# On the Comparison of Reliability Experiments Based on the Convolution Order

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In this article we study the comparison of experiments in system reliability theory when the component lifetimes are independent and identically distributed random variables that have a common two-parameter exponential distribution with a location parameter $\theta$. For this purpose, we define a new stochastic order, which we call the *convolution order*, and study some basic properties of it. We then focus our attention on the family of distribution functions that are mixtures of distributions of partial sums of independent exponential random variables, and derive results that identify several conditions under which members of this family are ordered in the convolution stochastic order. We apply the results to order lifetimes of coherent systems, and as a consequence we obtain information inequalities among various lifetimes of coherent systems. We find situations wherein high reliability decreases statistical information.

KEY WORDS: Coherent systems; Convolution order; Coxian distributions; Dispersive order; Information comparisons; Order statistics; Poles and zeros of rational functions; Signatures; Stochastic orders.

## 1. INTRODUCTION

The notion of comparison of experiments, as introduced by Blackwell (1951, 1953) and others, concerns a partial ordering of the information contained in the experiments (or in the distributions of the underlying random variables). A review of the basic ideas and related results has been given by Goel and DeGroot (1979) and Lehmann (1988), and a comprehensive treatment of this topic was given by Torgersen (1991).

Stepniak (1997c) nicely summarized the idea of the information contained in a statistical experiment as follows: "Any statistical experiment can be perceived as an information channel transforming a deterministic quantity (parameter) into a random quantity (observation)." In this article the "channels" are reliability systems, and we derive various results useful for the purpose of comparing such channels with respect to their information content.

For the purpose of completeness, we state a definition and a key result.

*Definition 1.* Let $\mathbf{X}$ and $\mathbf{Y}$ be two $m$-dimensional random vectors ($m \geq 1$) with distribution functions $F_\theta$ and $G_\theta$, where $\theta \in \Theta$ is the parameter of interest. The experiment $S_2$ associated with $\mathbf{Y}$ is said to be *at least as informative* for $\theta$ as the experiment $S_1$ associated with $\mathbf{X}$ (denoted by $\mathbf{X} \leq_{\mathrm{i}} \mathbf{Y}$ or $F_\theta \leq_{\mathrm{i}} G_\theta$ or $S_1 \leq_{\mathrm{i}} S_2$), if for every decision problem involving $\theta$, and every prior distribution on $\Theta$, the expected Bayes risk from $F_\theta$ is not less than that from $G_\theta$.

*Proposition 1.* The information inequality $\mathbf{X} \leq_{\mathrm{i}} \mathbf{Y}$ holds if there exists a function $\phi : \mathbb{R}^{m+r} \to \mathbb{R}^m$ and an $r$-dimensional random vector $\mathbf{W}$ ($r \geq 1$) that is independent of $\mathbf{Y}$ and has a distribution function that does not depend on $\theta$, such that $\mathbf{X} =_{\mathrm{st}} \phi(\mathbf{Y}, \mathbf{W})$.

Proposition 2 is well known in the literature and was given by Lehmann (1959); it is the basic technical tool used by researchers to obtain interesting results on the comparison of various types of experiments. For example, Hansen and Torgersen (1974) and Stepniak (1997b) considered the comparison of normal experiments, and Torgersen (1984), Stepniak (1997a), and others studied the comparison of linear experiments. Eaton (1992) discussed a group action on covariances with applications to the comparison of linear normal experiments (see also Hauke and Markiewicz 1994). Hollander, Proschan, and Sconing (1987) and Goel (1988) gave results comparing experiments with censored data, and Lehmann (1988) discussed the comparison of location parameter experiments. Shaked and Tong (1990), Stepniak (1994), and others considered comparison of experiments through dependence of normal variables with a common marginal distribution, and Greenshtein and Torgersen (1997) and others discussed comparisons of sequential experiments.

In this article we study the comparison of experiments in system reliability theory when the component lifetimes are independent and identically distributed random variables that have a common two-parameter exponential distribution with a location parameter $\theta$. Specifically, for fixed $n, m \geq 2$, let $Z_1, Z_2, \ldots, Z_{\max\{n,m\}}$ be independent random variables with a common density function given by

$$f_\theta(z) = \begin{cases} \lambda e^{-\lambda(z-\theta)} & \text{for } z \geq \theta \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where $\lambda$ is assumed known. Let $S_1$ and $S_2$ be two reliability systems (corresponding to two experiments) with lifetimes

$$X = \tau_1(Z_1, Z_2, \ldots, Z_n)$$

and

$$Y = \tau_2(Z_1, Z_2, \ldots, Z_m),$$

where $\tau_1$ and $\tau_2$ are the two corresponding coherent life functions. Suppose that the values of $X$ and $Y$ may be observable, but not the individual $Z_i$'s. Then the problem of interest is to find out what types of systems are more informative according to Definition 1.

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For this purpose, in Section 2 we define a new stochastic order, which we call the *convolution order*. There we also study some basic properties of that order. In Section 3 we focus our attention on the family of distribution functions which are mixtures of distributions of partial sums of independent exponential random variables. In that section we identify several conditions under which members of this family are ordered in the convolution stochastic order. In Section 4 we apply the results in Section 3 to order lifetimes of coherent systems. As an application of the convolution order, if the lifetime of a coherent system, $X$, is less than the lifetime of another system, $Y$, in the convolution order sense, then $X$ is more informative than $Y$ with respect to the location parameter. This is really interesting, because we find situations where high reliability decreases statistical information. (In this article, by "increasing" we mean "nondecreasing," and by "decreasing" we mean "nonincreasing.")

## 2. THE CONVOLUTION ORDER

Let $X$ and $Y$ be two random variables. If there exists a nonnegative random variable $U$, independent of $X$, such that

$$Y =_{\mathrm{st}} X + U, \tag{2}$$

then we say that $X$ is smaller than $Y$ in the *convolution order* (denoted by $X \leq_{\mathrm{conv}} Y$). Obviously, the convolution order is a partial order.

The reason that we are interested in the convolution order in this article is that the convolution order is a useful tool for the purpose of comparison of experiments when the underlying parameter is a location parameter. This is seen in Proposition 2. Let $X$ and $Y$ be two random variables with a location parameter $\theta$; that is,

$$X = W + \theta \quad \text{and} \quad Y = Z + \theta$$

for some random variables $W$ and $Z$. Then, from Proposition 1, we obtain the following.

*Proposition 2.* Let $X$, $Y$, $W$, and $Z$ be as before. If $X \leq_{\mathrm{conv}} Y$ or, equivalently, if $W \leq_{\mathrm{conv}} Z$, then $Y \leq_{\mathrm{i}} X$.

In fact, from example 10.B.2 of Torgersen (1994), it follows that if $X$, $Y$, $W$, and $Z$ are as before, then $Y \leq_{\mathrm{i}} X \Longleftrightarrow X \leq_{\mathrm{conv}} Y$ or, equivalently, $Y \leq_{\mathrm{i}} X \Longleftrightarrow Z \leq_{\mathrm{conv}} W$.

