrical method.

The aim of this note is to find the adjoints of linear fractional composition operators on $\mathcal{D}$. They will be expressed as a linear fractional composition operator plus a two rank operator acting on $\mathcal{D}$ itself. The proof has to be different to that of the Hardy space. The key point is that many linear fractional composition operators induce normal composition operators on the Dirichlet space modulo constant functions (recall that an operator T on a Hilbert space is said to be normal if T commutes with its adjoint). This is a striking fact if we compare with the situation in the Hardy space, in which C_φ is normal if and only if $\varphi(z) = \lambda z$ with $|\lambda| \leq 1$ (see [14] or [6, Chap. 8]). Normal operators are one of the best understood classes of operators because they are unitarily similar to multiplication operators on Hilbert spaces of measurable functions (see Conway's book [3, Chap. IX], for instance). We will also provide a direct proof of which are the multiplication operators unitarily similar to the normal linear fractional composition operators.

As an application of our results, we will give an explicit formula for the norm of each linear fractional composition operator on the Dirichlet space modulo constant functions. Moreover, we can easily find the spectrum and the essential spectrum of such operators acting on $\mathcal{D}$. The spectrum was computed, in a different way, by Higdon [9]. In the case that the symbol φ is a linear fractional map with exactly one fixed point of multiplicity one on $\partial\mathbb{D}$, the spectrum of C_φ was previously found in [10]. A complete characterization of the spectrum of linear fractional composition operators acting on the classical Hardy space of the unit disk was previously obtained by Cowen [5]. The methods we will use here to find the spectrum of C_φ are rather different from Cowen's.

In the last section we will show how all the work can also be done on the Hardy space of the upper half plane $\mathcal{H}^2(\Pi)$. This time the adjoint of a linear fractional composition operator will be a scalar multiple of another linear fractional composition operator. In particular, we also characterize which linear fractional composition operators on $\mathcal{H}^2(\Pi)$ are normal. As a particular application, we will find the spectra of these operators on $\mathcal{H}^2(\Pi)$.

2. Preliminaries

In this section we introduce the space $\mathcal{D}_0$, where part of our work is set, and some basic properties of linear fractional maps which will make the paper more readable.

2.1. The Dirichlet space modulo constant functions

All the questions we treat here become easier if we ignore the constant functions. Let $\mathcal{D}_0$ be the space that consists of functions in the Dirichlet space $\mathcal{D}$ modulo constant functions. In this case,

$$\|f\|_{\mathcal{D}_0}^2 = \int_{\mathbb{D}} |f'(z)|^2 \, dA(z)$$

becomes a norm in $\mathcal{D}_0$. We can also think of $\mathcal{D}_0$ as the subspace of Dirichlet functions that vanish at the origin.

Since constant functions are invariant under any bounded composition operator C_φ on $\mathcal{D}$, the operator

$$\widetilde{C}_\varphi f = C_\varphi f - (C_\varphi f)(0)$$

takes boundedly $\mathcal{D}_0$ into itself. The operator $\widetilde{C}_\varphi$ can also be seen as the compression of C_φ to $\mathcal{D}_0$. Since there is no risk of confusion, we still denote $\widetilde{C}_\varphi$ by C_φ.

2.2. Linear fractional maps

If a, b, c, and d are complex numbers with $ad - bc \neq 0$, then the linear fractional map

$$\varphi(z) = \frac{az + b}{cz + d}$$

is a one-to-one map from the extended complex plane $\mathbb{C}^\infty = \mathbb{C} \cup \{\infty\}$ onto itself. Indeed, it is enough to define $\varphi(\infty) = a/c$, and $\varphi(-d/c) = \infty$ if $c \neq 0$, while $\varphi(\infty) = \infty$ if $c = 0$.

A linear fractional map which is not the identity has one or two fixed points in the extended complex plane. Two linear fractional maps φ and ψ are said to be conjugate if there is another linear fractional map T such that $\varphi = T^{-1}\psi T$. If φ has only one fixed point α, then it is called parabolic and it is conjugate under $Tz = 1/(z-\alpha)$ to $\psi(z) = z + \tau$ with $\tau \neq 0$. Observe that the derivative at the fixed point is 1.

If φ has two distinct fixed points α and β, then φ is conjugate under $Tz = (z-\alpha)/(z-\beta)$ to $\psi(z) = \mu z$. In this case, the linear fractional map is called elliptic if $|\mu| = 1$; hyperbolic if $\mu > 0$ and loxodromic, otherwise (see [1] for more details). It is not difficult to show that the derivative at the fixed points satisfy $\varphi'(\alpha) = 1/\varphi'(\beta)$. For φ loxodromic or hyperbolic the *attractive* fixed point of φ is the one for which the modulus of the derivative is strictly less than one.

A necessary condition for C_φ to be defined is that φ must take $\mathbb{D}$ into itself. This fact imposes some restrictions on the location of the fixed points of φ. One can easily show that if $\varphi(\mathbb{D}) \subset \mathbb{D}$, then

(a) If φ is parabolic, then its fixed point is on $\partial\mathbb{D}$.
(b) If φ is hyperbolic, the attractive fixed point is in $\overline{\mathbb{D}}$ and the other fixed point outside of $\mathbb{D}$ and both fixed points are on $\partial\mathbb{D}$ if and only if φ is an automorphism of $\mathbb{D}$.
(c) If φ is loxodromic or elliptic, one fixed point is in $\mathbb{D}$ and the other fixed point outside of $\overline{\mathbb{D}}$. If φ is elliptic, then it is an automorphism of $\mathbb{D}$. If φ is loxodromic, the attractive fixed point must be in $\mathbb{D}$.

3. Adjoints of linear fractional composition operators

In this section we will find the adjoint of each linear fractional composition operator acting on $\mathcal{D}_0$ as well as on $\mathcal{D}$.

First of all, we will prove the following easy theorem in which unitary composition operators on $\mathcal{D}_0$ are characterized. Recall that an invertible operator is unitary if the adjoint $T^\star$ equals to T^{-1}.

