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AN ELLIPTIC EQUATION WITH BLOWING-UP DIFFUSION AND DATA IN L^1: EXISTENCE AND UNIQUENESS

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We establish some existence and uniqueness results for a nonlinear elliptic equation. The problem has a diffusion matrix $A(x, u)$ such that $A(x, s)\xi\xi \geq \beta(s)|\xi|^2$, with $\beta : (s_0, +\infty) \mapsto \mathbb{R}$ a continuous, strictly positive function which goes to infinity when s is near s_0. On the other hand, $\frac{A(x,s)}{\beta(s)} \in L^\infty(\Omega \times (s_0, +\infty))^{N\times N}$. Also, the right-hand side f belongs to $L^1(\Omega)$. We make use of the concept of renormalized solutions adapted to our problem.

Keywords: Nonlinear elliptic equations; renormalized solutions; singular elliptic equations; Sobolev spaces; comparison principle; Navier–Stokes equations.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ ($N > 1$) be a bounded domain. We are interested in the study of the following nonlinear elliptic problem:

$$\begin{cases} w\nabla u - \operatorname{div}[A(x, u)\nabla u] + g(u) = f & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.1}$$

where $w \in L^2(\Omega)^N$, $A(x, \cdot)$ blows up for a finite value $s_0 \in \mathbb{R}\backslash\{0\}$, $g : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ and $f \in L^1(\Omega)$ are given data.

This kind of problems is encountered in some physical models, such as the equation for the rate of turbulent energy dissipation, ε, of the k–ε turbulence model.[24] The so-called turbulence models derive from Navier–Stokes equations describing a the motions of a fluid in a region $\Omega \subset \mathbb{R}^N$. In the case of a viscous, incompressible fluid, the dimensionless Navier–Stokes equations can be written as

$$\begin{cases} \dfrac{\partial w}{\partial t} + (w \cdot \nabla) w - \dfrac{1}{\mathrm{Re}} \Delta w + \nabla p = f\,, \\ \operatorname{div} w = 0\,, \end{cases} \tag{1.2}$$

w and p being the velocity field and the pressure of the fluid, and Re the Reynolds number, a positive constant linked to the flow. When this value is high enough, the flow become unstable and turbulent structures involving both the velocity field and pressure may appear.

In this case, solving the equations numerically becomes an arduous task due to the large number of nodes of an appropriate mesh over the domain Ω. However, from a practical point of view, obtaining mean values of the flow is often sufficient for real problems. Then, it is usual to split the instantaneous velocity w into two terms: a mean part, $\bar{w}$ and a fluctuating part, w' (the same with p),

$$w = \bar{w} + w'\,,$$

$$p = \bar{p} + p'\,.$$

Therefore, (1.2) yields to the averaged equations

$$\begin{cases} \dfrac{\partial \bar{w}}{\partial t} + (\bar{w} \cdot \nabla) \bar{w} - \dfrac{1}{\mathrm{Re}} \Delta \bar{w} + \nabla \bar{p} = f - \operatorname{div}(\overline{w' \times w'})\,, \\ \operatorname{div} \bar{w} = 0\,, \end{cases} \tag{1.3}$$

where $R = \overline{w' \times w'}$ is the Reynolds stress tensor. This term needs to be modelled to avoid an open formulation of (1.3) and that is referred to as the closure problem.

Many authors agree to describe R as a diffusion tensor of the type

$$R = \nu_{\mathrm{turb}}(x,t)(\nabla \bar{w} + \nabla \bar{w}^{\mathrm{T}}) + qI\,, \tag{1.4}$$

so that ∇q appearing in (1.3) when introducing (1.4), is absorbed by the pressure term $\nabla \bar{p}$.

But in this formulation the coefficient $\nu_{\mathrm{turb}}(x,t)$ still remains undetermined. In 1972, Jones and Launder introduced the k–ε model.[21] They proposed to model this eddy viscosity as a function of two scalar variables

$$k = \frac{1}{2} |w'|^2\,, \quad \varepsilon = \frac{\nu}{2} |\nabla w' + \nabla w'^{\mathrm{T}}|^2\,,$$

and

$$\nu_{\mathrm{turb}} = c_\mu \frac{k^2}{\varepsilon}\,,$$

k being the turbulent kinetic energy, ε being the dissipation of k and c_μ obtained by experiment. In this situation, the equations for k and ε are convection-diffusion reaction type, with nonlinear coefficients, namely

$$\begin{aligned}&\frac{\partial k}{\partial t} + \bar{w}\nabla k - \operatorname{div}\left[\left(\nu_0 + c_\mu \frac{k^2}{\varepsilon}\right)\nabla k\right] - \frac{c_\mu}{2}\frac{k^2}{\varepsilon}|\nabla \bar{w} + \nabla \bar{w}^{\mathrm{T}}|^2 + \varepsilon = 0\,,\\ &\frac{\partial \varepsilon}{\partial t} + \bar{w}\nabla \varepsilon - \operatorname{div}\left[\left(\nu_0 + c_1 \frac{k^2}{\varepsilon}\right)\nabla \varepsilon\right] - c_2 k|\nabla \bar{w} + \nabla \bar{w}^{\mathrm{T}}|^2 + c_3\frac{\varepsilon^2}{k} = 0\,.\end{aligned} \tag{1.5}$$

These two equations are coupled with the Reynolds equations (1.3) and (1.4). The reader interested in a detailed description of the k–ε model is referred to Mohammadi and Pironneau.[24]

Problem (1.1) stands for the steady state of the equation for ε in (1.5), i.e.

$$\begin{cases} \bar{w}\nabla\varepsilon - \operatorname{div}\left[\left(\nu_0 + c_1\frac{k^2}{\varepsilon}\right)\nabla\varepsilon\right] + c_3\frac{\varepsilon^2}{k} = c_2 k|\nabla\bar{w} + \nabla\bar{w}^T|^2 & \text{in } \Omega\,,\\ \varepsilon = \bar{\varepsilon} & \text{on } \partial\Omega\,, \end{cases} \tag{1.6}$$

considering $\bar{w}$ and k input data in this equation. Generally, the regularity $w \in H^1(\Omega)^N$ is deduced and thus, at least if $k \in L^\infty(\Omega)$, we have $k|\nabla w + \nabla w^T|^2 \in L^1(\Omega)$.

Problem (1.6) is also meaningful when dealing with the numerical simulation of the transient k–ε system. For instance, let $M > 1$, $\tau = T/M$ and $t_j = j\tau$, considering $w^j \sim \bar{w}(\cdot, t_j)$, $k^j \sim k(\cdot, t_j)$, $\varepsilon^j \sim \varepsilon(\cdot, t_j)$ and $p^j \sim p(\cdot, t_j)$, the resulting equations would be of the type

$$\frac{w^{j+1} - w^j}{\tau} + (w^{j+1}\cdot\nabla)w^{j+1} + \nabla p^{j+1} - \operatorname{div}\left[\left(\nu_0 + c_\mu\frac{(k^j)^2}{\varepsilon^j}\right)\nabla w^{j+1}\right] = f^{j+1}\,,$$

$$\operatorname{div} w^{j+1} = 0$$

$$\begin{aligned}\frac{k^{j+1} - k^j}{\tau} &+ w^{j+1}\nabla k^{j+1} - \operatorname{div}\left[\left(\nu_0 + c_\mu\frac{(k^j)^2}{\varepsilon^j}\right)\nabla k^{j+1}\right] + \varepsilon^j\\ &= \frac{c_\mu}{2}\frac{(k^j)^2}{\varepsilon^j}|\nabla w^{j+1} + \nabla {w^{j+1}}^{\mathrm{T}}|^2\,,\end{aligned}$$

$$\begin{aligned}\frac{\varepsilon^{j+1} - \varepsilon^j}{\tau} &+ w^{j+1}\nabla\varepsilon^{j+1} - \operatorname{div}\left[\left(\nu_0 + c_1\frac{(k^{j+1})^2}{\varepsilon^{j+1}}\right)\nabla\varepsilon^{j+1}\right] + c_3\frac{(\varepsilon^{j+1})^2}{k^{j+1}}\\ &= c_2 k^{j+1}|\nabla w^{j+1} + \nabla {w^{j+1}}^{\mathrm{T}}|^2\,.\end{aligned}$$

Indeed, any numerical scheme based on finite differences for the approximation in time, such as forward or backward Euler, Crank–Nicolson or fractional step schemes,[22,28] may lead to a sequence of steady uncoupled problems like (1.6); in this sequence of steady problems, w and k have been computed in a previous stage so that they enter in (1.6) as data.