The convolution order is obviously closed under increasing linear transformations. That is,

$$X \leq_{\mathrm{conv}} Y \Longrightarrow a + bX \leq_{\mathrm{conv}} a + bY. \tag{3}$$

The convolution order is obviously also closed under convolutions. That is, let $X_1, X_2, \ldots, X_n$ be a set of independent random variables, and let $Y_1, Y_2, \ldots, Y_n$ be another set of independent random variables. Then

$$(X_j \leq_{\mathrm{conv}} Y_j,\ j = 1, 2, \ldots, n) \Longrightarrow X_1 + X_2 + \cdots + X_n \leq_{\mathrm{conv}} Y_1 + Y_2 + \cdots + Y_n. \tag{4}$$

For any nonnegative random variable $X$ we denote its classical Laplace transform by $L_X$; that is,

$$L_X(s) = E[e^{-sX}], \qquad s \geq 0.$$

Recall that a nonnegative function $\phi$ is a Laplace transform of a nonnegative measure on $(0, \infty)$ if and only if $\phi$ is completely monotone; that is, all of the derivatives $\phi^{(n)}$ of $\phi$ exist, and they satisfy $(-1)^n \phi^{(n)}(x) \geq 0$ for all $x \geq 0$ and $n = 1, 2, \ldots$. It follows that for nonnegative random variables, we have

$$X \leq_{\mathrm{conv}} Y \Longleftrightarrow \frac{L_Y(s)}{L_X(s)} \text{ is a completely monotone function in } s \geq 0. \tag{5}$$

The convolution order is a very strong order; we point out some of its implications. It is obvious from (2) that

$$X \leq_{\mathrm{conv}} Y \Longrightarrow X \leq_{\mathrm{st}} Y, \tag{6}$$

where (see, e.g., Shaked and Shanthikumar 1994) $X \leq_{\mathrm{st}} Y$ means that $E\psi(X) \leq E\psi(Y)$ for all increasing functions $\psi$ for which these expectations exist.

If for two nonnegative random variables $X$ and $Y$ it holds that $L_Y(s)/L_X(s)$ is decreasing in $s \geq 0$, then $X$ is said to be smaller than $Y$ in the Laplace transform ratio (denoted by $X \leq_{\mathrm{Lt\text{-}r}} Y$). Shaked and Wong (1997) studied this order and derived useful inequalities that follow from it. From (5), it is seen that

$$X \leq_{\mathrm{conv}} Y \Longrightarrow X \leq_{\mathrm{Lt\text{-}r}} Y \Longrightarrow X \leq_{\mathrm{Lt}} Y,$$

where $X \leq_{\mathrm{Lt}} Y$ means that $L_X(s) \geq L_Y(s)$, $s \geq 0$ (again see, e.g., Shaked and Shanthikumar 1994 for applications of the order $\leq_{\mathrm{Lt}}$).

Another useful order is the dispersive order $\leq_{\mathrm{disp}}$, which has been studied by for example, Shaked and Shanthikumar (1994). According to their theorem 2.B.3, a random variable $X$ satisfies $X \leq_{\mathrm{disp}} X + Y$ for any random variable $Y$ that is independent of $X$ if and only if $X$ has a log-concave density. Thus we have

$$(X \leq_{\mathrm{conv}} Y, \text{ and } X \text{ has a logconcave density}) \Longrightarrow X \leq_{\mathrm{disp}} Y. \tag{7}$$

In this article our interest in the order $\leq_{\mathrm{conv}}$ stems from the fact (mentioned earlier) that it is equivalent (with the inequality reversed) to the information order. However, it is worthwhile to mention that the convolution order sometimes also can be used as a realistic assumption in some statistical inferential applications. For example, consider the problem of the nonparametric estimation of two life distributions, $F$ and $G$, in a two-sample problem. Suppose that any (observed) lifetime of interest here is a sum of two (unobserved) independent nonnegative random variables, such that the distribution of one variable (generically denoted by $U$) is determined by the environment in which the lifetime is observed, and the distribution of the second random variable (generically denoted by $X$) is independent of that environment. In such a case, if $F$ is the distribution of the lifetimes observed in a certain environment in which $U$ is essentially 0, and $G$ is the distribution of the lifetimes observed in another environment in which $U$ is positive, then it is realistic to assume that $F \leq_{\mathrm{conv}} G$. (Here and later, the notation $F \leq_{\mathrm{conv}} G$ means that the two underlying random variables are ordered with respect to $\leq_{\mathrm{conv}}$.)

For example, let $G$ be the distribution of the time from an exposure to some bacteria until the bacteria cause a mouse in a certain geographical region to expire. Suppose that this time comprises an incubation period, $U$, of the bacteria, and the period, $X$, that it takes after incubation to cause the mouse's death. Similarly, let $F$ be the distribution of the time from the exposure to that bacteria until the bacteria cause a similar mouse to expire in another geographical region, in which incubation period $U$ is negligible. Then it is reasonable to assume that $F \leq_{\text{conv}} G$, and to proceed with statistical inference (say, estimating $F$ and $G$, or testing $H_0 : F = G$) under the constraint $F \leq_{\text{conv}} G$. Note that here a weaker order (e.g., $F \leq_{\text{st}} G$) fails to exploit the full extent of the intuition available about this application.

In this article we do not develop statistical inference procedures under the constraint $F \leq_{\text{conv}} G$.

## 3. MIXTURES OF PARTIAL SUMS OF EXPONENTIAL RANDOM VARIABLES

In this section, all of the exponential random variables considered have a location parameter 0. The lifetime of every $k$-out-of-$n$ system, with components that have independent and identically distributed exponential lifetimes, is a sum of independent (but not identically distributed) exponential random variables (see Sec. 4 for details). Furthermore, the lifetime distribution of every coherent system with such components is a mixture of distributions of partial sums of independent exponential random variables (again, see Sec. 4 for details). Thus, to compare lifetimes of coherent systems, it is useful to obtain some comparison results for the class $\mathcal{PH}_C$ defined in the next paragraph.

The family of distributions that are mixtures of distributions of partial sums of independent exponential random variables (which is a subset of the family of phase-type distributions) is called the class of the *Coxian* distributions and was denoted by Asmussen (1987, p. 74) as $\mathcal{PH}_C$. Formally, the distribution function of a nonnegative random variable $T$ belongs to $\mathcal{PH}_C$ if and only if its Laplace transform is of the form

$$L_T(s) = \sum_{k=1}^{n} p_k \prod_{i=1}^{k} \frac{\delta_i}{\delta_i + s}, \qquad s \geq 0, \tag{8}$$

where $\delta_1, \delta_2, \ldots, \delta_n$ are some positive parameters, $p_k \geq 0$ for $k = 1, 2, \ldots, n$, and $\sum_{k=1}^{n} p_k = 1$.

Let us now fix $n$ and $\delta_1, \delta_2, \ldots, \delta_n$. Let $T_{\mathbf{p}}$ denote the random variable with Laplace transform given in (8), where $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ is a probability vector; that is,

$$T_{\mathbf{p}} = S_k \equiv \exp(\delta_1) + \exp(\delta_2) + \cdots + \exp(\delta_k) \quad \text{with probability } p_k, \qquad k = 1, 2, \ldots, n, \tag{9}$$

where $\exp(\delta)$ denotes an exponential random variable with rate $\delta$ and the random variables $\exp(\delta_1), \exp(\delta_2), \ldots, \exp(\delta_k)$ are independent. Note we describe some results that yield comparisons of members in $\mathcal{PH}_C$ according to the order $\leq_{\text{conv}}$.

*Theorem 1.* For some $1 \leq i \leq n$, let

$$\mathbf{p} = \Big(0, \ldots, 0, \overset{i}{1}, 0, \ldots, 0\Big)$$

where the $i$ above the 1 denotes the location of the 1, and

$$\mathbf{q} = (0, \ldots, 0, q_i, q_{i+1}, \ldots, q_n),$$

where $\mathbf{q}$ is a probability vector. Then $T_{\mathbf{p}} \leq_{\text{conv}} T_{\mathbf{q}}$.

*Proof.* Write the ratio of the Laplace transform as

$$\begin{aligned}\frac{L_{T_{\mathbf{q}}}(s)}{L_{T_{\mathbf{p}}}(s)} &= \frac{q_i L_{S_i}(s) + q_{i+1} L_{S_{i+1}}(s) + \cdots + q_n L_{S_n}(s)}{L_{S_i}(s)} \\ &= q_i + q_{i+1} \frac{L_{S_{i+1}}(s)}{L_{S_i}(s)} + \cdots + q_n \frac{L_{S_n}(s)}{L_{S_i}(s)} \\ &= q_i + q_{i+1} L_{\exp(\delta_{i+1})}(s) + \cdots + q_n L_{\sum_{j=i+1}^{n} \exp(\delta_j)}(s),\end{aligned}$$

and note that it is a convex combination of Laplace transforms of sums of independent exponential random variables. Therefore, $T_{\mathbf{p}} \leq_{\text{conv}} T_{\mathbf{q}}$ by (5).