Theorem 3.1. *Let φ be a holomorphic self-map of $\mathbb{D}$. Then C_φ acting on $\mathcal{D}_0$ is a unitary operator if and only if φ is an automorphism of $\mathbb{D}$.*

Proof. Since only composition operators induced by automorphisms are invertible [6, Thm. 1.6], the condition is clearly necessary. Now, let φ be an automorphism of $\mathbb{D}$. For $f, g \in \mathcal{D}_0$ the change of variables $w = \varphi(z)$ in the third equality below yields

$$
\begin{aligned}
\langle C_\varphi^\star f, g\rangle &= \langle f, C_\varphi g\rangle \\
&= \int_{\mathbb{D}} f'(z)\overline{g'(\varphi(z))\varphi'(z)}\, dA(z) \\
&= \int_{\mathbb{D}} f'(\varphi^{-1}(z))(\varphi^{-1}(z))'\overline{g'(z)}\, dA(z) \\
&= \langle C_{\varphi^{-1}} f, g\rangle.
\end{aligned}
$$

Thus $C_\varphi^\star = C_{\varphi^{-1}} = C_\varphi^{-1}$ and the result follows. □

Next theorem says that the adjoint of a linear fractional composition operator on $\mathcal{D}_0$ is another linear fractional composition operator. Unlike the adjoint in the Hardy space, Toeplitz operators do not appear (see [4]). The argument below is different to the one of the Hardy space.

Theorem 3.2. *Let $\varphi(z) = (az + b)(cz + d)^{-1}$ be a linear fractional self map of $\mathbb{D}$ and consider C_φ acting on $\mathcal{D}_0$. Then $C_\varphi^\star = C_\psi$, where $\psi(z) = (\bar{a}z - \bar{c})(-\bar{b}z + \bar{d})^{-1}$.*

Proof. Since φ takes $\mathbb{D}$ into itself, then d must be different from zero. So we can set $p = \varphi(0) = b/d$ that belongs to $\mathbb{D}$. Now, we consider the automorphism of the unit disk

$$\alpha_p(z) = \frac{p - z}{1 - \bar{p}z}$$

that interchanges p with the origin and satisfies $\alpha_p^{-1} = \alpha_p$. In this way, $\phi = \alpha_p \circ \varphi$ fixes the origin. Indeed,

$$\phi(z) = \frac{\alpha z}{\gamma z + \delta},$$

where

$$\alpha = (bc - ad)\bar{d}; \qquad \gamma = (c\bar{d} - \bar{b}a)d \quad \text{and} \quad \delta = (|d|^2 - |b|^2)d.$$

We will compute the adjoint of C_ϕ. Let $f(z) = \sum_{k=1}^{\infty} a_k z^k$ be any function in $\mathcal{D}_0$. For each positive integer we set $u_n(z) = z^n/\sqrt{n}$. We have

$$
\begin{aligned}
\langle C_\phi^\star f, u_n\rangle &= \langle f, C_\phi u_n\rangle \\
&= \langle f, u_n \circ \phi\rangle \\
&= \frac{1}{\sqrt{n}}\langle f, \phi^n\rangle \\
&= \frac{1}{\sqrt{n}} \int_{\mathbb{D}} f'(z)\overline{(\phi^n(z))'}\, dA(z).
\end{aligned}
$$

We write

$$g(z) = (\phi^n(z))' = \frac{n\delta\alpha^n z^{n-1}}{(\gamma z + \delta)^{n+1}} = \sum_{k=1}^{\infty} b_k z^{k-1}.$$

The orthogonality of the monomials with respect to $dA(z) = r dr d\theta/\pi$ allows us to compute

$$\begin{aligned}\frac{1}{\sqrt{n}}\int_{\mathbb{D}} f'(z)\overline{(\phi^n(z))'}\,dA(z) &= \frac{1}{\sqrt{n}}\int_{\mathbb{D}} f'(z)\overline{g(z)}\,dA(z)\\ &= \frac{2}{\sqrt{n}}\int_0^1 \sum_{k=1}^{\infty} k a_k \bar{b}_k r^{2k-1}\,dr\\ &= \frac{1}{\sqrt{n}}\sum_{k=1}^{\infty} a_k\bar{b}_k. \qquad (3.1)\end{aligned}$$

We can recover the integral with respect to θ and perform the change of variables $z = e^{i\theta}$. We obtain that (3.1) equals to

$$\begin{aligned}\frac{1}{2\pi\sqrt{n}}\int_{-\pi}^{\pi}\frac{f(e^{i\theta})}{e^{i\theta}}\overline{g(e^{i\theta})}\,d\theta &= \frac{n}{2\pi i\sqrt{n}}\int_{|z|=1}\frac{f(z)}{z^2}\frac{\bar{\delta}\bar{\alpha}^n\bar{z}^{n-1}}{(\bar{\gamma}\bar{z}+\bar{\delta})^{n+1}}\,dz\\ &= \frac{\sqrt{n}}{2\pi i}\int_{|z|=1}\frac{f(z)\bar{\delta}\bar{\alpha}^n}{(\bar{\gamma}+\bar{\delta}z)^{n+1}}\,dz. \qquad (3.2)\end{aligned}$$

Consider $\eta = \psi \circ \alpha_p$, that is, $\eta(z) = (\bar{\alpha}/\bar{\delta})z - \bar{\gamma}/\bar{\delta}$ that also applies $\mathbb{D}$ into itself. Using the Cauchy integral formula for the n-th derivative we see that (3.2) equals to

$$\frac{\sqrt{n}}{n!}\frac{\bar{\alpha}^n}{\bar{\delta}^n}f^{n)}\left(-\frac{\bar{\gamma}}{\bar{\delta}}\right) = \frac{\sqrt{n}}{n!}(f\circ\eta)^{n)}(0).$$

Observe that the last quantity is the n-th coefficient in the Taylor development of $f \circ \eta$ around the origin and multiplied by $\sqrt{n}$. Thus, we have

$$\langle C_\phi^\star f, u_n\rangle = \sqrt{n}\langle f\circ\eta, z^n/n\rangle = \langle f\circ\eta, z^n/\sqrt{n}\rangle = \langle C_\eta f, u_n\rangle.$$

Therefore $C_\phi^\star = C_\eta$. Since $C_\varphi = C_\phi C_{\alpha_p}$, we may apply Theorem 3.1 in the second equality below

$$C_\varphi^\star = C_{\alpha_p}^\star C_\phi^\star = C_{\alpha_p^{-1}} C_\eta = C_{\alpha_p} C_\eta = C_{\eta\circ\alpha_p} = C_\psi.$$

The proof is finished. □

Remark 3.1. Some of the arguments in the proof above are also used in [7, Chap. 2], to obtain an expression for $C_\varphi^\star$ acting on the Bergman space for φ with an interior and a boundary fixed point. However, in [7] the operator $C_\varphi^\star$ is not expressed as an operator acting on the Bergman space itself.