In this work, we study the difficulties involving the equation for ε considering that both w and k are known; in this way, we try to analyze the properties which can be deduced for the solutions; this may be useful for the complete resolution of the k–ε model which has not been done yet.

Blanchard and Redwane have studied some existence and uniqueness results for a similar problem but with right-hand side in $L^2(\Omega)$.[5] Orsina has analyzed the case when f is a bounded Radon measure on Ω,[27] but his model leads to L^∞-estimates for the solutions.

Due to the singularity of the diffusion coefficient, problem (1.1) contains a term which becomes an indetermination on the set $\{u = s_0\}$, which in turn may not be negligible. Besides, we do not assume any hypothesis on the asymptotic behavior of $\beta(s)$ for $s \to +\infty$. These two considerations imply that the idea of a weak solution is not well-suited for this setting. For that reason, we introduce the concept of renormalized solution for problem (1.1), which is an adaptation of the one introduced by Blanchard and Redwane.[5]

The notion of renormalized solutions was firstly introduced by DiPerna and Lions in the study of the Fokker–Planck–Boltzmann equations.[13,14] Later on, this concept has been adapted to other situations; for instance, in the analysis of linear and nonlinear elliptic equations by Boccardo, Diaz, Giachetti, Murat and Puel,[6,7] Murat,[25,26] Dal Maso, Murat, Orsina and Prignet,[12] and Gómez and Ortegón[19,20]; it has also been applied to the study of linear and nonlinear parabolic equations by Blanchard,[2] Blanchard and Murat,[3] Blanchard and Redwane,[4] P. L. Lions,[23] Climent and Fernández,[10,11] and Gómez.[19] On the other hand, Andreu, Mazón, Segura de León and Toledo[1] have made use of this technique for a degenerate parabolic equation.

The main results of this paper were first announced in a previous work[16] but in the case of a diffusion matrix $B_1(x) + B_2(x)\beta(s)$, with B_1, $B_2 \in L^\infty(\Omega)^{N\times N}$, and $\beta : (s_0, +\infty) \mapsto \mathbb{R}$, a continuous, strictly positive function which exploses for a finite, negative value s_0, $\lim_{s\to s_0^+} \beta(s) = +\infty$. Here, we consider $A(x, s)$ a general diffusion matrix satisfying several hypotheses which extend in some way the particular structure referred above. Moreover, we fully develop here the proofs of the existence and uniqueness results for (1.1).

The paper is organized as follows. First, we describe the main difficulties of the problem, then we enumerate the assumptions on data and this will lead us to the introduction of a renormalized solution to problem (1.1). The existence result is based on approximate problems whose solutions satisfy Boccardo–Gallouët estimates.[8] We end with a uniqueness result under more restrictive assumptions. For full details the reader should refer to García.[15]

2. Setting of the Problem

We will assume the following hypothesis

$(H1)$ $A : \Omega \times (s_0, +\infty) \mapsto \mathbb{R}^{N\times N}$ is a Carathedory function (i.e. $A(x, \cdot)$ is a continuous function a.e. $x \in \Omega$, and $A(\cdot, s)$ is a measurable function, for all

$s \in (s_0, +\infty))$ such that

$$A(x,s)\xi\xi \geq \beta(s)|\xi|^2 \text{ a.e. } x \in \Omega\,, \quad \text{for all } s \in (s_0, +\infty)\,,$$

and $\beta : (s_0, +\infty) \mapsto \mathbb{R}$ is a continuous function, $s_0 \in \mathbb{R},\ s_0 < 0$, and there exists a constant $\tilde{\beta} > 0$ such that

$$\beta(s) \geq \tilde{\beta}\,, \quad \text{for all } s > s_0 \quad \text{and} \quad \lim_{s \to s_0^+} \beta(s) = +\infty\,.$$

$(H2)$ $\dfrac{A(x,s)}{\beta(s)} \in L^\infty(\Omega \times (s_0, +\infty))^{N \times N}$.

$(H3)$ $g : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is a Caratheodory function such that

$$g(x,s)s \geq 0,\ \text{a.e. } x \in \Omega\,, \quad \text{for all } s \in \mathbb{R}\,;$$

$\forall\, m > 0$, $\exists\, \gamma_m \in L^1(\Omega)$, such that $\sup_{|s| \leq m} |g(x,s)| \leq \gamma_m(x)$, a.e. $x \in \Omega$, for all $s \in \mathbb{R}$.

$(H4)$ $w \in L^2(\Omega)^N$, $\operatorname{div} w = 0$ in Ω and $w \cdot n = 0$ on $\partial\Omega$, $\partial\Omega$ being a Lipschitz continuous boundary ($n = n(x)$ is the unitary, normal and outward vector on $x \in \partial\Omega$).

$(H5)$ $f \in L^1(\Omega)$.

Remark. By $A(u)$ (or $g(u)$) we mean the functions $x \mapsto A(x, u(x))$, (similarly $x \mapsto g(x, u(x))$), $x \in \Omega$.

Remark The Lipschitz continuous regularity on $\partial\Omega$ is assumed in $(H4)$ in order for the constraint $w \cdot n = 0$ to make sense in $H^{-1/2}(\partial\Omega)$. We may avoid any regularity on $\partial\Omega$ and any constraint on the normal component of w if we assume, instead of $(H4)$, the following hypothesis:

$(H4)'$ $w \in L^2(\Omega)^N$, $\operatorname{div} w = 0$ in Ω and $\exists\,(w_\delta)_\delta \subset L^\infty(\Omega)^N$ with $\operatorname{div} w_\delta = 0$ in Ω, such that $w_\delta \to w$, in $L^2(\Omega)^N$-strongly.

Obviously, under $(H4)$ we have $w \in H_0(\operatorname{div}, \Omega)$,

$$H_0(\operatorname{div}, \Omega) = \{v \in L^2(\Omega)^N, \operatorname{div} v = 0 \text{ in } \Omega,\ v \cdot n = 0 \text{ on } \partial\Omega\}\,,$$

and it is well known that $\mathcal{D}(\Omega)^N$ is dense in such a space with the $L^2(\Omega)^N$ norm,[18] and so there is a sequence w_δ satisfying $(H4)'$.

Remark. The nonlinear terms together with the singularity on s_0 and $(H5)$ constitute the main difficulties of this problem.

In fact, for a solution $u \in W_0^{1,q}(\Omega)$, $q < 2$, the term $A(u)\nabla u$ is not determined on the set $\{u = s_0\}$, which may not be negligible. On the other hand, since $f \in L^1(\Omega)$, a solution u of problem (1.1) cannot be chosen as a test function in the variational formulation. Indeed it is expected that a solution belongs to a Sobolev space $W_0^{1,q}(\Omega)$, for some $q < 2$.