Note that in Theorem 1, some of the $q_j$, $i \leq j \leq n$, could be 0. In light of Theorem 1, one may conjecture that $T_{\mathbf{p}} \leq_{\text{conv}} T_{\mathbf{q}}$ when $\mathbf{p} = (p_1, \ldots, p_i, 0, \ldots, 0)$ and $\mathbf{q} = (0, \ldots, 0, \overset{i}{1}, 0, \ldots, 0)$. But this is not true in general. For example, take $n = 3$, $\delta_i = 1$ for $i = 1, 2, 3$, $\mathbf{p} = (1/10, 0, 9/10)$, and $\mathbf{q} = (0, 0, 1)$. Then the ratio of the Laplace transforms is

$$\frac{L_{T_{\mathbf{q}}}(s)}{L_{T_{\mathbf{p}}}(s)} = \frac{10}{s^2 + 2s + 10}, \qquad s \geq 0.$$

It can be easily checked that $L_{T_{\mathbf{q}}}(s)/L_{T_{\mathbf{p}}}(s)$ is not convex, and therefore $T_{\mathbf{p}} \not\leq_{\text{conv}} T_{\mathbf{q}}$. It is worth noting that the fact that $L_{T_{\mathbf{q}}}/L_{T_{\mathbf{p}}}$ is not a Laplace transform of a probability distribution essentially follows from the fact that the polynomial in the foregoing denominator does not have real roots. Explicitly, if $L_{T_{\mathbf{q}}}/L_{T_{\mathbf{p}}}$ were the Laplace transform of a probability distribution, then

$$\frac{L_{T_{\mathbf{q}}}(-it)}{L_{T_{\mathbf{p}}}(-it)} = \frac{10}{(-it)^2 + 2(-it) + 10} = \frac{10}{(it)^2 - 2(it) + 10}$$

would have been the characteristic function of a probability distribution. From theorem 3.1 of Takano (1951), it follows that in such a case the polynomial $z^2 - 2z + 10$ [or, equivalently, the denominator of $L_{T_{\mathbf{q}}}(s)/L_{T_{\mathbf{p}}}(s)$] would have had at least one real root. Because this is not the case, we see that $T_{\mathbf{p}} \not\leq_{\text{conv}} T_{\mathbf{q}}$. Nonetheless, we provide a sufficient condition for the foregoing conjecture in Theorem 2. This condition will be useful for the purpose of bounding from above, in the order $\leq_{\text{conv}}$, the lifetime of a coherent system by the lifetime of a $k$-out-of-$n$ system. To state and prove Theorem 2 and the results that follow it, we need to introduce some terminology.

Note from (8) that $L_{T_{\mathbf{p}}}$ can be expressed as a ratio,

$$L_{T_{\mathbf{p}}}(s) = \frac{Q(s)}{R(s)}, \qquad s \geq 0, \tag{10}$$

where $Q$ and $R$ are polynomials. The roots of the denominator $R$ are all included in the set $\{-\delta_1, -\delta_2, \ldots, -\delta_n\}$. The real roots of the numerator $Q$, if any exist, must be negative, because $Q(s) > 0$ for any $s \geq 0$. The roots of $Q$ are called the "zeros" of $L_{T_{\mathbf{p}}}$, and the roots of $R$ are called the "poles" of $L_{T_{\mathbf{p}}}$.

*Theorem 2.* For some $1 \leq i \leq n$, let

$$\mathbf{p} = (p_1, p_2, \ldots, p_i, 0, \ldots, 0) \quad \text{and}$$

$$\mathbf{q} = \Big(0, \ldots, 0, \overset{i}{1}, 0, \ldots, 0\Big),$$

where $\mathbf{p}$ is a probability vector. If $L_{T_{\mathbf{p}}}$ has only real 0s, then $T_{\mathbf{p}} \leq_{\text{conv}} T_{\mathbf{q}}$.

To prove Theorem 2, we need the following results, which involve completely monotone functions. The first lemma is inspired by lemma 1 of Zemanian (1959).

*Lemma 1.* Let $H$ be a function such that $H(s) > 0$ for all $s \geq 0$, and let $-G$ be the logarithmic derivative of $H$, that is,

$$G(s) = -\frac{d}{ds}\log H(s) = -\frac{H'(s)}{H(s)}.$$

If $G(s)$ is completely monotone in $s \geq 0$, then $H(s)$ is also completely monotone in the same interval.

*Proof.* Because $H(s) > 0$, and $G$ is decreasing in $s \geq 0$, it follows that $-H'(s) \geq 0$ when $s \geq 0$. Differentiating $-H'(s)$ $n$ times, we get

$$(-1)^{n+1}H^{(n+1)}(s) = \sum_{k=0}^{n}\binom{n}{k}[(-1)^{n-k}H^{(n-k)}(s)][(-1)^{k}G^{(k)}(s)].$$

The lemma is now easily proven by induction.

The next lemma is also inspired by the work of Zemanian (1959).

*Lemma 2.* Let $H$ be a function defined as

$$H(s) = \frac{\prod_{i=1}^{h}(s-\eta_i)}{\prod_{j=1}^{m}(s-\rho_j)}, \qquad s \geq 0,$$

for some integers $h$ and $m$ and some constants $\eta_i$, $i = 1, 2, \ldots, h$, and $\rho_j$, $j = 1, 2, \ldots, m$. If $m \geq h$ and $0 > \rho_i > \eta_i$ for $i = 1, 2, \ldots, h$, then $H$ is completely monotone in $s \geq 0$.

*Proof.* The negative of the logarithmic derivative, $G$, of $H$ can be expressed as

$$G(s) = \sum_{j=1}^{m}\frac{1}{s-\rho_j} - \sum_{i=1}^{h}\frac{1}{s-\eta_i}$$

for $s \geq 0$. Differentiating $G$ $n$ times, we get

$$(-1)^{n}G^{(n)}(s) = n!\left\{\sum_{j=1}^{m}\frac{1}{(s-\rho_j)^{n+1}} - \sum_{i=1}^{h}\frac{1}{(s-\eta_i)^{n+1}}\right\}, \qquad s \geq 0.$$

The stated assumptions now imply that $G(s)$ is completely monotone in $s \geq 0$, and the stated result follows from Lemma 1.

It is worthwhile to mention that a further study of functions of the form of $H$ in Lemma 2 was given by Sumita and Masuda (1987).

*Proof of Theorem 2.* Using expression (10), write

$$\frac{L_{T_{\mathbf{q}}}(s)}{L_{T_{\mathbf{p}}}(s)} = \frac{\prod_{j=1}^{i}\frac{\delta_j}{\delta_j+s}}{\sum_{k=1}^{i}p_k\prod_{j=1}^{k}\frac{\delta_j}{\delta_j+s}} = \frac{(\prod_{j=1}^{i}\delta_j)/R_2(s)}{Q_1(s)/R_1(s)}, \qquad s \geq 0;$$

here $Q_1$ is a polynomial, and $R_1(s) = R_2(s) = \prod_{j=1}^{i}(\delta_j + s)$. Simplifying, we obtain

$$\frac{L_{T_{\mathbf{q}}}(s)}{L_{T_{\mathbf{p}}}(s)} = \frac{\prod_{j=1}^{i}\delta_j}{Q_1(s)}, \qquad s \geq 0.$$

By assumption, $Q_1$ has only real roots, and these have to be negative; see the discussion after (10). Therefore, by Lemma 2, $L_{T_{\mathbf{q}}}(s)/L_{T_{\mathbf{p}}}(s)$ is completely monotone in $s \geq 0$, and hence $T_{\mathbf{p}} \leq_{\text{conv}} T_{\mathbf{q}}$ by (5).