Theorem 3.2 will allow us to find the adjoints of linear fractional composition operators on the Dirichlet space. First recall that, for each $w \in \mathbb{D}$, the function $K_w(z) = 1 + \log(1-\bar{w}z)^{-1}$ is the reproducing kernel at w in the Dirichlet space, that is, for $f \in \mathcal{D}$ we have $\langle f, K_w\rangle = f(w)$. If φ is an analytic self map of $\mathbb{D}$, then it is easy to see how $C_\varphi^\star$ acts on the reproducing kernels. Indeed, for $f \in \mathcal{D}$, we have

$$\langle C_\varphi^\star K_w, f\rangle = \langle K_w, C_\varphi f\rangle = \langle K_w, f \circ \varphi\rangle = f(\varphi(w)) = \langle K_{\varphi(w)}, f\rangle.$$

Thus $C_\varphi^\star K_w = K_{\varphi(w)}$. We have

Theorem 3.3. *Let $\varphi(z) = (az + b)(cz + d)^{-1}$ be a linear fractional self map of $\mathbb{D}$. Then, for $f \in \mathcal{D}$ we have*

$$C_\varphi^\star f = f(0)K_{\varphi(0)} + C_\psi f - f(\psi(0)),$$

where $\psi(z) = (\bar{a}z - \bar{c})(-\bar{b}z + \bar{d})^{-1}$.

Proof. For all $f, g \in \mathcal{D}$ we have

$$\begin{aligned}\langle C_\varphi^\star f, g\rangle &= \langle f(0)C_\varphi^\star K_0 + C_\varphi^\star(f - f(0)), g\rangle \\ &= \langle f(0)K_{\varphi(0)}, g\rangle + \langle C_\varphi^\star(f - f(0)), g\rangle.\end{aligned}$$

Now, upon applying Theorem 3.2 in the third equality below

$$\begin{aligned}\langle C_\varphi^\star(f - f(0)), g\rangle &= \langle f - f(0), C_\varphi g\rangle \\ &= \langle f - f(0), C_\varphi g\rangle_{\mathcal{D}_0} \\ &= \langle C_\psi(f - f(0)), g\rangle_{\mathcal{D}_0} \\ &= \langle C_\psi(f - f(0)), g\rangle - \langle (C_\psi(f - f(0)))(0), g\rangle \\ &= \langle C_\psi f - f(\psi(0)), g\rangle.\end{aligned}$$

Therefore, the result follows. □

The following corollary follows immediately from the above theorem.

Corollary 3.4. *Let φ be an analytic self-map of $\mathbb{D}$. Then C_φ is unitary on $\mathcal{D}$ if and only if $\varphi(z) = \mu z$ and $|\mu| = 1$.*

4. Normal linear fractional composition operators

In this section we characterize which linear fractional composition operators are normal on $\mathcal{D}_0$ and $\mathcal{D}$. Unitary composition operators are normal, but they are not the only ones on $\mathcal{D}_0$. We have

Theorem 4.1. *A linear fractional composition operator C_φ is normal on $\mathcal{D}_0$ if and only if one of the following holds*

(a) *The symbol φ is an automorphism.*
(b) *The symbol φ is parabolic.*

(c) *The symbol φ has an interior and an exterior fixed point and φ is conjugate to $z \to \mu z$ with* $0 < |\mu| < 1$.

Proof. Clearly, Theorem 3.1 implies that if (a) is true, then C_φ is normal. Suppose that φ is parabolic. The unique fixed point α of a parabolic self map of the disk must be on $\partial\mathbb{D}$. Upon conjugating with $\eta(z) = \alpha z$, we have $\eta^{-1} \circ \varphi \circ \eta$ fixes the point 1. Since $C_{\eta\circ\varphi\circ\eta^{-1}} = C_{\eta^{-1}} C_\varphi C_\eta$ and normality is preserved under unitary similarities, we may suppose that φ fixes the point 1 from the beginning.

Now, conjugating with $\sigma(z) = i(1+z)/(1-z)$, we see that the upper half plane version of φ is

$$\tau(z) = z + a,$$

where $\Im a \geq 0$ (of course, if $\Im a = 0$, then φ is an automorphism and we already know that C_φ is normal). Coming back to the unit circle

$$\varphi(z) = \frac{(2-a)z + a}{-az + 2 + a}.$$

By Theorem 3.2, $C_\varphi^\star = C_\psi$, where

$$\psi(z) = \frac{(2-\bar{a})z + \bar{a}}{-\bar{a}z + 2 + \bar{a}}.$$

It is easy to check that $\varphi \circ \psi = \psi \circ \varphi$. Thus,

$$C_\varphi C_\varphi^\star = C_\varphi C_\psi = C_{\psi\circ\varphi} = C_{\varphi\circ\psi} = C_\psi C_\varphi = C_\varphi^\star C_\varphi,$$

which means that C_φ is normal.

Suppose, now, that φ has an interior fixed point q (this configuration includes the hyperbolic non automorphism with an interior and a boundary fixed point). Consider the involutive automorphism α_q that interchanges q with the origin. By Theorem 3.1, C_{α_q} is a unitary operator. Hence, C_φ is normal if and only if $C_{\alpha_q}^{-1} C_\varphi C_{\alpha_q} = C_{\alpha_q\circ\varphi\circ\alpha_q}$ is normal. Thus we may suppose from the beginning that φ fixes the origin. In addition, φ has another fixed point p.

Case 1. $p = \infty$. In this case φ must be of the form

$$\varphi(z) = \mu z \qquad \text{with } 0 < |\mu| \leq 1.$$

By Theorem 3.2, $C_\varphi^\star = C_\psi$, where $\psi(z) = \bar{\mu} z$. Obviously C_φ commutes with C_ψ. Therefore, C_φ is normal.

Case 2. $p \neq \infty$. In this case φ must be of the form

$$\varphi(z) = \frac{\mu z}{1 - ((1-\mu)/p)z} \qquad \text{with } 0 < |\mu| < 1.$$

Thus, by Theorem 3.2, $C_\varphi^\star = C_\psi$, where

$$\psi(z) = \bar{\mu} z + (1-\bar{\mu})/\bar{p} \qquad \text{with } 0 < |\mu| < 1. \tag{4.1}$$

It is easy to check that φ and ψ do not commute. Therefore, the same is true of C_φ and C_ψ. Thus, C_φ is not normal and, neither is C_ψ.