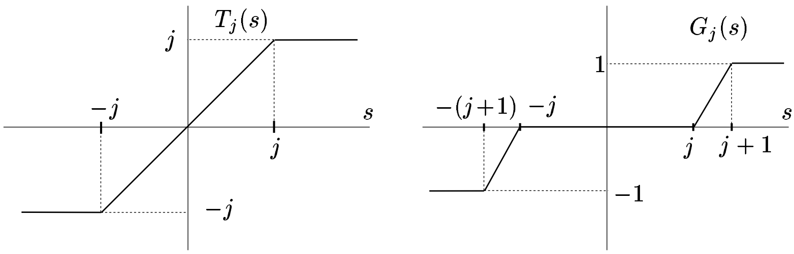

Fig. 1. Functions $T_j(s)$ and $G_j(s)$ given in (2.1) and (2.2) respectively.

These remarks show that the notion of weak solution to problem (1.1) is not well-suited; consequently, it is necessary to introduce a new functional frame adapted to this situation.

This kind of elliptic problems having a diffusion coefficient which blows up for a finite value of the unknown have been firstly studied by Blanchard and Redwane[5]; in these references it is assumed $f \in L^2(\Omega)$, $w = 0$, $g(x, u) = \lambda u$ and $A(x, s) = \beta(s)$ with $\beta(s)$ a diagonal matrix. However, in our setting we add another essential difficulty since $f \in L^1(\Omega)$.

Throughout this paper, for any non-negative real number j, we denote $T_j(s)$ the truncation function at height j (see Fig. 1), i.e.

$$T_j(s) = \operatorname{sgn} s \min(j, |s|)\,, \quad \operatorname{sgn} s = \begin{cases} \dfrac{s}{|s|} & \text{if } s \neq 0\,, \\ 0 & \text{if } s = 0\,. \end{cases} \tag{2.1}$$

We also introduce the function $G_j(s)$ defined as follows:

$$G_j(s) = T_{j+1}(s) - T_j(s) = \begin{cases} 0\,, & \text{if } |s| < j\,, \\ \operatorname{sgn} s\,, & \text{if } |s| \geq j+1\,, \\ s - j \operatorname{sgn} s\,, & \text{if } j \leq |s| \leq j+1\,. \end{cases} \tag{2.2}$$

The following lemma, due to Boccardo and Gallouët, will be used in the sequel; it is a very useful compactness result in nonlinear elliptic equations with the right-hand side in $L^1(\Omega)$ or a measure.

Lemma 1. *Let $(u_\delta)_\delta$ be a family of measurable functions such that*

(i) $T_j(u_\delta) \in H_0^1(\Omega)$, *for all* $j > 0$, *and*
(ii) $\forall\, \delta > 0$, $\exists\, C > 0$ (*independent of j and δ*) *such that* $\int_\Omega |\nabla G_j(u_\delta)|^2 \leq C$, *for all* $j > 0$.

Then $(u_\delta)_\delta$ is bounded in $W_0^{1,q}(\Omega)$ for all q such that $1 \leq q < N/(N-1)$.

Finally, we introduce the space $W_c^{1,\infty}(\mathbb{R}) = \{\varphi \in W^{1,\infty}(\mathbb{R}),\ \operatorname{supp} \varphi \text{ is compact}\}$. Now, we are ready to define the notion of renormalized solution adapted to our setting.

Definition 1. A measurable function u defined on Ω is a renormalized solution of problem (1.1) provided that

$(R1)$ $u \in L^1(\Omega)$; $u(x) \geq s_0$, a.e. $x \in \Omega$; $A(u)\nabla u \chi_{\{u>s_0\}} \in L^1(\Omega)^N$; $g(u) \in L^1(\Omega)$.
$(R2)$ $T_j(u) \in H_0^1(\Omega)$, for all $j > 0$; $A^{S/2}(u)\nabla T_j(u)\chi_{\{u>s_0\}} \in L^2(\Omega)^N$, for all $j > 0$.
$(R3)$ $\forall\, G \in W^{1,\infty}(\mathbb{R})$, with $\operatorname{supp} G'$ compact is

$$\lim_{\eta\to 0} \frac{1}{\eta} \int_{\{s_0+\eta<u<s_0+2\eta\}} A(u)\nabla u \nabla u\, G(u) = \int_{\{u=s_0\}} (g(u) - f)\, G(u)\,;$$

$(R4)$ $\displaystyle\lim_{n\to\infty} \frac{1}{n} \int_{\{s_0<u\leq n\}} A(u)\nabla u \nabla u = 0.$

$(R5)$ $\forall\, h \in W_c^{1,\infty}(\mathbb{R})$ with $h(s_0) = 0$ we have

$$\int_\Omega w\nabla u\, h(u)v + \int_\Omega A(u)\nabla u \nabla v\, h(u)\chi_{\{u>s_0\}} + \int_\Omega A(u)\nabla u\nabla u\, h'(u)\, v\chi_{\{u>s_0\}} + \int_\Omega g(u)\, h(u)v = \int_\Omega f\, h(u)v\,, \quad \text{for all } v \in \mathcal{D}(\Omega)\,. \tag{2.3}$$

Remark $A^S = (A + A^{\mathrm{T}})/2$ is the symmetric part of A; from $(H1)$ A^S turns out to be positive definite. Also, we denote $A^{S/2}$ the unique positive-definite square root of A^S.

Remark. Formulation (2.3) has been formally obtained through pointwise multiplication of (1.1) by a function $h(u)$, and then considering the resulting equality in the sense of distributions. Now every term in this last expression makes sense, thanks to the conditions assumed on both solution and data.

Remark. Condition $(R3)$ describes the behavior of u near s_0. This information was lost when substituting (1.1) by (2.3); one should notice that $G = 1$ is a possible choice. On the other hand, condition $(R4)$ yields an asymptotic behavior of the energy for great values of u. We may have $A(u)\nabla u\nabla u \notin L^1(\Omega)$ not only because of $f \notin L^2(\Omega)$, but also due to the lack of a hypothesis more restrictive on the growth of $A(s)$ for large values of s.

Remark. It is important to point out the fact that if the function h in the variational formulation (2.3) has compact support in $(s_0, +\infty)$, then we can choose any $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ as test functions in (2.3).

Remark. When dealing with a problem like (1.1) having this kind of singularitiy in the diffusion coefficient (even with regular data), special care should be taken, since it may yield a situation where a solution (in a usual sense) does not exist. For instance, consider the following problem in $N = 1$ (it is easy to construct similar examples for $N > 1$)

$$\begin{cases} -\dfrac{d}{dx}\left[\left(2 + \dfrac{1}{\sqrt{u+2}}\right)\dfrac{du}{dx}\right] = 2\alpha \quad \text{in } (-1,1)\,, \\ u(-1) = u(1) = 0\,, \end{cases} \tag{2.4}$$

α being a constant value. By applying Kirchoff's transformation

$$C(s) = \int_0^s (2 + (t+2)^{-1/2})\, dt = 2s + 2(s+2)^{1/2} - 2\sqrt{2}\,,$$

problem (2.4) becomes

$$\begin{cases} -\dfrac{d^2}{dx^2}\,[C(u)] = 2\alpha \quad \text{in } (-1,1)\,, \\ C(u(-1)) = C(u(1)) = 0\,. \end{cases}$$

Hence, the problem has only a solution which can be written as

$$C(u) = \alpha(1 - x^2)\,. \tag{2.5}$$

But on the other hand, since $C(-2) = -4 - 2\sqrt{2}$, u cannot be retrieved from (2.5) as soon as $\alpha < -4 - 2\sqrt{2}$.

Nevertheless, Theorem 1 claims that problem (1.1) has a solution in the sense of Definition 1.

3. The Existence Result

The existence of a renormalized solution to problem (1.1) is given in the following:

Theorem 1. *Under the assumptions* $(H1)$–$(H5)$ (*or* $(H4)'$ *instead of* $(H4)$), *there exists a renormalized solution* u *of problem* (1.1) *such that*

$$u \in W_0^{1,q}(\Omega)\,, \quad A(u)\nabla u \chi_{\{u>s_0\}} \in L^q(\Omega)^N\,, \quad \textit{for all } q < \frac{N}{N-1}\,.$$

Moreover, every renormalized solution u *satisfies*

$$u \in W_0^{1,q}(\Omega)\,, \quad \textit{for all } 1 \le q < \frac{N}{N-1}\,. \tag{3.1}$$

The proof of this theorem will be developed below.