Another situation in which members of $\mathcal{PH}_C$ can be compared in the convolution order is described in the following theorem.

*Theorem 3.* For some $1 \leq i \leq n-1$, let

$$\mathbf{p} = (0, \ldots, 0, p_i, p_{i+1}, 0, \ldots, 0) \quad \text{and}$$
$$\mathbf{q} = (0, \ldots, 0, q_i, q_{i+1}, 0, \ldots, 0),$$

where $\mathbf{p}$ and $\mathbf{q}$ are probability vectors. If $p_i \geq q_i$ then $T_{\mathbf{p}} \leq_{\text{conv}} T_{\mathbf{q}}$.

*Proof.* Write

$$\frac{L_{T_{\mathbf{q}}}(s)}{L_{T_{\mathbf{p}}}(s)} = \frac{q_i\prod_{j=1}^{i}\frac{\delta_j}{\delta_j+s} + q_{i+1}\prod_{j=1}^{i+1}\frac{\delta_j}{\delta_j+s}}{p_i\prod_{j=1}^{i}\frac{\delta_j}{\delta_j+s} + p_{i+1}\prod_{j=1}^{i+1}\frac{\delta_j}{\delta_j+s}}$$
$$= \frac{q_i + q_{i+1}\frac{\delta_{i+1}}{\delta_{i+1}+s}}{p_i + p_{i+1}\frac{\delta_{i+1}}{\delta_{i+1}+s}} = \frac{\delta_{i+1} + q_i s}{\delta_{i+1} + p_i s}, \qquad s \geq 0.$$

The roots of the denominator and the numerator are $-\delta_{i+1}/q_i$ and $-\delta_{i+1}/p_i$. The assumption yields $0 > -\delta_{i+1}/p_i \geq -\delta_{i+1}/q_i$. Therefore, by Lemma 2, $L_{T_{\mathbf{q}}}(s)/L_{T_{\mathbf{p}}}(s)$ is completely monotone in $s \geq 0$, and hence $T_{\mathbf{p}} \leq_{\text{conv}} T_{\mathbf{q}}$ by (5).

In Theorems 1, 2, and 3, the probability vectors $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ and $\mathbf{q} = (q_1, q_2, \ldots, q_n)$ satisfy $\mathbf{p} \leq_{\text{st}} \mathbf{q}$; that is, $\sum_{j=i}^{n} p_j \leq \sum_{j=i}^{n} q_j$, $i = 1, 2, \ldots, n$. Using theorem 1.A.6 of Shaked and Shanthikumar (1994), it is easy to prove that

$$\mathbf{p} \leq_{\text{st}} \mathbf{q} \Longrightarrow T_{\mathbf{p}} \leq_{\text{st}} T_{\mathbf{q}}. \tag{11}$$

It is of interest to note that from (12) and (13) in Section 4, it follows that Theorem 3 of Kochar, Mukerjee, and Samaniego (1999) is a special case of (11). Thus, in light of (6), one may wonder whether in general it is true that $\mathbf{p} \leq_{\text{st}} \mathbf{q} \Longrightarrow T_{\mathbf{p}} \leq_{\text{conv}} T_{\mathbf{q}}$. The example after Theorem 1 shows that this is not the case.

The following example and proposition will be needed in the sequel.

*Example 1.* If $\beta_1 > \beta_2$ then $\exp(\beta_1) \leq_{\text{conv}} \exp(\beta_2)$. To see this, note that the ratio of the Laplace transforms of $\exp(\beta_2)$ and $\exp(\beta_1)$ at $s$ is equal to $(\beta_2/\beta_1)((s+\beta_1)/(s+\beta_2))$, and by Lemma 2, this ratio is completely monotone.

*Proposition 3.* For some $1 \leq i \leq n$, let

$$\mathbf{p} = (0, \ldots, 0, p_i, p_{i+1}, \ldots, p_n) \quad \text{and}$$
$$\mathbf{q} = (0, \ldots, 0, q_i, q_{i+1}, \ldots, q_n),$$

where $\mathbf{p}$ and $\mathbf{q}$ are probability vectors. If $p_i \leq q_i$ and $T_{\mathbf{p}}$ and $T_{\mathbf{q}}$ do not have the same distribution, then $T_{\mathbf{p}} \not\leq_{\text{conv}} T_{\mathbf{q}}$.

*Proof.* Write

$$\frac{L_{T_{\mathbf{q}}}(s)}{L_{T_{\mathbf{p}}}(s)} = \frac{q_i \prod_{j=1}^{i} \frac{\delta_j}{\delta_j+s} + q_{i+1} \prod_{j=1}^{i+1} \frac{\delta_j}{\delta_j+s} + \cdots + q_n \prod_{j=1}^{n} \frac{\delta_j}{\delta_j+s}}{p_i \prod_{j=1}^{i} \frac{\delta_j}{\delta_j+s} + p_{i+1} \prod_{j=1}^{i+1} \frac{\delta_j}{\delta_j+s} + \cdots + p_n \prod_{j=1}^{n} \frac{\delta_j}{\delta_j+s}}$$

$$= \frac{q_i + q_{i+1} \prod_{j=i+1}^{i+1} \frac{\delta_j}{\delta_j+s} + \cdots + q_n \prod_{j=i+1}^{n} \frac{\delta_j}{\delta_j+s}}{p_i + p_{i+1} \prod_{j=i+1}^{i+1} \frac{\delta_j}{\delta_j+s} + \cdots + p_n \prod_{j=i+1}^{n} \frac{\delta_j}{\delta_j+s}},$$

$s \geq 0.$

Note that the numerator and the denominator in the last fraction are Laplace transforms, say of $\widetilde{T}_{\mathbf{p}}$ and of $\widetilde{T}_{\mathbf{q}}$. If the ratio is completely monotone, then there exists a random variable $U$, independent of $\widetilde{T}_{\mathbf{p}}$, such that $\widetilde{T}_{\mathbf{q}} =_{\text{st}} \widetilde{T}_{\mathbf{p}} + U$. It follows that $q_i = P\{\widetilde{T}_{\mathbf{p}} + U = 0\} - P\{\widetilde{T}_{\mathbf{p}} = 0\}P\{U = 0\}$. Therefore, $P\{U = 0\} = q_i/p_i \geq 1$, a contradiction.

So far we have compared members of $\mathcal{PH}_C$ that have the same set of parameters $\delta_1, \delta_2, \ldots, \delta_n$. In the next result, the compared variables have slightly different sets of parameters, and we indicate this by making the set of parameters an argument of $T$.

*Theorem 4.* Let $\delta_0, \delta_1, \ldots, \delta_n$ be some positive constants, and let $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a probability vector. If

$$\mathbf{p} = (\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n) \quad \text{and}$$

$$\mathbf{q} = (0, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n),$$

then $T_{\mathbf{p}}(\delta_1, \delta_2, \ldots, \delta_n) \leq_{\text{conv}} T_{\mathbf{q}}(\delta_0, \delta_1, \delta_2, \ldots, \delta_n)$.

*Proof.* A straightforward computation yields $T_{\mathbf{q}}(\delta_0, \delta_1, \delta_2, \ldots, \delta_n) =_{\text{st}} T_{\mathbf{p}}(\delta_1, \delta_2, \ldots, \delta_n) + \exp(\delta_0)$, where $\exp(\delta_0)$ is independent of $T_{\mathbf{p}}(\delta_1, \delta_2, \ldots, \delta_n)$. The stated result thus follows from (2).