Finally, since composition operators induced by hyperbolic symbols with an exterior and boundary fixed point are similar under a unitary operator to C_ψ for some ψ of the form in (4.1), it follows from the argument in the paragraph above that they are not normal either. This completes the proof of the theorem. □

The following corollary follows easily from Theorems 3.3 and 4.1.

Corollary 4.2. *Let φ be a linear fractional self-map of $\mathbb{D}$. Then C_φ is normal on $\mathcal{D}$ if and only if $\varphi(z) = \mu z$ with* $0 < |\mu| \leq 1$.

4.1. Multiplication operators

Let (X, μ) be a measure space and $L^2(X, \mu)$ the space of square integrable complex-valued functions on X. For each bounded complex-valued measurable function ϕ on X we may consider the *multiplication operator* $M_\phi : L^2(X, \mu) \to L^2(X, \mu)$ defined by pointwise multiplication

$$\left(M_\phi f\right)(x) = \phi(x) f(x) \qquad\qquad (x \in X).$$

As mentioned in the introduction, normal operators are unitarily similar to multiplication operators. But also every multiplication operator is clearly normal.

In this subsection we will exhibit those multiplication operators which are similar to the normal linear fractional composition operators on $\mathcal{D}_0$. In order to do this we need a Theorem of Paley-Wiener.

Let Π denote the upper half plane of the complex plane. The Hardy space of the upper half plane $\mathcal{H}^2(\Pi)$ is the space of functions analytic on Π for which the norm

$$\|f\|^2_{\mathcal{H}^2(\Pi)} = \sup_{0<y<\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(x+iy)|^2\, dx$$

is finite.

Now, recall that Plancherel's Theorem states that the Fourier transform

$$\mathcal{F}(f)(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ixt}\, dx$$

defines an isometric isomorphism from $L^2(\mathbb{R}, dt/(2\pi))$ onto $L^2(\mathbb{R}, dt)$. The corresponding result for $\mathcal{H}^2(\Pi)$ is a theorem of Paley and Wiener (see [12], p. 372], for instance).

A Paley-Wiener Theorem. *The space $\mathcal{H}^2(\Pi)$ is isometrically isomorphic, under the Fourier transform, to the space $L^2(\mathbb{R}^+, dt)$.*

There is also a version of the Paley-Wiener Theorem for the Dirichlet space of the upper half plane. We consider the following linear fractional map

$$\sigma(z) = i\frac{1+z}{1-z}$$

that takes the unit disk onto Π. Observe that if $f \in \mathcal{D}_0$ and $F = f \circ \sigma^{-1}$, then

$$\frac{1}{\pi}\int_{\Pi} |F'(x+iy)|^2\, dx\, dy = \int_{\mathbb{D}} |f'(z)|^2\, dA(z).$$

This allows to define the Dirichlet space of the upper half plane $\mathcal{D}_\pi$ consisting of those analytic functions F on Π for which the integral in the left-hand side of the above display is finite. If we identify functions that differ by a constant, then $\mathcal{D}_\pi$ becomes a Hilbert space which makes of $C_\sigma : \mathcal{D}_\pi \to \mathcal{D}_0$ an isometric isomorphism. A proof of the following theorem can be found in [9].

A Paley-Wiener Theorem for $\mathcal{D}_\pi$. *The space $\mathcal{D}_\pi$ is isometrically isomorphic, under the Fourier transform, to $L^2(\mathbb{R}^+, tdt)$.*

Now we can prove the following Theorem.

Theorem 4.3. *Let C_φ be a linear fractional composition operator acting on $\mathcal{D}_0$. Then*

(a) *If φ is conjugate to $\eta(z) = \mu z$, with $0 < |\mu| \leq 1$, then C_φ is unitarily similar to a diagonal operator.*
(b) *If φ is parabolic which is conjugate to $\tau(z) = z + a$, then C_φ is unitarily similar to multiplication by $\phi(t) = e^{iat}$ on $L^2(\mathbb{R}^+, tdt)$.*
(c) *If φ is a hyperbolic automorphism conjugated to $\eta(z) = \lambda z$, then C_φ is unitarily similar to multiplication by $\phi(t) = \lambda^{-it}$ on $L^2(\mathbb{R}, 2\pi dt)$.*
(d) *If φ is hyperbolic with just one fixed point on $\partial\mathbb{D}$, then C_φ is unitarily similar to the product of a unitary operator and a normal operator or viceversa.*

Remark 4.1. Theorem 4.3 provides a different argument of the fact that normal linear fractional composition operators are indeed normal. Furthermore, it can also be used to give an alternative proof of Theorem 3.2.

Proof of Theorem 4.3. To prove (a) we observe that C_φ is unitarily similar to C_η. Set $u_n(z) = z^n/\sqrt{n}$. We have $C_\eta u_n = \mu^n u_n$. Thus C_η is a diagonal operator with the sequence $\{\mu^n\}_{n\geq 1}$ on the main diagonal.

In case (b), note that C_φ is unitarily similar, under $C_\sigma : \mathcal{D}_\pi \to \mathcal{D}_0$, to $C_\tau : \mathcal{D}_\pi \to \mathcal{D}_\pi$. Therefore, it is enough to prove the result for C_τ. Let $F \in \mathcal{H}^2(\Pi) \cap \mathcal{D}_\pi$. Then the properties of the Fourier transform show that $(F(z+a))\hat{} = e^{iat}\hat{F}(t)$. Since $\mathcal{H}^2(\Pi) \cap \mathcal{D}_\pi$ is dense in $\mathcal{D}_\pi$, the above equality holds for all $F \in \mathcal{D}_\pi$. Consequently, the operator C_τ acting on $\mathcal{D}_\pi$ is unitarily similar under the Fourier

Transform to the multiplication operator $M_\phi : L^2(\mathbb{R}^+, t\,dt) \longrightarrow L^2(\mathbb{R}^+, t\,dt)$, where $\phi(t) = e^{iat}$.