3.1. *Approximate problems*

Thanks to $(H4)$ there is a sequence $(w_\delta)_\delta \subset L^\infty(\Omega)^N$, $\operatorname{div} w_\delta = 0$ in Ω and $\lim_{\delta\to 0} w_\delta = w$ strongly in $L^2(\Omega)^N$.

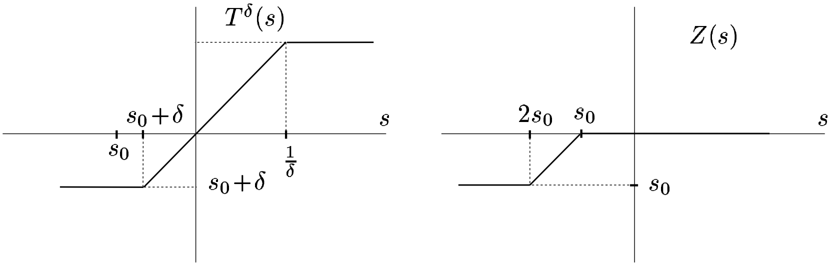

Fig. 2. Functions T^δ and $Z(s)$ given in (3.2) and (3.12) respectively.

In order to deal with the singularity at s_0, a new truncation function (see Fig. 2) will be introduced, namely

$$T^\delta(s) = \begin{cases} \dfrac{1}{\delta}\,, & \text{if } s \geq \dfrac{1}{\delta} \\ s\,, & \text{if } s_0 + \delta < s < \dfrac{1}{\delta} \\ s_0 + \delta\,, & \text{if } s \leq s_0 + \delta \end{cases} \tag{3.2}$$

then we put $A_\delta(x, s) = A(x, T^\delta(s))$; $0 < \delta < |s_0|$. Note that $A_\delta(x, s)\xi\xi \geq \beta(T^\delta(s))\xi\xi$ and $\beta(T^\delta(s)) \in L^\infty(\mathbb{R})$. We also put $f_\delta = T_{1/\delta}(f)$ and $g_\delta(s) = T_{1/\delta}(g(T_{1/\delta}(s)))$. The approximate problem is

$$\begin{cases} u_\delta \in H_0^1(\Omega) \\ w_\delta \nabla u_\delta - \operatorname{div}[A_\delta(u_\delta)\nabla u_\delta] + g_\delta(u_\delta) = f_\delta\,, \quad \text{in } \Omega\,. \end{cases} \tag{3.3}$$

The existence of a solution to (3.3) can be readily stated by a straightforward application of Schauder's fixed point theorem. In fact, $u_\delta \in H_0^1(\Omega)$ verifies the variational formulation of problem (3.3), namely

$$\int_\Omega w_\delta \nabla u_\delta \phi + \int_\Omega A_\delta(u_\delta)\,\nabla u_\delta\,\nabla\phi + \int_\Omega g_\delta(u_\delta)\phi = \int_\Omega f_\delta \phi\,, \quad \text{for all } \phi \in H_0^1(\Omega)\,. \tag{3.4}$$

Remark. Let $V \in L^\infty(\mathbb{R})$ and $\tilde{V}(s) = \int_0^s V(t)\,dt$. It is well known that (see Theorem 7.8 of Ref. 17) if $v \in H_0^1(\Omega)$ then $\tilde{V}(v) \in H_0^1(\Omega)$, and $\nabla\tilde{V}(v) = V(v)\nabla v$. In particular,

$$\int_\Omega w_\delta \nabla u_\delta V(u_\delta) = \int_\Omega w_\delta \nabla \tilde{V}(u_\delta) = -\int_\Omega \operatorname{div} w_\delta \tilde{V}(u_\delta) = 0\,. \tag{3.5}$$

This property will be continually used throughout this paper for some different choices of V.

3.2. *A priori estimates*

If $f \in L^2(\Omega)$, we can simply take u_δ as a test function in (3.4) to conclude that there exists a subsequence of $(u_\delta)_\delta$ which converges in $H_0^1(\Omega)$ weakly. Since we just assume $f \in L^1(\Omega)$, we cannot expect weak convergence of solutions in $H_0^1(\Omega)$. Taking $\phi = T_j(u_\delta)$, $j > 0$, as a test function in (3.4) we have

$$\int_\Omega A_\delta(u_\delta)\, \nabla u_\delta\, \nabla T_j(u_\delta) + \int_\Omega g_\delta(u_\delta) T_j(u_\delta) = \int_\Omega f_\delta\, T_j(u_\delta)\,. \tag{3.6}$$

Using $(H1)$ and $(H3)$, (3.6) becomes

$$\tilde{\beta} \int_\Omega \nabla T_j(u_\delta) \nabla T_j(u_\delta) \leq \int_\Omega f_\delta T_j(u_\delta)\,,$$

hence,

$$\|\nabla T_j(u_\delta)\|^2_{L^2(\Omega)} \leq \frac{j}{\tilde{\beta}} \|f\|_{L^1(\Omega)}\,. \tag{3.7}$$

Let $B_j^\delta = \{x \in \Omega : j \leq |u_\delta| < j+1\}$ and take $\phi = G_j(u_\delta)$, defined on (2.2) (see Fig. 1), in (3.4). We have

$$\int_\Omega A_\delta(u_\delta)\, \nabla u_\delta\, \nabla G_j(u_\delta) + \int_\Omega g_\delta(u_\delta) G_j(u_\delta) = \int_\Omega f_\delta G_j(u_\delta)\,.$$

Thus

$$\tilde{\beta} \int_{B_j^\delta} |\nabla u_\delta|^2 + \int_\Omega g_\delta(u_\delta) G_j(u_\delta) \leq \int_\Omega f G_j(u_\delta)\,,$$

hence

$$\tilde{\beta} \int_{B_j^\delta} |\nabla u_\delta|^2 \leq \int_{\{|u_\delta| \geq j\}} |f|\,, \tag{3.8}$$

and

$$\int_{\{|u_\delta| \geq j+1\}} |g_\delta(u_\delta)| \leq \int_{\{|u_\delta| \geq j\}} |f| \leq \|f\|_{L^1(\Omega)}\,. \tag{3.9}$$

The inequalities (3.7) and (3.8) tell us that the sequence $(u_\delta)_\delta$ satisfies Boccardo–Gallouët's estimates (Lemma 1), which lead us to the existence of some constants $C_q > 0$ such that

$$\|u_\delta\|_{W_0^{1,q}(\Omega)} \leq C_q\,, \quad \text{for all } q\text{: } 1 \leq q < \frac{N}{N-1}\,.$$

Therefore there is a subsequence, which will be denoted in the same way, and a function $u \in W_0^{1,q}(\Omega)$, such that

$$\begin{aligned} &u_\delta \rightharpoonup u\,, \quad \text{in } W_0^{1,q}(\Omega)\text{-weakly}\,, \quad \text{for all } q\text{: } 1 \leq q < \frac{N}{N-1} \\ &u_\delta \to u\,, \quad \text{in } L^r(\Omega)\text{-strongly}\,, \quad \text{for all } r\text{: } 1 \leq r < \frac{N}{N-2} \\ &u_\delta \to u\,, \quad \text{a.e. in } \Omega\,. \end{aligned} \tag{3.10}$$

Also, by the estimate (3.7) and the convergences (3.10), we have

$$T_j(u_\delta) \rightharpoonup T_j(u)\,, \quad \text{in } H_0^1(\Omega)\text{-weakly}\,, \quad \text{for all } j > 0\,.$$
$$T_{1/\delta}(u_\delta) \to u\,, \quad g_\delta(u_\delta) \to g(u)\,, \quad \text{a.e. } x \in \Omega\,. \tag{3.11}$$

3.3. *Passing to the limit*

The analysis of passing to the limit on every term of (3.4), and the proof that the limit function u appearing in (3.10) is a renormalized solution, will be divided in eight steps. We have already verified the first conditions of $(R1)$ and $(R2)$.