## 4. INFORMATION COMPARISONS OF COHERENT SYSTEMS

Let $Z_1, Z_2, \ldots, Z_{\max\{n,m\}}$ be independent and identically distributed random lifetimes with a location parameter $\theta$. Consider a reliability system of $n$ components with lifetimes $Z_1, Z_2, \ldots, Z_n$. Let $\tau_1$ be the coherent life function of the system. (See Esary and Marshall 1970 for the definition and properties of coherent life functions.) Then the lifetime of the system is $X = \tau_1(Z_1, Z_2, \ldots, Z_n)$. From the minimal path or cut set representations of $\tau_1$ [see (4.2) or (4.3) in Esary and Marshall 1970], it follows that $\theta$ is a location parameter of $X$. Thus, if $\theta$ is unknown and $X$ is observed, then some information about $\theta$ is obtained. Similarly, if $\tau_2$ is another coherent life function of $m$ components, then $\theta$ is also a location parameter of its lifetime, $Y = \tau_2(Z_1, Z_2, \ldots, Z_m)$. In this section we obtain some results that compare the information content of $X$ with that of $Y$. Using Proposition 2, we do this by identifying conditions under which $X \leq_{\text{conv}} Y$.

Note that because the order $\leq_{\text{conv}}$ is preserved under shifts [see (3)], for the purpose of obtaining $X \leq_{\text{conv}} Y$, we assume that $\theta = 0$ (this causes no loss of generality). However, the inequality $X \leq_{\text{conv}} Y$, whenever it is obtained, indicates, by Proposition 2, that $Y \leq_{\text{i}} X$ with respect to the parameter $\theta$.

Samaniego (1985) introduced, and Kochar, Mukerjee, and Samaniego (1999) further studied, a useful concept that can be used to express the lifetime of a coherent system with independent and identically distributed component lifetimes. They observed that the lifetime distribution of any coherent system, say $X = \tau_1(Z_1, Z_2, \ldots, Z_n)$, can be expressed as a mixture of the distributions of the order statistics $Z_{(1)} \leq Z_{(2)} \leq \cdots \leq Z_{(n)}$ associated with $Z_1, Z_2, \ldots, Z_n$. Explicitly, they defined the "signature" of $\tau_1$ as the probability vector $\mathbf{p}$ with elements

$$p_k = (\text{number of orderings of } Z_1, Z_2, \ldots, Z_n \text{ for which the } k\text{th failure causes system failure})/n!, \quad k = 1, 2, \ldots, n,$$

and noted that

$$X = Z_{(k)} \text{ with probability } p_k, \qquad k = 1, 2, \ldots, n. \tag{12}$$

If the $Z_i$'s have the two-parameter exponential distribution given in (1), then it is well known that the foregoing order statistics (when $\theta = 0$, which we assume without loss of generality) can be expressed as

$$Z_{(k)} = \sum_{i=1}^{k} \exp((n-i+1)\lambda), \qquad k = 1, 2, \ldots, n, \tag{13}$$

where the exponential random variables in (13) are independent. Thus we see from (12), (13), and (9) that $X = \tau_1(Z_1, Z_2, \ldots, Z_n)$ has a distribution in $\mathcal{PH}_C$. Now, using the results in Section 3 we can obtain a host of comparisons of pairs of lifetimes of coherent systems in the convolution order. By Proposition 2, these comparisons are equivalent to comparisons in the information order, with respect to the location parameter $\theta$, as defined in Definition 1.

Let $\tau_{k:n}$ denote the life function of a $k$-out-of-$n$ system. Recall that $\tau_{k:n}(Z_1, Z_2, \ldots, Z_n) = Z_{(n-k+1)}$. From Theorem 1, we get the following.

*Theorem 5.* Consider a reliability system, with coherent life function $\tau$, having $n$ components with independent two-parameter exponential lifetimes $Z_1, Z_2, \ldots, Z_n$ whose common density is given in (1). If for some $1 \leq i \leq n$, the signature of $\tau$ is of the form $\mathbf{p} = (0, \ldots, 0, p_i, p_{i+1}, \ldots, p_n)$, then

$$\tau(Z_1, Z_2, \ldots, Z_n) \leq_{\text{i}} \tau_{n-i+1:n}(Z_1, Z_2, \ldots, Z_n).$$

Roughly speaking, the inequality in Theorem 5 says that in many instances, the better (from a reliability theory standpoint) a coherent system is, the less informative it is. In light of (7), this observation is not really surprising. The density function of any $k$-out-of-$n$ system with independent and identically distributed two-parameter exponential component lifetimes is log-concave. This follows from, for example, (13) and the preservation of the log-concavity property under convolutions. Therefore, from (7) and Theorem 1, for $\tau$ of Theorem 5, we get that

$$\tau_{n-i+1:n}(Z_1, Z_2, \ldots, Z_n) \leq_{\text{disp}} \tau(Z_1, Z_2, \ldots, Z_n). \tag{14}$$

Intuitively, it is clear that the more dispersed a random variable is, the less informative it should be about a location parameter. The conclusion of Theorem 5 agrees with this intuition.

It is worthwhile to note that for exponential random variables, (14) extends the conclusion of theorem 2.1 of Khaledi and Kochar (2000). These authors studied only $k$-out-of-$n$ systems (i.e., order statistics), but under the weaker assumption that the component lifetimes have a decreasing failure rate (DFR) common distribution.

From Theorem 2, we get the following.

*Theorem 6.* Consider a reliability system, with coherent life function $\tau$, having $n$ components with independent two-parameter exponential lifetimes $Z_1, Z_2, \ldots, Z_n$ whose common density is given in (1). If for some $1 \le i \le n$, the signature of $\tau$ is of the form $\mathbf{p} = (p_1, p_2, \ldots, p_i, 0, \ldots, 0)$, and if $L_{\tau(Z_1, Z_2, \ldots, Z_n)}$ has only real 0s, then

$$\tau_{n-i+1:n}(Z_1, Z_2, \ldots, Z_n) \le_{\mathrm{i}} \tau(Z_1, Z_2, \ldots, Z_n) \le_{\mathrm{i}} \tau_{n:n}(Z_1, Z_2, \ldots, Z_n).$$

In the next result we see the influence, in the sense of information content, of adding a component to a $k$-out-of-$n$ system.

*Theorem 7.* Let $Z_1, Z_2, \ldots, Z_{n+1}$ be independent identically distributed random variables whose common density is given in (1). Then

$$\tau_{k:n+1}(Z_1, Z_2, \ldots, Z_{n+1}) \le_{\mathrm{i}} \tau_{k:n}(Z_1, Z_2, \ldots, Z_n) \le_{\mathrm{i}} \tau_{k+1:n+1}(Z_1, Z_2, \ldots, Z_{n+1}), \quad 1 \le k \le n. \tag{15}$$

*Proof.* The signature of $\tau_{k:n}$ is

$$\mathbf{p} = \Big(0, \ldots, 0, \overset{n-k+1}{1}, 0, \ldots, \overset{n}{0}\Big),$$

whereas the signature of $\tau_{k:n+1}$ is

$$\mathbf{q} = \Big(0, \ldots, 0, \overset{n-k+2}{1}, 0, \ldots, \overset{n+1}{0}\Big).$$

Therefore, by Theorem 4, $\tau_{k:n}(Z_1, Z_2, \ldots, Z_n) \le_{\mathrm{conv}} \tau_{k:n+1}(Z_1, Z_2, \ldots, Z_{n+1})$ and the first inequality of Theorem 7 is obtained from Proposition 2.