To show (c) we may use similarities under unitary composition operators and assume that φ fixes the points -1 and 1. Now, C_φ is unitarily similar, under C_σ, to $C_\eta : \mathcal{D}_\pi \to \mathcal{D}_\pi$. Therefore, it is enough to prove the result for $C_\eta : \mathcal{D}_\pi \to \mathcal{D}_\pi$. By the properties of the Fourier transform, for $F \in \mathcal{H}^2(\Pi) \cap \mathcal{D}_\pi$ the formula $(F(\lambda z))\hat{} = (1/\lambda)\hat{F}(t/\lambda)$ holds. Again, the fact that $\mathcal{H}^2(\Pi) \cap \mathcal{D}_\pi$ is dense in $\mathcal{D}_\pi$ shows that $C_\eta : \mathcal{D}_\pi \to \mathcal{D}_\pi$ is unitarily similar to $T_\lambda : L^2(\mathbb{R}^+, tdt) \longrightarrow L^2(\mathbb{R}^+, tdt)$ defined by $T_\lambda f(t) = (1/\lambda)f(t/\lambda)$. Now, the identity

$$\int_0^\infty |f(t)|^2\, tdt = \int_0^\infty |tf(t)|^2\, \frac{dt}{t}$$

shows that the map $f(t) \to tf(t)$ induces an isometry from the space $L^2(\mathbb{R}^+, tdt)$ onto $L^2(\mathbb{R}^+, dt/t)$. Under the latter map T_λ is unitarily similar to the composition operator $C_{t/\lambda} : L^2(\mathbb{R}^+, dt/t) \longrightarrow L^2(\mathbb{R}^+, dt/t)$ defined by $C_{t/\lambda}f(t) = f(t/\lambda)$. Now, observe that dt/t is the Haar measure corresponding to the multiplicative locally compact Abelian group $\mathbb{R}^+$, that is, the measure μ that is uniquely defined except for a positive scalar multiple and that satisfies $\mu(rA) = \mu(A)$ for any positive r and any measurable set A. The group of characters corresponding to this measure is formed by

$$\gamma_t(x) = x^{it} = e^{it\log x} \qquad (t \in \mathbb{R})$$

and thus the dual group is the additive group of real numbers (see [13, Section 1.2]). This time the Fourier transform for $f \in L^2(\mathbb{R}^+, dt/t)$ is defined as

$$\hat{f}(t) = \frac{1}{2\pi}\int_0^\infty f(x)x^{-it}\,\frac{dx}{x} \qquad (t \in \mathbb{R}).$$

By Plancherel's Theorem (see [13, Thm. 1.6.1], for instance) the Fourier transform defines an isometry from $L^2(\mathbb{R}^+, dt/t)$ onto $L^2(\mathbb{R}, 2\pi dt)$. In addition, for $g(x) \in L^2(\mathbb{R}^+, dt/t)$, the following identity

$$\int_0^\infty g(x/\lambda)x^{-it}\,\frac{dx}{x} = \lambda^{-it}\int_0^\infty g(x)x^{-it}\,\frac{dx}{x}$$

shows that $C_{t/\lambda}$ acting on $L^2(\mathbb{R}^+, dt/t)$ is unitarily similar under the last Fourier transform to the multiplication operator M_ϕ acting on $L^2(\mathbb{R}, 2\pi dt)$, where $\phi(t) = \lambda^{-it}$.

It remains to prove (d). First assume that φ has an interior fixed point and a boundary fixed point, that we may suppose that is 1. Then φ is conjugate under σ to

$$\psi(z) = \lambda z + a \quad \text{with} \quad 0 < \lambda < 1 \quad \text{and} \quad \Im a > 0.$$

Therefore, C_φ is unitarily similar to $C_\eta C_\tau$, where $\tau(z) = z + a$ and $\eta(z) = \lambda z$. Thus C_φ is unitarily similar to the product of a unitary operator and a normal operator. If φ has a boundary and an exterior fixed point, then $C_\varphi^\star = C_\psi$, where ψ has an interior and a boundary fixed point. It follows that C_φ is unitarily similar to the product of a normal operator and a unitary operator. The proof is complete. □

Remark 4.2. The arguments of the proof of (b) and (c) are similar to part of the arguments in Theorems 3.13 and 3.14 in [7], which assert the non cyclicity of composition operators induced by automorphisms on the Dirichlet space. To compute the spectrum of some linear fractional composition operators Higdon [9] also uses, in a different way, the Fourier transform (not the one with respect to the Haar measure). However, he never states that some of the linear fractional composition operators are normal or even similar to multiplication ones. The computation of the spectrum in [9] is done by direct calculation in a rather complicated way and mainly based on Cowen's methods.

5. The spectra

Now, we can easily deduce the spectra of each linear fractional composition operator on $\mathcal{D}_0$ as well as on $\mathcal{D}$. The proof below is much simpler than that in [9]. The case (iv) in theorem below appeared first in [10].

Theorem 5.1. *Let C_φ be a linear fractional composition operator acting on $\mathcal{D}_0$. Then*

(i) *If φ is an elliptic automorphism and the derivative $\varphi'(\alpha)$ at its interior fixed point is an n-th root of the unity, then $\sigma(C_\varphi) = \{\varphi'(\alpha)^k : k = 0, 1, \ldots, n-1\}$.*
(ii) *If φ is an automorphism which is not conjugate to a rotation through a rational multiple of π, then $\sigma(C_\varphi) = \{z \in \mathbb{C} : |z| = 1\}$.*
(iii) *If φ is a parabolic non automorphism which is conjugate to $\tau(z) = z + a$, then $\sigma(C_\varphi) = \{e^{iat} : t \geq 0\} \cup \{0\}$.*
(iv) *If φ is hyperbolic with just one boundary fixed point, then $\sigma(C_\varphi) = \overline{\mathbb{D}}$.*
(v) *If φ is not elliptic and has an exterior and an interior fixed point and $\varphi'(\alpha)$ is the derivative at the latter point, then $\sigma(C_\varphi) = \{\varphi'(\alpha)^n : n = 1, 2, \ldots\} \cup \{0\}$.*

Proof. Parts (i) through (iii) follows immediately from Theorem 4.3 and the following three well known and easy to prove results (see [8] for instance): the spectrum is invariant under similarities; the spectrum of a diagonal operator is the closure of the elements in the diagonal and the spectrum of a multiplication operator is the essential range of the multiplier.