3.3.1. *Step 1*: $g(u) \in L^1(\Omega)$

Since u is measurable and g a Caratheodory function, we have that $g(u)$ is also measurable. It remains to show that $\int_\Omega |g(u)| < +\infty$. To do so, we write $g(u) = g(u)\chi_{\{|u|<1\}} + g(u)\chi_{\{|u|\geq 1\}}$. Using $(H3)$, we have $|g(u)|\chi_{\{|u|<1\}} \leq \gamma_1$; by virtue of (3.9), (3.11) and Fatou's lemma, we have

$$\int_\Omega |g(u)|\chi_{\{|u|\geq 1\}} \leq \varliminf_{\delta\to 0} \int_\Omega |g_\delta(u_\delta)|\chi_{\{|u_\delta|\geq 1\}} \leq \|f\|_{L^1(\Omega)}\,,$$

and so $g(u) \in L^1(\Omega)$.

3.3.2. *Step 2*: $g_\delta(u_\delta) \to g(u)$, *in* $L^1(\Omega)$*-strongly*

Obviously we can write

$$\int_\Omega |g_\delta(u_\delta) - g(u)| = \int_{\{|u_\delta|<j+1\}} |g_\delta(u_\delta) - g(u)| + \int_{\{|u_\delta|\geq j+1\}} |g_\delta(u_\delta) - g(u)|\,.$$

By Lebesgue's theorem we obtain

$$\lim_{\delta\to 0} \int_{\{|u_\delta|<j+1\}} |g_\delta(u_\delta) - g(u)| = 0\,.$$

For the second integral, we have

$$\begin{aligned}\int_{\{|u_\delta|\geq j+1\}} |g_\delta(u_\delta) - g(u)| &\leq \int_{\{|u_\delta|\geq j+1\}} |g_\delta(u_\delta)| + \int_{\{|u_\delta|\geq j+1\}} |g(u)| \\ &\leq \int_{\{|u_\delta|\geq j\}} |f| + \int_{\{|u_\delta|\geq j+1\}} |g(u)|\,.\end{aligned}$$

Therefore

$$\varlimsup_{\delta\to 0} \int_{\{|u_\delta|\geq j+1\}} |g_\delta(u_\delta) - g(u)| \leq \int_{\{|u|\geq j\}} |f| + \int_{\{|u|\geq |j+1\}} |g(u)|\,, \quad \text{for all } j > 0$$

and then $g_\delta(u_\delta) \to g(u)$ in $L^1(\Omega)$-strongly.

3.3.3. *Step 3*: meas($\{u < s_0\}$) = 0

Define $Z(s)$ as (see Fig. 2)

$$Z(s) = \begin{cases} s_0\,, & \text{if } s < 2s_0\,, \\ s - s_0\,, & \text{if } 2s_0 < s < s_0\,, \\ 0\,, & \text{if } s > s_0\,. \end{cases} \tag{3.12}$$

and take $\phi = Z(u_\delta)$ as a test function in (3.4). Then we have

$$\int_\Omega A_\delta(u_\delta)\,\nabla u_\delta\,\nabla Z(u_\delta) + \int_\Omega g_\delta(u_\delta) Z(u_\delta) = \int_\Omega f_\delta Z(u_\delta)\,.$$

Consequently

$$\int_{\{2s_0<u_\delta<s_0\}} \beta(T^\delta(u_\delta))\nabla Z(u_\delta)\nabla Z(u_\delta) \le |s_0|\,\|f\|_{L^1(\Omega)}\,,$$

and thus

$$\beta(s_0+\delta)\int_\Omega |\nabla Z(u_\delta)|^2 \le |s_0|\,\|f\|_{L^1(\Omega)}\,,$$

and from Poincaré's inequality

$$\int_\Omega |Z(u_\delta)|^2 \le \frac{C}{\beta(s_0+\delta)}\,,$$

with $C = C(\|f\|_{L^1(\Omega)}, |s_0|, \alpha)$. Now, letting $\delta \to 0$, we deduce $Z(u) = 0$, a.e. in Ω, which means that $u \ge s_0$, a.e. in Ω.

3.3.4. *Step 4*: $A_\delta^{S/2}(u_\delta)\,\nabla T_j(u_\delta) \rightharpoonup A^{S/2}(u)\,\nabla T_j(u)\,\chi_{\{u>s_0\}}$, *in* $L^2(\Omega)$*-weakly, for all* $j > 0$

In this step we are going to make use of hypothesis ($H1$). In fact, the argument is based on a certain Kirchoff's transformation.

Let $B_\delta(s) = \int_0^s \beta(T^\delta(t))\,dt$. Now we prove that the sequence $(z_\delta)_\delta = (B_\delta(u_\delta))_\delta$ verifies Boccardo–Gallouët's estimates (Lemma 1). Indeed, taking $\phi = T_j(B_\delta(u_\delta))$, $j > 0$, in (3.4) we obtain

$$\int_\Omega A_\delta(u_\delta)\nabla u_\delta \nabla T_j(B_\delta(u_\delta)) + \int_\Omega g_\delta(u_\delta) T_j(B_\delta(u_\delta)) = \int_\Omega f_\delta T_j(B_\delta(u_\delta))$$

hence,

$$\int_\Omega |\nabla T_j(B_\delta(u_\delta))|^2 \le \frac{j}{\alpha}\,\|f\|_{L^1(\Omega)}\,. \tag{3.13}$$

Now putting $\phi = G_j(B_\delta(u_\delta))$ in (3.4) we deduce

$$\int_\Omega A_\delta(u_\delta)\nabla u_\delta \nabla G_j(B_\delta(u_\delta)) + \int_\Omega g_\delta(u_\delta) G_j(B_\delta(u_\delta)) = \int_\Omega f_\delta\, G_j(B_\delta(u_\delta))\,.$$

Therefore,

$$\int_\Omega |\nabla G_j(B_\delta(u_\delta))|^2 \leq \frac{\|f\|_{L^1(\Omega)}}{\alpha}\,. \tag{3.14}$$

In conclusion, there is a subsequence (still denoted in the same way), and a limit function $z \in W_0^{1,q}(\Omega)$, such that

$$\begin{aligned}
z_\delta &\rightharpoonup z\,, && \text{in } W_0^{1,q}(\Omega)\text{-weakly}\,, && \text{for all } q\colon 1 < q < \frac{N}{N-1}\,.\\
z_\delta &\to z\,, && \text{in } L^r(\Omega)\text{-strongly}\,, && \text{for all } r\colon 1 < r < \frac{N}{N-2}\,.\\
z_\delta &\to z\,, && \text{a.e. } x \in \Omega\,, && \\
T_j(z_\delta) &\rightharpoonup T_j(z)\,, && \text{in } H_0^1(\Omega)\text{-weakly}\,, && \text{for all } j > 0\,.
\end{aligned}$$

Notice that, thanks to the everywhere convergence of both u_δ and z_δ, we have $z = B(u)$, where $B(s) = \int_0^s \beta(t)\,dt$; in particular, this shows that in the case $\int_0^{s_0} \beta(t)\,dt = -\infty$, then $\operatorname{meas}(\{u = s_0\}) = 0$.