Next, the signature of $\tau_{k+1:n+1}$ is

$$\mathbf{r} = \Big(0, \ldots, 0, \overset{n-k+1}{1}, 0, \ldots, \overset{n+1}{0}\Big).$$

From (12) and (13), we see that $\tau_{k:n}(Z_1, Z_2, \ldots, Z_n)$ has its distribution in $\mathcal{PH}_C$ with the parameters $\delta_i = (n-i+1)\lambda$, $i = 1, 2, \ldots, n$, and the foregoing probability vector $\mathbf{p}$. Similarly, $\tau_{k+1:n+1}(Z_1, Z_2, \ldots, Z_{n+1})$ has its distribution in $\mathcal{PH}_C$ with the parameters $\delta_i = (n-i+2)\lambda$, $i = 1, 2, \ldots, n+1$, and foregoing the probability vector $\mathbf{r}$. That is,

$$\tau_{k:n}(Z_1, Z_2, \ldots, Z_n) = \exp(n\lambda) + \exp((n-1)\lambda) + \cdots + \exp(k\lambda),$$

and

$$\tau_{k+1:n+1}(Z_1, Z_2, \ldots, Z_{n+1}) = \exp((n+1)\lambda) + \exp(n\lambda) + \cdots + \exp((k+1)\lambda).$$

From Example 1, we see that $\exp((n+1)\lambda) \le_{\mathrm{conv}} \exp(k\lambda)$. From the closure property (4) of the order $\le_{\mathrm{conv}}$, it follows that $\tau_{k+1:n+1}(Z_1, Z_2, \ldots, Z_{n+1}) \le_{\mathrm{conv}} \tau_{k:n}(Z_1, Z_2, \ldots, Z_n)$, and the second inequality of Theorem 7 follows from Proposition 2.

The inequality

$$\tau_{k:n}(Z_1, Z_2, \ldots, Z_n) \le_{\mathrm{i}} \tau_{k+1:n}(Z_1, Z_2, \ldots, Z_n), \quad 2 \le k \le n \tag{16}$$

(which follows from Theorem 5), together with the inequalities (15) can be summarized as follows.

*Theorem 8.* Let $Z_1, Z_2, \ldots,$ be independent identically distributed random variables whose common density is given in (1). Then

$$\tau_{i:m}(Z_1, Z_2, \ldots, Z_m) \le_{\mathrm{i}} \tau_{j:n}(Z_1, Z_2, \ldots, Z_n), \text{ whenever } i \le j \text{ and } m-i \ge n-j. \tag{17}$$

*Proof.* Note that the two inequalities in (15), and the inequality (16), easily follow from (17) by a proper choice of $i$, $j$, $m$, and $n$. To show the converse, assume that (15) and (16) hold. If $m \ge n$, then

$$\tau_{i:m}(Z_1, Z_2, \ldots, Z_m) \le_{\mathrm{i}} \tau_{i:n}(Z_1, Z_2, \ldots, Z_n) \le_{\mathrm{i}} \tau_{j:n}(Z_1, Z_2, \ldots, Z_n),$$

where the first inequality follows from the left side inequality in (15) and $m \ge n$, and the second inequality follows from (16) and $i \le j$. And if $m < n$, then

$$\tau_{i:m}(Z_1, Z_2, \ldots, Z_m) \le_{\mathrm{i}} \tau_{i+n-m:n}(Z_1, Z_2, \ldots, Z_n) \le_{\mathrm{i}} \tau_{j:n}(Z_1, Z_2, \ldots, Z_n),$$

where here the first inequality follows from the right side inequality in (15) and $m < n$, and the second inequality follows from (16) and $j \ge i + n - m$.

Results of the type (17), but for other stochastic orders, have been given by Lillo, Nanda, and Shaked (2001), Nanda and Shaked (2001), Boland, Hu, Shaked, and Shanthikumar (2002), and others.

In the following two examples we illustrate the use of almost all of the results derived in this section and the previous section. In these examples, the component lifetimes are two-parameter exponential with location parameter $\theta$ and rate $\lambda = 1$.

*Example 2.* The five coherent systems of order 3 with component lifetimes $Z_1$, $Z_2$, and $Z_3$, together with their signatures, are described in Table 1. The coherent life functions of these systems are indexed by the corresponding signatures. From Theorem 3 and Proposition 2, we obtain that the lifetimes of these five systems are totally ordered with respect to their information content as follows:

$$\tau_{(1,0,0)}(Z_1, Z_2, Z_3) \ge_{\mathrm{i}} \tau_{(\frac{1}{3}, \frac{2}{3}, 0)}(Z_1, Z_2, Z_3) \ge_{\mathrm{i}} \tau_{(0,1,0)}(Z_1, Z_2, Z_3) \ge_{\mathrm{i}} \tau_{(0, \frac{2}{3}, \frac{1}{3})}(Z_1, Z_2, Z_3) \ge_{\mathrm{i}} \tau_{(0,0,1)}(Z_1, Z_2, Z_3).$$

In fact, we may refine the foregoing sequence of inequalities by adding to it the lifetimes of the two coherent systems of size 2 as follows:

$$\begin{aligned}\tau_{(1,0,0)}(Z_1,Z_2,Z_3) &\ge_{\mathrm{i}} \tau_{(1,0)}(Z_1,Z_2) \ge_{\mathrm{i}} \tau_{(\frac{1}{3},\frac{2}{3},0)}(Z_1,Z_2,Z_3)\\ &\ge_{\mathrm{i}} \tau_{(0,1,0)}(Z_1,Z_2,Z_3)\\ &\ge_{\mathrm{i}} \tau_{(0,\frac{2}{3},\frac{1}{3})}(Z_1,Z_2,Z_3)\\ &\ge_{\mathrm{i}} \tau_{(0,1)}(Z_1,Z_2) \ge_{\mathrm{i}} \tau_{(0,0,1)}(Z_1,Z_2,Z_3).\end{aligned}$$

The first and last inequalities above follow from Theorem 7; the second and the second to last inequalities follow by direct computation. However, the system that comprises only one component, $\tau(Z_1)=Z_1$, cannot be added to the foregoing sequence; it can be shown that it is not comparable with, for instance, $\tau_{(\frac{1}{3},\frac{2}{3},0)}(Z_1,Z_2,Z_3)$ in the order $\le_{\mathrm{i}}$ or, equivalently, the order $\le_{\mathrm{conv}}$.

The following lemma is used in the next example. It is a restatement of a result of Zemanian (1959).

*Lemma 3.* Let $H$ be a function of a complex variable, $z=s+i\omega$, defined as

$$H(z)=\frac{\prod_{i=1}^{h}(z-\eta_i)\prod_{i=1}^{g}(z-\nu_i)}{\prod_{i=1}^{m}(z-\rho_i)\prod_{i=1}^{q}(z-\xi_i)},$$

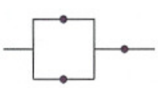
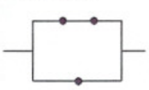
for some integers $h$, $g$, $m$, and $q$ and some real constants $\eta_i$, $i=1,2,\ldots,h$, and $\rho_i$, $i=1,2,\ldots,m$, and some complex constants $\nu_i$, $i=1,2,\ldots,g$, and $\xi_i$, $i=1,2,\ldots,q$. Suppose that the real poles are numbered according to their decreasing values, that is, $\rho_1\ge\rho_2\ge\cdots\ge\rho_m$. Denote the real parts of the complex poles, and of all the 0s, by $\alpha_i$, and number them according to their decreasing values, that is, $\alpha_1\ge\alpha_2\ge\cdots\ge\alpha_\ell$, where $\ell=h+g+q$. If $m\ge\ell$, if $0>\rho_1\ge\alpha_1$, and if

$$\sum_{i=1}^{\ell}\alpha_i\le\rho_1+(\ell-1)\min\{\rho_\ell,\alpha_\ell\}, \tag{18}$$

then $H$ is completely monotone in $s\ge0$.

*Example 3.* The 20 coherent systems of order 4, with component lifetimes $Z_1$, $Z_2$, $Z_3$, and $Z_4$, together with their signatures, are described in Table 2. Note that the 20 systems correspond to only 17 different signatures. As in Example 2, we index the coherent life functions of the systems by the corresponding signatures.