To prove (iv) we observe that C_φ or $C_\varphi^\star$ is equal to C_ϕ, where ϕ has an exterior and a boundary fixed point. Since $\sigma(C_\varphi^\star) = \overline{\sigma(C_\varphi)}$, it is enough to compute $\sigma(C_\phi)$. Since, in $\mathcal{D}_0$, any composition operator with univalent symbol has norm less than or equal to 1, the spectral radius of C_φ is less than or equal to 1. On the other hand, we may suppose that $\phi(z) = \mu z + (1-\mu)/\bar{p}$ where $0 < \mu < 1$ and $|p| = 1$. Since $f_\lambda(z) = (z-p)^\lambda$ (with $\Re\lambda \geq 0$ and $\lambda \neq 0$) are eigenfunctions corresponding to μ^λ, it must be $\sigma(C_\phi) = \overline{\mathbb{D}}$.

It remains to prove (v). In this case, the result is well known for power compact composition operators (see [6, Chapter 7]). Now, since the interior fixed point of φ is attractive, it follows that $\varphi(\mathbb{D})$ or $\varphi \circ \varphi(\mathbb{D})$ is a relatively compact subset of $\mathbb{D}$. Therefore, C_φ is a power compact operator. The proof is concluded. □

Remark 5.1. As in the Hardy space (see [5]), in case (iii), $\sigma(C_\varphi)$ is a spiral or line segment joining 0 and 1 because the imaginary part of a is positive.

Recall that the essential spectrum $\sigma_e(T)$ of an operator T is the set of complex numbers λ for which $T-\lambda$ is not invertible modulo compact operators, that is, the spectrum of the projection of T onto the Calkin algebra. The essential spectrum $\sigma_e(T)$ is always a compact subset contained in $\sigma(T)$. For instance, it is easy to check that eigenvalues of infinite multiplicity are always in the essential spectrum. As a corollary of Theorem 5.1 we have

Corollary 5.2. *Let C_φ be a linear fractional composition operator acting on $\mathcal{D}_0$. Then $\sigma_e(C_\varphi) = \sigma(C_\varphi)$, except if φ is not elliptic and has an exterior and an interior fixed point, in which case $\sigma_e(C_\varphi) = \{0\}$.*

Proof. In case (i) of Theorem 5.1, each point in $\sigma(T)$ is an eigenvalue of infinite multiplicity. Therefore, $\sigma_e(C_\varphi) = \sigma(C_\varphi)$. In case (ii) or (iii) of Theorem 5.1 the operator C_φ is normal and the essential spectrum of a normal operator contains all the non-isolated points of the spectrum ([3, Chap. XI, Prop. 4.6]). Thus, $\sigma_e(C_\varphi) = \sigma(C_\varphi)$. In case (iv) of Theorem 5.1, the operator C_φ or $C_\varphi^\star$ has each point of the open unit disk as an eigenvalue of infinite multiplicity. Therefore, $\sigma_e(C_\varphi) = \overline{\sigma_e(C_\varphi^\star)} = \overline{\mathbb{D}}$. Finally, in case (v) of Theorem 5.1, we know that C_φ^n is compact for n at most 2 and, therefore, $\sigma_e(C_\varphi^n) = \{0\}$. From the Spectral Theorem it follows that $\sigma_e(C_\varphi) = \{0\}$. □

Remark 5.2. Since the constant functions are invariant under any composition operator, the matrix of C_φ acting on $\mathcal{D}$ is of the form

$$\begin{pmatrix} 1 & X \\ 0 & Y \end{pmatrix}$$

where Y is the matrix of C_φ acting on $\mathcal{D}_0$. Therefore, as pointed out in [9] it is enough to add the point 1 to case (v) in Theorem 5.1 to obtain the spectrum of C_φ acting on $\mathcal{D}$. Of course, the essential spectrum of C_φ acting on $\mathcal{D}$ coincides with the essential spectrum of C_φ acting on $\mathcal{D}_0$.

6. The norm

Since the norm of a diagonal operator is the supremum of the elements in the diagonal and the norm of a multiplication operator is the supremum of the essential range of the multiplier, the following corollary follows immediately from Theorem 4.3.

Corollary 6.1. *Let φ be a linear fractional self-map of $\mathbb{D}$. If φ is elliptic or it has a boundary fixed point, then $\|C_\varphi\|_{\mathcal{D}_0} = 1$.*

If φ has an exterior and an interior fixed point and is not conjugate to $z \to \mu z$, then the formula for the norm of C_φ becomes more complicated.

Theorem 6.2. *Let $\varphi(z) = (az+b)(cz+d)^{-1}$ be a linear fractional self-map of $\mathbb{D}$. Then*

$$\|C_\varphi\|_{\mathcal{D}_0} = \frac{1}{2}\left[\frac{|a|^2+|d|^2-|b|^2-|c|^2}{|ad-bc|^2} - \sqrt{\left[\frac{|a|^2+|d|^2-|b|^2-|c|^2}{|ad-bc|^2}\right]^2 - 4}\,\right].$$

Proof. Let ψ denote the linear fractional map furnished by Theorem 3.2. We set $\phi = \varphi \circ \psi$. We have

$$\phi(z) = \frac{(|a|^2-|b|^2)z + b\bar{d} - a\bar{c}}{(\bar{a}c - \bar{b}d)z + |d|^2 - |c|^2}.$$

As in [4], we have the equalities

$$\|C_\varphi\|^2 = \|C_\varphi^\star C_\varphi\| = \lim_{n\to\infty} \|(C_\psi C_\varphi)^n\|^{1/n} = \lim_{n\to\infty} \|C_\phi^n\|^{1/n}.$$

Thus $\|C_\varphi\|$ is the square root of the spectral radius of C_ϕ.

If φ is an automorphism of $\mathbb{D}$, then $\psi = \varphi^{-1}$ and ϕ is the identity. In this case, the formula for $\|C_\varphi\|_{\mathcal{D}_0}$ trivially holds.

If φ is not an automorphism, then φ and ψ take $\mathbb{D}$ into disks strictly contained in $\mathbb{D}$. Therefore, ϕ cannot be an automorphism of $\mathbb{D}$. By Theorem 4.1, as C_ϕ is normal, the map ϕ is a parabolic non automorphism or it is conjugate to $\eta(z) = \mu z$ with $0 < |\mu| < 1$. If ϕ is parabolic, and p is the boundary fixed point, then $\phi'(p) = 1$ is the spectral radius of ϕ. If ϕ is conjugate to η, then by Theorem 5.1, the spectral radius of C_ϕ is $|\phi'(p)|$, where p is the interior fixed point of ϕ. Thus in any case, the spectral radius of C_ϕ is $|\phi'(p)|$, where p is the fixed point of ϕ in $\overline{\mathbb{D}}$.