Let us now study the relationship between ∇z and $\beta(u)\nabla u$. Since

$$\nabla u_\delta = \frac{1}{\beta(T^\delta(u_\delta))} \nabla z_\delta \tag{3.15}$$

and by virtue of being $\frac{1}{\beta(T^\delta(u_\delta))} \in L^\infty(\Omega)$, we readily obtain the convergence

$$\beta(T^\delta(u_\delta))^{-1} \to \beta(u)^{-1}\,, \quad \text{in } L^\infty(\Omega) \text{ weakly-}*\,,$$

and this implies, together with (3.10) and (3.15) that $\nabla u = \beta(u)^{-1}\nabla z$ (here, we are considering $\beta(u)^{-1} = 0$ in the set $\{u = s_0\}$). Finally,

$$\nabla z = \beta(u)\nabla u\,, \quad \text{on } \{u > s_0\}\,. \tag{3.16}$$

Now we are going to check $A_\delta^S(u_\delta)\,\nabla u_\delta \in L^q(\Omega)^N$. In fact, this is an immediate consequence from $(H2)$ and the definition of a symmetric matrix because ($\|\cdot\|$ denotes the spectral matrix norm)

$$|A_\delta^S(u_\delta)\,\nabla u_\delta|^q \leq \|A_\delta^S(u_\delta)\|^q\,|\nabla u_\delta|^q \leq C\|A_\delta(u_\delta)\|_\infty^q\,|\nabla u_\delta|^q \leq \tilde{C}\beta(T^\delta(u_\delta))^q\,|\nabla u_\delta|^q\,,$$

and $\nabla z_\delta = \beta(T^\delta(u_\delta))\,\nabla u_\delta$ is a bounded sequence in $L^q(\Omega)^N$. Consequently, there is at least a subsequence such that

$$A_\delta^S(u_\delta)\,\nabla u_\delta \rightharpoonup \xi\,, \quad \text{in } L^q(\Omega)^N\text{-weakly, } 1 \leq q < \frac{N}{N-1}\,. \tag{3.17}$$

Repeating the argument above $\nabla u_\delta = A_\delta^{-S}(u_\delta)\,A_\delta^S(u_\delta)\nabla u_\delta$, (here we are identifying $A^{-S}(u)$ as the null matrix when u reaches the value s_0) and then $\xi = A^S(u)\nabla u$ on $\{u > s_0\}$. We also notice that this also shows that the sequence $(A(u_\delta)\nabla u_\delta)_\delta$ is bounded in $L^q(\Omega)^N$ and that $A(u)\nabla u\chi_{\{u>s_0\}} \in L^q(\Omega)^N$ for $1 \leq q < N/(N-1)$.

At this point, we introduce a new sequence $(e_j^\delta)_\delta$, namely

$$e_j^\delta = A_\delta^{S/2}(u_\delta)\,\nabla T_j(u_\delta)\,.$$

From (3.6), we deduce that $(e_j^\delta)_\delta$ is bounded in $L^2(\Omega)^N$. Then there is a subsequence (still denoted in the same way) such that

$$e_j^\delta \rightharpoonup e_j\,, \quad \text{in } L^2(\Omega)^N\text{-weakly, for all } j>0\,. \tag{3.18}$$

Now, we try to identify this limit function e_j. To do so, we rewrite

$$e_j^\delta = A_\delta^{-S/2}(u_\delta)\, A_\delta^S(u_\delta)\nabla u_\delta\, \chi_{\{|u_\delta|<j\}}\,.$$

On one hand, it is obvious that

$$A_\delta^{-S/2}(u_\delta)\,\chi_{\{|u_\delta|<j\}} \to A^{-S/2}(u)\,\chi_{\{|u|<j\}}\,, \quad \text{in } L^\infty(\Omega) \text{ weakly-}*\,,$$

and on the other hand, we have (3.18) and (3.17). Thus, passing to the limit when $\delta\to 0$ on both sides of the equality, we get $e_j = A(u)^{-S/2}\,\xi\,\chi_{\{|u|<j\}})$. Moreover, the presence of the term $A^{-S/2}(u)$ allows us to deduce $e_j\equiv 0$ on the level set $\{u=s_0\}$ and then

$$\begin{aligned} e_j &= A(u)^{-S/2}\,A^S(u)\nabla u\chi_{\{u>s_0\}}\,\chi_{\{|u|<j\}} \\ &= A(u)^{S/2}\nabla T_j(u)\,\chi_{\{u>s_0\}}\,. \end{aligned}$$

In particular,

$$A(u)\nabla T_j(u)\nabla T_j(u)\chi_{\{u>s_0\}} \in L^1(\Omega)\,, \quad \text{for all } j>0\,. \tag{3.19}$$

3.3.5. *Step 5: For every $j>0$, $A_\delta(u_\delta)^{S/2}\,\nabla T_j(u_\delta) \to A(u)^{S/2}\,\nabla T_j(u)\,\chi_{\{u>s_0\}}$, in $L^2(\Omega)^N$-strongly*

This step is the key of the approximating procedure. Indeed, this strong convergence in $L^2(\Omega)^N$ will allow us to pass to the limit in certain terms which, in turn, will lead to $(R3)$ and $(R5)$. Let $\phi = T_j(u)P_l(u_\delta)H_l(z_\delta)$ be a new test function in the variational formulation (3.4), with $P_l(s)$ and $H_l(s)$ (see Fig. 3) given as follows:

$$P_l(s) = \begin{cases} 0\,, & \text{if } s \le s_0 + \dfrac{1}{l} \\ l\left(s - s_0 - \dfrac{1}{l}\right), & \text{if } s_0+\dfrac{1}{l} < s < s_0 + \dfrac{2}{l} \\ 1\,, & \text{if } s > s_0 + \dfrac{2}{l} \end{cases} \tag{3.20}$$

$$H_l(s) = \begin{cases} 2 - |s|/l\,, & \text{if } l \le |s| \le 2l \\ 1\,, & \text{if } |s| \le l \\ 0\,, & \text{if } |s| \ge 2l\,. \end{cases} \tag{3.21}$$

Note that there is a constant $M>0$, independent of δ, such that $\phi(x)=0$ almost everywhere in the set $C_M^\delta = \{x\in\Omega : |u_\delta(x)| \ge M\}$. This is true since $|z_\delta| \ge \tilde\beta|u_\delta|$

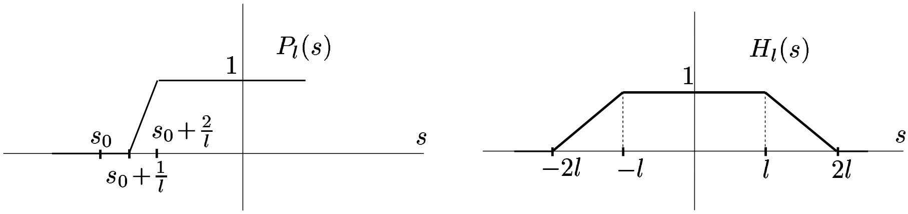

Fig. 3. Functions $P_l(s)$ and $H_l(s)$ defined in (3.20) and (3.21) respectively.

and $H_l(s)$ is zero outside the interval $[-2l, 2l]$. The support of the function ϕ is then included in the set

$$\left\{x \in \Omega: \quad \max\left(\frac{-2l}{\tilde{\beta}}, s_0 + \frac{1}{l}\right) \le u_\delta(x) \le \frac{2l}{\tilde{\beta}}\right\}. \tag{3.22}$$

The expression (3.4) for this ϕ becomes

$$\begin{aligned} &\int_\Omega w_\delta \nabla u_\delta T_j(u) P_l(u_\delta) H_l(z_\delta) + \int_\Omega A_\delta(u_\delta) \nabla u_\delta \nabla T_j(u) P_l(u_\delta) H_l(z_\delta) \\ &\quad + \int_\Omega A_\delta(u_\delta) \nabla u_\delta \nabla u_\delta P_l'(u_\delta) T_j(u) H_l(z_\delta) \\ &\quad + \int_\Omega A_\delta(u_\delta) \nabla u_\delta \nabla z_\delta H_l'(z_\delta) P_l(u_\delta) T_j(u) \\ &= \int_\Omega (f_\delta - g_\delta(u_\delta)) T_j(u) P_l(u_\delta) H_l(z_\delta). \end{aligned} \tag{3.23}$$