*Table 1. Coherent Systems of Size 3*

| *System* | $\tau(Z_1,Z_2,Z_3)$ | *Signature* |
|---|---|---|
| Series | $\min\{Z_1,Z_2,Z_3\}=Z_{(1:3)}$ | $(1,0,0)$ |
|  | $\min\{\max\{Z_1,Z_3\},Z_2\}$ | $(\frac{1}{3},\frac{2}{3},0)$ |
| 2-out-of-3 | $Z_{(2:3)}=\tau_{2:3}(Z_1,Z_2,Z_3)$ | $(0,1,0)$ |
|  | $\max\{\min\{Z_1,Z_3\},Z_2\}$ | $(0,\frac{2}{3},\frac{1}{3})$ |
| Parallel | $\max\{Z_1,Z_2,Z_3\}=Z_{(3:3)}$ | $(0,0,1)$ |

Thirteen of the 17 signatures have only at most 2 adjacent nonzero coordinates. Applying Theorem 3, we thus obtain

$$\begin{aligned}&\tau_{(1,0,0,0)}(Z_1,Z_2,Z_3,Z_4)\\ &\quad\ge_{\mathrm{i}}\tau_{(\frac{1}{2},\frac{1}{2},0,0)}(Z_1,Z_2,Z_3,Z_4)\ge_{\mathrm{i}}\tau_{(\frac{1}{4},\frac{3}{4},0,0)}(Z_1,Z_2,Z_3,Z_4)\\ &\quad\ge_{\mathrm{i}}\tau_{(0,1,0,0)}(Z_1,Z_2,Z_3,Z_4)\ge_{\mathrm{i}}\tau_{(0,\frac{5}{6},\frac{1}{6},0)}(Z_1,Z_2,Z_3,Z_4)\\ &\quad\ge_{\mathrm{i}}\tau_{(0,\frac{2}{3},\frac{1}{3},0)}(Z_1,Z_2,Z_3,Z_4)\ge_{\mathrm{i}}\tau_{(0,\frac{1}{2},\frac{1}{2},0)}(Z_1,Z_2,Z_3,Z_4)\\ &\quad\ge_{\mathrm{i}}\tau_{(0,\frac{1}{3},\frac{2}{3},0)}(Z_1,Z_2,Z_3,Z_4)\ge_{\mathrm{i}}\tau_{(0,\frac{1}{6},\frac{5}{6},0)}(Z_1,Z_2,Z_3,Z_4)\\ &\quad\ge_{\mathrm{i}}\tau_{(0,0,1,0)}(Z_1,Z_2,Z_3,Z_4)\ge_{\mathrm{i}}\tau_{(0,0,\frac{3}{4},\frac{1}{4})}(Z_1,Z_2,Z_3,Z_4)\\ &\quad\ge_{\mathrm{i}}\tau_{(0,0,\frac{1}{2},\frac{1}{2})}(Z_1,Z_2,Z_3,Z_4)\ge_{\mathrm{i}}\tau_{(0,0,0,1)}(Z_1,Z_2,Z_3,Z_4).\end{aligned}$$

Four of the 17 signatures have 3 nonzero coordinates. The Laplace transforms of the corresponding systems are

$$L_{\tau_{(\frac{1}{4},\frac{1}{4},\frac{1}{2},0)}(Z_1,Z_2,Z_3,Z_4)}(s)=\frac{(s+4-2\sqrt{2}i)(s+4+2\sqrt{2}i)}{(s+4)(s+3)(s+2)},\qquad s\ge0,$$

$$L_{\tau_{(\frac{1}{4},\frac{7}{12},\frac{1}{6},0)}(Z_1,Z_2,Z_3,Z_4)}(s)=\frac{(s+6+2\sqrt{3})(s+6-2\sqrt{3})}{(s+4)(s+3)(s+2)},\qquad s\ge0, \tag{19}$$

$$L_{\tau_{(0,\frac{1}{2},\frac{1}{4},\frac{1}{4})}(Z_1,Z_2,Z_3,Z_4)}(s)=\frac{6(s+2)}{(s+4)(s+3)(s+1)},\qquad s\ge0, \tag{20}$$

and

$$L_{\tau_{(0,\frac{1}{6},\frac{7}{12},\frac{1}{4})}(Z_1,Z_2,Z_3,Z_4)}(s)=\frac{2(s+5+\sqrt{13})(s+5-\sqrt{13})}{(s+4)(s+3)(s+2)(s+1)},\qquad s\ge0. \tag{21}$$

Note that the Laplace transforms in (19)–(21) have only real zeros. Thus, from Theorems 5 and 6, we obtain

$$\begin{aligned}\tau_{2:4}(Z_1,Z_2,Z_3,Z_4)&\le_{\mathrm{i}}\tau_{(\frac{1}{4},\frac{7}{12},\frac{1}{6},0)}(Z_1,Z_2,Z_3,Z_4)\\ &\le_{\mathrm{i}}\tau_{4:4}(Z_1,Z_2,Z_3,Z_4),\\ \tau_{1:4}(Z_1,Z_2,Z_3,Z_4)&\le_{\mathrm{i}}\tau_{(0,\frac{1}{2},\frac{1}{4},\frac{1}{4})}(Z_1,Z_2,Z_3,Z_4)\\ &\le_{\mathrm{i}}\tau_{3:4}(Z_1,Z_2,Z_3,Z_4),\end{aligned}$$

and

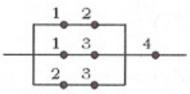
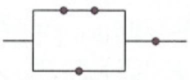
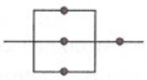
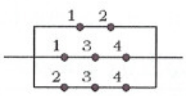
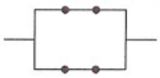
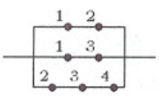
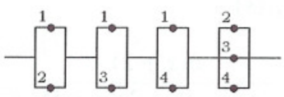
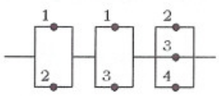
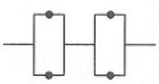
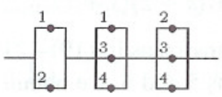
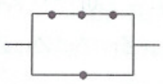
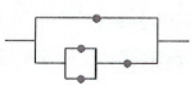
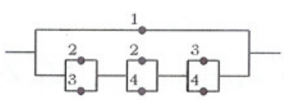
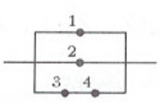
$$\begin{aligned}\tau_{1:4}(Z_1,Z_2,Z_3,Z_4)&\le_{\mathrm{i}}\tau_{(0,\frac{1}{6},\frac{7}{12},\frac{1}{4})}(Z_1,Z_2,Z_3,Z_4)\\ &\le_{\mathrm{i}}\tau_{3:4}(Z_1,Z_2,Z_3,Z_4).\end{aligned}$$

Furthermore, from Theorem 5, we also get

$$\tau_{(\frac{1}{4},\frac{1}{4},\frac{1}{2},0)}(Z_1,Z_2,Z_3,Z_4)\le_{\mathrm{i}}\tau_{4:4}(Z_1,Z_2,Z_3,Z_4).$$

With the aid of Lemma 3, we can obtain some other interesting comparisons. For example, consider the ratio of the Laplace