Let $\mathcal{T}$ denote the trace of the matrix representation of ϕ in which all coefficients of ϕ are divided by $|ad-bc|^2$. We have

$$\mathcal{T} = \frac{|a|^2+|d|^2-|b|^2-|c|^2}{|ad-bc|^2}.$$

One easily checks (see [1] for instance) that

$$\phi'(p) = \frac{1}{4}(\mathcal{T} \pm \sqrt{\mathcal{T}^2 - 4})^2. \tag{6.1}$$

If $\phi'(p)$ were positive, then by just extracting square roots we would obtain the formula for $\|C_\varphi\|_{\mathcal{D}_0}$, except that there would be an ambiguity because the plus-minus sign. Since $|\phi'(p)| \le 1$, it is enough to prove that $\mathcal{T} \ge 2$ to show that $\phi'(p)$ is indeed positive and to rule out the plus sign. It follows from (6.1) that

$$\phi'(p)^{-1} + \phi'(p) = \mathcal{T}^2 - 2.$$

Since $\mathcal{T}$ is real, the map ϕ is not loxodromic. Therefore, ϕ must be hyperbolic or parabolic. It follows that $|\mathcal{T}| \ge 2$. Thus it is suffices to show that $\mathcal{T}$ is positive. To prove this, we observe that we have the expression

$$\mathcal{T} = \frac{|ad - bc|^2 - |a\bar{b} - c\bar{d}|^2 + (|d|^2 - |b|^2)^2}{(|d|^2 - |b|^2)|ad - bc|^2}.$$

Now, consider the involutive automorphism α_p, where $p = \varphi(0) = b/d$ and define $\eta = \alpha_p \circ \varphi$ that fixes the origin. We have

$$\eta(z) = \frac{(bc - ad)\bar{d}z}{(c\bar{d} - a\bar{b})dz + (|d|^2 - |b|^2)d}.$$

Since $\eta(z)$ must take $\mathbb{D}$ into itself and $\eta((|b|^2 - |d|^2)/(c\bar{d} - a\bar{b})) = \infty$ and $|b| < |d|$, it follows that $|d|^2 - |b|^2 > |a\bar{b} - c\bar{d}|$ and, therefore, $\mathcal{T} > 0$. The proof is finished. □

Remark 6.1. If φ fixes the origin, then $\|C_\varphi\|_{\mathcal{D}} = 1$. But, in the general case, to obtain an exact formula for $\|C_\varphi\|_{\mathcal{D}}$ seems to be difficult.

7. The Hardy space of the upper half plane

In the previous sections, we have seen that the Dirichlet space modulo constants is quite a natural setting for studying linear fractional composition operators. Another natural space of analytic functions without constant functions is the Hardy space of the upper half plane that we already defined in section 4. The situation here is much simpler than that in the Dirichlet space. One of the reasons for this is that only linear fractional transformations

$$\varphi(z) = az + b \qquad \text{with } a > 0 \text{ and } \Im b \ge 0$$

induce bounded composition operators on $\mathcal{H}^2(\Pi)$ (see [11]).

7.1. Adjoints

This time the adjoint of a linear fractional composition operator on $\mathcal{H}^2(\Pi)$ is a scalar multiple of a linear fractional one. We have

Theorem 7.1. *Let $\varphi(z) = az+b$ be such that $a > 0$ and $\Im b \geq 0$ and consider C_φ acting on $\mathcal{H}^2(\Pi)$. Then $C_\varphi^\star = a^{-1}C_\psi$, where $\psi(z) = a^{-1}z - a^{-1}\bar{b}$. Furthermore, we have*

(a) *If φ is parabolic, then C_φ is unitarily similar to multiplication by e^{ibt} on $L^2(\mathbb{R}^+, dt)$. In particular, if φ is a parabolic automorphism, then C_φ is unitary.*

(b) *If φ is a hyperbolic automorphism, then C_φ is unitarily similar to multiplication by $a^{-it-1/2}$ on $L^2(\mathbb{R}, 2\pi dt)$.*

Proof. To prove the result we first prove (a) and (b). This time, as in the proof of Theorem 4.3, we will use Fourier transforms. First, suppose that $\varphi(z) = z + b$ with $\Im b \geq 0$. Then the properties of the Fourier transform show that $(f(z+b))\hat{}= e^{ibt}\hat{f}(t)$ for any $f \in \mathcal{H}^2(\Pi)$. Consequently, C_φ is unitarily similar under the Fourier transform to the multiplication operator $M_\phi : L^2(\mathbb{R}^+, dt) \longrightarrow L^2(\mathbb{R}^+, dt)$, where $\phi(t) = e^{ibt}$. Therefore, (a) is proved. In addition, since the adjoint of M_ϕ is $M_{\bar{\phi}}$, where $\bar{\phi}(t) = \overline{e^{ibt}} = e^{-i\bar{b}t}$, we find that $C_\varphi^\star = C_\psi$, where $\psi = z - \bar{b}$.

Second, suppose that φ is a hyperbolic automorphism. This implies that $a \neq 1$ and $\Im b = 0$. Thus $\tau(z) = z + b/(1-a)$ is a parabolic automorphism. In addition, $C_\tau^{-1}C_\varphi C_\tau = C_\eta$, where $\eta(z) = az$. Thus, by (a), C_φ is unitarily similar to C_η. Now, the properties of the Fourier transform imply that for $f \in \mathcal{H}^2(\Pi)$ the formula $(F(az))\hat{}= (1/a)\hat{F}(t/a)$ holds. Thus C_φ is unitarily similar under the Fourier transform to $T_a : L^2(\mathbb{R}^+, dt) \longrightarrow L^2(\mathbb{R}^+, dt)$ defined by $T_a f(t) = (1/a)f(t/a)$. Now, the identity

$$\int_0^\infty |f(t)|^2 dt = \int_0^\infty |\sqrt{t} f(t)|^2 \frac{dt}{t}$$

shows that the map $f(t) \to \sqrt{t} f(t)$ induces an isometry from $L^2(\mathbb{R}^+, dt)$ onto $L^2(\mathbb{R}^+, dt/t))$. Under the latter isometry T_a is similar to the following scalar multiple of a composition operator

$$a^{-1/2}C_{t/a} : L^2(\mathbb{R}^+, dt/t) \longrightarrow L^2(\mathbb{R}^+, dt/t).$$