The next step in our analysis is to deduce the passing to the limit in (3.23); we do it by studying each integral separately:

(A) Behavior of $\int_\Omega (f_\delta - g_\delta(u_\delta)) T_j(u) P_l(u_\delta) H_l(z_\delta)$

It is straightforward by Lebesgue's theorem that

$$\lim_{\delta\to 0} \int_\Omega (f_\delta - g_\delta(u_\delta)) T_j(u) P_l(u_\delta) H_l(z_\delta) = \int_\Omega (f - g(u)) T_j(u) P_l(u) H_l(z)$$

and

$$\lim_{l\to\infty} \left\{\lim_{\delta\to 0} \int_\Omega (f_\delta - g_\delta(u_\delta)) T_j(u) P_l(u_\delta) H_l(z_\delta)\right\} = \int_\Omega (f - g(u)) T_j(u) \chi_{\{u>s_0\}}.$$

(B) Behavior of $\int_\Omega w_\delta \nabla u_\delta T_j(u) P_l(u_\delta) H_l(z_\delta)$

Putting $M = M(l) = 2l/\tilde{\beta}$ we have

$$\int_\Omega w_\delta \nabla u_\delta T_j(u) P_l(u_\delta) H_l(z_\delta) = \int_\Omega w_\delta \nabla T_M(u_\delta) T_j(u) P_l(u_\delta) H_l(z_\delta).$$

Since $\nabla T_M(u_\delta) \rightharpoonup \nabla T_M(u)$ weakly in $L^2(\Omega)$, we deduce that

$$\nabla T_M(u_\delta) T_j(u) P_l(u_\delta) H_l(z_\delta) \rightharpoonup \nabla T_M(u) T_j(u) P_l(u) H_l(B(u)) \quad \text{weakly in } L^2(\Omega).$$

On the other hand, $w_\delta \to w$ strongly in $L^2(\Omega)$. Thus,

$$\lim_{\delta\to 0}\int_\Omega w_\delta\,\nabla u_\delta\,T_j(u)\,P_l(u_\delta)\,H_l(z_\delta) = \int_\Omega w\,\nabla T_M(u)\,T_j(u)\,P_l(u)\,H_l(B(u))\,.$$

This last integral is equal to zero since (see (3.5) and the associated remark) $T_M(u) \in H_0^1(\Omega)$ and choosing $V \in L^\infty(\mathbb{R})$ as

$$V(s) = T_j(T_M(s))\,P_l(T_M(s))\,H_l(B(T_M(s)))\,,\quad \tilde{V}(s) = \int_0^s V(t)\,dt\,,$$

we have

$$\int_\Omega w\nabla T_M(u)\,T_j(u)P_l(u)H_l(A(u)) = \int_\Omega w\nabla\tilde{V}(T_M(u)) = -\langle\nabla\cdot w, \tilde{V}(T_M(u))\rangle = 0\,,$$

where $\langle\cdot\,,\cdot\rangle$ stands for the duality pair between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

(C) Behavior of $\int_\Omega A_\delta(u_\delta)\,\nabla u_\delta\,\nabla T_j(u)\,P_l(u_\delta)\,H_l(z_\delta)$

Obviously, when $\delta \to 0$ in such an expression

$$A_\delta(u_\delta)\,\nabla T_j(u)\,P_l(u_\delta)\,H_l(z_\delta) \to A(u)\,\nabla T_j(u)\,P_l(u)\,H_l(z)\,,\quad \text{in } L^2(\Omega)\text{-strongly}\,,$$

since $u_\delta \to u$ almost everywhere $x \in \Omega$, the support of u_δ is included in a compact as (3.22), and A is a Caratheodory function. On the other hand, recall that $M = M(l) = 2l/\tilde{\alpha}$,

$$\nabla T_M(u_\delta) \rightharpoonup \nabla T_M(u)\,,\quad \text{in } L^2(\Omega)\text{-weakly}\,.$$

Hence

$$\lim_{\delta\to 0}\int_\Omega A_\delta(u_\delta)\nabla u_\delta\,\nabla T_j(u)\,P_l(u_\delta)\,H_l(z_\delta) = \int_\Omega A(u)\nabla T_M(u)\,\nabla T_j(u)\,P_l(u)\,H_l(B(u))\,.$$

If l is large enough, say $l \geq \frac{\tilde{\beta}j}{2}$,

$$\nabla T_M(u)\,\nabla T_j(u) = \nabla T_M(T_j(u))\,\nabla T_j(u)\,,\quad \text{for all } M \geq j\,.$$

Consequently, (3.19) and Lebesgue's theorem yield

$$\lim_{l\to\infty}\left\{\lim_{\delta\to 0}\int_\Omega A_\delta(u_\delta)\nabla u_\delta\,\nabla T_j(u)\,P_l(u_\delta)\,H_l(z_\delta)\right\} = \int_\Omega A(u)\nabla T_j(u)\,\nabla T_j(u)\,\chi_{\{u>s_0\}}\,.$$

(D) Behavior of $\int_\Omega A_\delta(u_\delta)\,\nabla u_\delta\,\nabla u_\delta\,P_l'(u_\delta)\,H_l(z_\delta)T_j(u)$

In order to study this integral we begin to notice that from (3.20) $P_l'(u_\delta) = l\,\chi_{\{s_0+\frac{1}{l}<u_\delta<s_0+\frac{2}{l}\}}$. Then, we take a new test function in (3.4), namely $\phi = T_j(u)K_l(u_\delta)H_l(z_\delta)$, where (see Fig. 4)

$$K_l(s) = P_l(s) - 1\,,\quad \text{for all } s \in \mathbb{R}\,. \tag{3.24}$$

In this situation, formulation (3.4) leads to the analysis of some new integrals.

Firstly, it is immediate to remark that

$$\lim_{l\to\infty}\left\{\lim_{\delta\to 0}\int_\Omega (f_\delta - g_\delta(u_\delta))\,T_j(u)\,K_l(u_\delta)\,H_l(z_\delta)\right\} = -\int_{\{u=s_0\}}(f - g(u))T_j(u)\,.$$

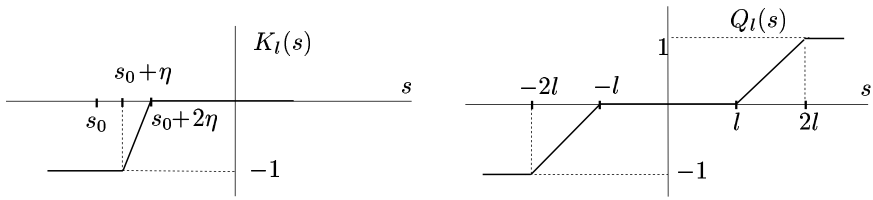

Fig. 4. Functions $K_l(s)$ and $Q_l(s)$ defined in (3.24) and (3.28) respectively.