*Table 2. Coherent Systems of Size 4*

| *System* | $\tau(Z_1, Z_2, Z_3, Z_4)$ | *Signature* |
|---|---|---|
| Series | $\min\{Z_1, Z_2, Z_3, Z_4\} = Z_{(1:4)} = \tau_{4:4}(Z_1, Z_2, Z_3, Z_4)$ | $(1, 0, 0, 0)$ |
| Consecutive 3-out-of-4 | $\max\{\min\{Z_1, Z_2, Z_3\}, \min\{Z_2, Z_3, Z_4\}\}$ | $(\frac{1}{2}, \frac{1}{2}, 0, 0)$ |
|  | $\min\{\tau_{2:3}(Z_1, Z_2, Z_3), Z_4\}$ | $(\frac{1}{4}, \frac{3}{4}, 0, 0)$ |
|  | $\min\{\max\{Z_1, Z_2\}, \max\{Z_1, Z_3\}, Z_4\}$ | $(\frac{1}{4}, \frac{7}{12}, \frac{1}{6}, 0)$ |
|  | $\min\{\max\{Z_1, Z_2, Z_3\}, Z_4\}$ | $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0)$ |
| 3-out-of-4 | $Z_{(2:4)} = \tau_{3:4}(Z_1, Z_2, Z_3, Z_4)$ | $(0, 1, 0, 0)$ |
|  | $\max\{\min\{Z_1, Z_2\}, \min\{Z_1, Z_3, Z_4\}, \min\{Z_2, Z_3, Z_4\}\}$ | $(0, \frac{5}{6}, \frac{1}{6}, 0)$ |
|  | $\max\{\min\{Z_1, Z_2\}, \min\{Z_3, Z_4\}\}$ | $(0, \frac{2}{3}, \frac{1}{3}, 0)$ |
|  | $\max\{\min\{Z_1, Z_2\}, \min\{Z_1, Z_3\}, \min\{Z_2, Z_3, Z_4\}\}$ | $(0, \frac{2}{3}, \frac{1}{3}, 0)$ |
| Consecutive 2-out-of-4 | $\max\{\min\{Z_1, Z_2\}, \min\{Z_2, Z_3\}, \min\{Z_3, Z_4\}\}$ | $(0, \frac{1}{2}, \frac{1}{2}, 0)$ |
|  | $\max\{\min\{Z_1, \max\{Z_2, Z_3, Z_4\}\}, \min\{Z_2, Z_3, Z_4\}\}$ | $(0, \frac{1}{2}, \frac{1}{2}, 0)$ |
|  | $\max\{\min\{Z_1, \max\{Z_2, Z_3, Z_4\}\}, \min\{Z_2, Z_3\}\}$ | $(0, \frac{1}{3}, \frac{2}{3}, 0)$ |
|  | $\min\{\max\{Z_1, Z_2\}, \max\{Z_3, Z_4\}\}$ | $(0, \frac{1}{3}, \frac{2}{3}, 0)$ |
|  | $\min\{\max\{Z_1, Z_2\}, \max\{Z_1, Z_3, Z_4\}, \max\{Z_2, Z_3, Z_4\}\}$ | $(0, \frac{1}{6}, \frac{5}{6}, 0)$ |
| 2-out-of-4 | $Z_{(3:4)} = \tau_{2:4}(Z_1, Z_2, Z_3, Z_4)$ | $(0, 0, 1, 0)$ |
|  | $\max\{Z_1, \min\{Z_2, Z_3, Z_4\}\}$ | $(0, \frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ |
|  | $\max\{Z_1, \min\{Z_2, Z_4\}, \min\{Z_3, Z_4\}\}$ | $(0, \frac{1}{6}, \frac{7}{12}, \frac{1}{4})$ |
|  | $\max\{\tau_{2:3}(Z_1, Z_2, Z_3), Z_4\}$ | $(0, 0, \frac{3}{4}, \frac{1}{4})$ |
|  | $\max\{Z_1, Z_2, \min\{Z_3, Z_4\}\}$ | $(0, 0, \frac{1}{2}, \frac{1}{2})$ |
| Parallel | $\max\{Z_1, Z_2, Z_3, Z_4\} = Z_{(4:4)} = \tau_{1:4}(Z_1, Z_2, Z_3, Z_4)$ | $(0, 0, 0, 1)$ |

transforms,

$$\frac{L\tau_{(\frac{1}{4},\frac{1}{4},\frac{1}{2},0)}(Z_1,Z_2,Z_3,Z_4)(s)}{L\tau_{(\frac{1}{2},\frac{1}{2},0,0)}(Z_1,Z_2,Z_3,Z_4)(s)} = \frac{(s+4-2\sqrt{2}i)(s+4+2\sqrt{2}i)}{2(s+6)(s+2)}, \qquad s \geq 0.$$

It is easy to see that (18) and the other conditions of Lemma 3 hold, and, therefore,

$$\tau_{(\frac{1}{4},\frac{1}{4},\frac{1}{2},0)}(Z_1, Z_2, Z_3, Z_4) \leq_{\mathrm{i}} \tau_{(\frac{1}{2},\frac{1}{2},0,0)}(Z_1, Z_2, Z_3, Z_4).$$

Similarly,

$$\frac{L\tau_{(0,\frac{1}{2},\frac{1}{4},\frac{1}{4})}(Z_1,Z_2,Z_3,Z_4)(s)}{L\tau_{(\frac{1}{4},\frac{7}{12},\frac{1}{6},0)}(Z_1,Z_2,Z_3,Z_4)(s)} = \frac{6(s+2)^2}{(s+1)(s+6+2\sqrt{3})(s+6-2\sqrt{3})}, \qquad s \geq 0.$$

Again, it can be seen that (18) and the other conditions of Lemma 3 hold, and, therefore,

$$\tau_{(0,\frac{1}{2},\frac{1}{4},\frac{1}{4})}(Z_1, Z_2, Z_3, Z_4) \leq_{\mathrm{i}} \tau_{(\frac{1}{4},\frac{7}{12},\frac{1}{6},0)}(Z_1, Z_2, Z_3, Z_4).$$

With the aid of Lemma 2, we get

$$\tau_{(\frac{1}{4},\frac{7}{12},\frac{1}{6},0)}(Z_1, Z_2, Z_3, Z_4) \leq_{\mathrm{i}} \tau_{(\frac{1}{2},\frac{1}{2},0,0)}(Z_1, Z_2, Z_3, Z_4)$$

and

$$\tau_{(0,0,\frac{1}{2},\frac{1}{2})}(Z_1, Z_2, Z_3, Z_4) \leq_{\mathrm{i}} \tau_{(0,\frac{1}{6},\frac{7}{12},\frac{1}{4})}(Z_1, Z_2, Z_3, Z_4).$$

Finally, from Proposition 3, it is seen that $\tau_{(\frac{1}{4},\frac{3}{4},0,0)}(Z_1, Z_2, Z_3, Z_4)$, $\tau_{(\frac{1}{4},\frac{7}{12},\frac{1}{6},0)}(Z_1, Z_2, Z_3, Z_4)$, and $\tau_{(\frac{1}{4},\frac{1}{4},\frac{1}{2},0)}(Z_1, Z_2, Z_3, Z_4)$ are not comparable in the order $\leq_{\mathrm{i}}$.

## 5. DISCUSSION

In this article we have formalized a new type of stochastic order—the convolution order—and derive some basic useful properties of it. This order happens to be equivalent to the (reverse of the) of the well-known information ordering of statistical experiments when the unknown parameter of interest is a location one. The convolution order also has a statistical relevance in some two-sample nonparametric inference settings.

The main application of the convolution order in this article is the establishment of various information comparisons of lifetimes of different reliability coherent systems with components that have an unknown location parameter. To do this, we first obtained a number of mathematical results on properties of the convolution order for the class of the Coxian distributions, which is a subclass of the phase-type distributions. These results were applied to yield a host of information comparisons of reliability systems that have identical components with two-parameter exponential lifetimes.

The information comparisons that we obtained may be practically useful, and they also throw a new light on the meaning of reliability coherent structures in the context of statistical information theory. However, a serious practical shortcoming of the results in Section 4 is that they apply only to independent and identically distributed two-parameter exponential lifetimes. It would be nice if the exponential assumption throughout Section 4 could be weakened (that is, generalized) to other distributions. However, our method of deriving the information inequalities depends heavily on the exponential lack-of-memory property that yields (13), and consequently, using the notion of signatures, it yields the Coxian distributions of the system lifetimes. Thus we feel that there is little hope (at least using our methods) of obtaining similar nontrivial information comparisons for coherent systems with lifetimes that are not two-parameter exponential or are not identically distributed. Of course, in the trivial case of one-component "systems," the convolution ordering of the components yields at once (see Prop. 2) the information ordering of such "systems."

The statistical relevance of the convolution order in some two-sample nonparametric settings was indicated in Section 2. In fact, the convolution order arises naturally whenever any observed lifetime is a *sum* of two (quantitative) factors: the influence of the environment and the individuality inherent in the particular observation. Thus it may be useful, and of interest, to develop some statistical inference procedures under the constraint of the convolution order. Such a development has not been done in this article.

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