The Fourier transform with respect the multiplicative group of positive real numbers, defines a unitary isometry from $L^2(\mathbb{R}^+, dt/t)$ onto $L^2(\mathbb{R}, 2\pi dt)$. In this way, $a^{-1/2}C_{t/a}$ becomes unitarily similar to M_ϕ acting on $L^2(\mathbb{R}, 2\pi dt)$, where $\phi(t) = a^{-it-1/2}$. Therefore, (b) is also proved. Since $M_\phi^\star = M_{\bar{\phi}}$, where $\bar{\phi}(t) = \overline{a^{-it-1/2}} = a^{it-1/2}$, as in the paragraph above, it follows that $C_{az}^\star = a^{-1}C_{a^{-1}z}$.

For the general case $\varphi(z) = az + b$ we write $\psi(z) = a^{-1}z - a^{-1}\bar{b}$. We have $\varphi = \varphi_1 \circ \varphi_2$, where $\varphi_1(z) = z + b$ and $\varphi_2(z) = az$ and $\psi = \psi_2 \circ \psi_1$, where $\psi_1(a) = z - \bar{b}$ and $\psi_2(z) = a^{-1}z$. Hence,

$$C_\varphi^\star = (C_{\varphi_2} C_{\varphi_1})^\star = C_{\varphi_1}^\star C_{\varphi_2}^\star = a^{-1} C_{\psi_1} C_{\psi_2} = a^{-1} C_\psi .$$

The result is proved. □

Now, the following corollary follows immediately.

Corollary 7.2. *Let $\varphi(z) = az + b$ be such that $a > 0$ and $\Im b \geq 0$. Then C_φ acting on $\mathcal{H}^2(\Pi)$ is normal if and only if φ is an automorphism of Π or φ is parabolic. In particular, C_φ is unitary if and only if φ is a parabolic automorphism.*

Of course, if $\Im b = 0$ the following corollary follows by a simple change of variables.

Corollary 7.3. *Let $\varphi(z) = az + b$ be such that $a > 0$ and $\Im b \geq 0$. Then $\|C_\varphi\|_{\mathcal{H}^2(\Pi)} = a^{-1/2}$.*

7.2. The spectra

We can also obtain easily the spectrum and the essential spectrum of C_φ.

Theorem 7.4. *Let $\varphi(z) = az + b$ be such that $a > 0$ and $\Im b \geq 0$ and let C_φ act on $\mathcal{H}^2(\Pi)$. We have*

(a) *If φ is an automorphism, then $\sigma(C_\varphi) = \{z \in \mathbb{C} : |z| = a^{-1/2}\}$.*
(b) *If φ is a parabolic non automorphism, then $\sigma(C_\varphi) = \{e^{ibt} : t \geq 0\} \cup \{0\}$.*
(c) *If φ is a hyperbolic non automorphism, then $\sigma(C_\varphi) = \{z \in \mathbb{C} : |z| \leq a^{-1/2}\}$.*

Furthermore, the spectrum and the essential spectrum of C_φ coincide.

Proof. Since the spectrum of a multiplication operator is the essential range of the multiplier (a) and (b) follow immediately from Theorem 7.1. Since C_φ is normal and there is no isolated points in the spectrum, it also follows that $\sigma_e(C_\varphi) = \sigma(C_\varphi)$.

To prove (c), observe that the spectral radius of C_φ is $a^{-1/2}$. In addition, since $\Im b > 0$, the functions $f_\lambda(z) = (z-b)^\lambda \in \mathcal{H}^2(\Pi)$ if and only if $\Re\lambda < -1/2$. Therefore, since $C_\varphi f_\lambda(z) = a^\lambda f_\lambda(z)$, each of the points in the open disk $|z| < a^{-1/2}$ is an eigenvalue of infinite multiplicity. Thus, it follows that $\sigma_e(C_\varphi) = \sigma(C_\varphi) = \{z \in \mathbb{C} : |z| \leq a^{-1/2}\}$. The proof is finished. □

Acknowledgement. The authors would like to thank the referee, María José Martín and Dragan Vukotić for helpful comments and suggestions.

References

1. Ahlfors, L.V.: Complex Analysis. McGraw-Hill, New York, 1979
2. Appel, M.J., Bourdon, P.S., Thrall, J.J.: Norms of composition operators in the Hardy space. Experiment. Math. **5**, 111–117 (1996)
3. Conway, J.B.: A Course in Functional Analysis. Springer-Verlag, New York, 1985
4. Cowen, C.C.: Composition operators on $\mathcal{H}^2$. J. Operator Theory **9**, 77–106 (1983)
5. Cowen, C.C.: Linear fractional composition operators on $\mathcal{H}^2$. Integral equations and operator Theory, **11**, 151–160 (1988)
6. Cowen, C.C., MacCluer, B.D.: Composition Operators on Spaces of Analytic Functions. CRC Press, 1995
7. Gallardo-Gutiérrez, E.A., Montes-Rodríguez, A.: The role of the spectrum of the cyclic behavior of composition operators. Memoirs of the A.M.S., (to appear)
8. Halmos, P.R.: A Hilbert space problem book. Van Nostrand Company, Inc., 1967
9. Higdon, W.M.: Spectra of composition operators on the Dirichlet space. Preprint
10. Hurst, P.R.: Relating composition operators on different weighted Hardy spaces. Arch. Math. **68**, 503–513 (1997)
11. Matache, V.: Composition operators on Hardy spaces of a half plane. Proc. Amer. Math. Soc., **127**, 1483–1491 (1999)
12. Rudin, W.: Real and complex analysis. Third edn., McGraw Hill, New York, 1987
13. Rudin, W.: Fourier analysis on groups. John Wiley & Sons, Inc. 1962
14. Schwartz, H.J.: Composition Operators on $\mathcal{H}^p$. Thesis, University of Toledo, 1969
15. Vukotić, D.: Norms and essential norms of some univalently induced composition operators in Bergman spaces. Preprint