Secondly, (3.15) yields the equality

$$\int_\Omega A_\delta(u_\delta)\,\nabla u_\delta\,\nabla T_j(u)\,K_l(u_\delta)\,H_l(z_\delta)$$

$$= \int_\Omega \frac{A_\delta(u_\delta)}{\beta(T^\delta(u_\delta))}\,\nabla T_{2l}(z_\delta)\,\nabla T_j(u)\,K_l(u_\delta)\,H_l(z_\delta)\,\chi_{\{u>s_0\}}\,,$$

since $\nabla T_j(u) = \nabla T_j(u)\,\chi_{\{u>s_0\}}$. Thanks to $(H2)$, we have

$$\frac{A_\delta(u_\delta)}{\beta(T^\delta(u_\delta))}\chi_{\{u>s_0\}} \to \frac{A(u)}{\beta(u)}\chi_{\{u>s_0\}}\,, \quad \text{a.e. } x\in\Omega, \text{ and *-weakly in } L^\infty(\Omega)\,.$$

Therefore, using the weak convergence in $L^2(\Omega)^N$ for the sequence $\nabla T_{2l}(z_\delta)$

$$\lim_{\delta\to 0}\int_\Omega A_\delta(u_\delta)\,\nabla u_\delta\,\nabla T_j(u)\,K_l(u_\delta)\,H_l(z_\delta)$$

$$= \int_{\{u>s_0\}} \frac{A(u)}{\beta(u)}\,\nabla B(u)\,\nabla T_j(u)\,K_l(u)\,H_l(z)\,,$$

and remarking (3.19) we consider now the limit of that expression when l goes to infinity,

$$\lim_{l\to\infty}\int_{\{u>s_0\}} A(u)\,\nabla u\,\nabla T_j(u)\,K_l(u)\,H_l(z)\,\chi_{\{u>s_0\}}$$

$$= -\int_\Omega \frac{A(u)}{\beta(u)}\,\nabla B(u)\,\nabla T_j(u)\,\chi_{\{u=s_0\}}\,\chi_{\{u>s_0\}}\,.$$

But $\chi_{\{u=s_0\}}\,\chi_{\{u>s_0\}} = 0$, almost everywhere in Ω, then

$$\lim_{l\to\infty}\left\{\lim_{\delta\to 0}\int_\Omega A_\delta(u_\delta)\,\nabla u_\delta\,\nabla T_j(u)\,K_l(u_\delta)\,H_l(z_\delta)\right\} = 0\,.$$

The treatment of the transport term follows as in (B). In order to finish (D), it only remains to study the term

$$I_{\delta l} = \int_\Omega A_\delta(u_\delta)\nabla u_\delta\nabla z_\delta T_j(u)K_l(u_\delta)H_l'(z_\delta)\,. \tag{3.25}$$

In fact, we will show that

$$\lim_{l\to\infty}\left\{\overline{\lim_{\delta\to 0}}\,|I_{\delta l}|\right\} = 0\,. \tag{3.26}$$

Indeed, we have

$$|I_{\delta l}| \leq \frac{j}{l} \int_{\{l<|z_\delta|<2l\}} A_\delta(u_\delta)\, \nabla u_\delta\, \nabla z_\delta \tag{3.27}$$

because $A_\delta(u_\delta)\nabla u_\delta \nabla z_\delta \geq 0$. This last integral can be obtained taking in (3.4) $\phi = Q_l(z_\delta) \in H_0^1(\Omega)$ (see Fig. 4), with

$$Q_l(s) = \begin{cases} \operatorname{sgn} s\,, & \text{if } |s| > 2l \\ \dfrac{s}{l} - \operatorname{sgn} s & \text{if } l < |s| < 2l \\ 0 & \text{if } |s| < l \end{cases} \tag{3.28}$$

then we obtain,

$$\lim_{\delta\to 0} \frac{1}{l} \int_{\{l<|z_\delta|<2l\}} A_\delta(u_\delta)\, \nabla u_\delta\, \nabla z_\delta = \int_\Omega (f - g(u))\, Q_l(z)\,,$$

which yields,

$$\lim_{l\to\infty} \left\{ \lim_{\delta\to 0} \frac{1}{l} \int_{\{l<|z_\delta|<2l\}} A_\delta(u_\delta)\nabla u_\delta \nabla z_\delta \right\} = 0 \tag{3.29}$$

and this implies (3.26). So, summing up, we have shown that

$$\lim_{l\to\infty} \left\{ \overline{\lim_{\delta\to 0}} \int_\Omega A_\delta(u_\delta)\nabla u_\delta\, P_l'(u_\delta) H_\delta(z_\delta) T_j(u) \right\} = \int_{\{u=s_0\}} (g(u) - f) T_j(u)\,.$$

(E) Behavior of $II_{\delta l} = \int_\Omega A_\delta(u_\delta)\, \nabla u_\delta\, \nabla z_\delta\, H_l'(z_\delta)\, P_l(u_\delta) T_j(u)$

This is identical to $I_{\delta l}$ and it is straightforward that $\lim_{l\to\infty}\{\overline{\lim}_{\delta\to 0} |II_{\delta l}|\} = 0$.

(F) Conclusion

By virtue of (A)–(E), we deduce that

$$\int_\Omega A(u)\, \nabla T_j(u)\, \nabla T_j(u)\, \chi_{\{u>s_0\}} = \int_\Omega (f - g(u))\, T_j(u)\,, \quad \text{for all } j > 0\,. \tag{3.30}$$

On the other hand, taking $\phi = T_j(u_\delta)$ in (3.4), we readily find that

$$\lim_{\delta\to 0} \int_\Omega A_\delta(u_\delta)\, \nabla u_\delta\, \nabla T_j(u_\delta) = \int_\Omega (f - g(u))\, T_j(u)\,, \tag{3.31}$$

and comparing (3.30) and (3.31), we get for every $j > 0$,

$$\lim_{\delta\to 0} \int_\Omega A_\delta(u_\delta)\, \nabla T_j(u_\delta)\, \nabla T_j(u_\delta) = \int_\Omega A(u)\, \nabla T_j(u)\, \nabla T_j(u) \chi_{\{u>s_0\}}\,. \tag{3.32}$$

By virtue of the weak convergence in $L^2(\Omega)$ of the sequence $(e_j^\delta)_\delta$ to e_j, step 4 together with (3.32), we must conclude that $(e_j^\delta)_\delta$ converges in $L^2(\Omega)$-strongly to e_j.

3.3.6. *Step 6: Behavior of u near s_0*

Now we take $\phi = G(u_\delta)K_l(u_\delta)$ in (3.4), with $K_l(s)$ defined in (3.24) and any $G \in W^{1,\infty}(\mathbb{R})$, $\operatorname{supp} G'$ compact. It must be noticed that $G(s) = 1$ could be a right choice. Then,

$$l \int_{\{s_0+\frac{1}{l}<u_\delta<s_0+\frac{2}{l}\}} A_\delta(u_\delta)\,\nabla u_\delta\,\nabla u_\delta\,G(u_\delta) + \int_\Omega A_\delta(u_\delta)\,\nabla u_\delta\,\nabla G(u_\delta)\,K_l(u_\delta)$$
$$= \int_\Omega (f_\delta - g_\delta(u_\delta))\,K_l(u_\delta)\,G(u_\delta)\,. \tag{3.33}$$

Let $R > 0$ such that $\operatorname{supp} G' \subset [-R, R]$, then $G'(u_\delta)\nabla u_\delta = G'(u_\delta)\nabla T_R(u_\delta)$. Moreover, as far as the first integral of (3.33) is concerned, we get for l large enough,

$$\nabla u_\delta\,\chi_{\{s_0+\frac{1}{l}<u_\delta<s_0+\frac{2}{l}\}} = \nabla T_r(u_\delta)\,\chi_{\{s_0+\frac{1}{l}<u_\delta<s_0+\frac{2}{l}\}} \quad \text{with } r = \left|s_0 + \frac{1}{l}\right|.$$

Passing to the limit when $\delta \to 0$ in (3.33), according to the strong convergence deduced in the previous step, we have

$$\int_\Omega A(u)\,\nabla u\,\nabla G(u)\,K_l(u)\chi_{\{u>s_0\}} + l\int_{\{s_0+\frac{1}{l}<u<s_0+\frac{2}{l}\}} A(u)\,\nabla u\,\nabla u\,G(u)$$
$$= \int_\Omega (f - g(u))\,G(u)\,K_l(u)\,.$$

Finally

$$\lim_{l\to\infty} l\int_{\{s_0+\frac{1}{l}<u<s_0+\frac{2}{l}\}} A(u)\nabla u\nabla u\,G(u) = \int_{\{u=s_0\}} (g(u) - f)G(u)\,,$$

and then condition $(R3)$ in Definition 1.

3.3.7. *Step 7: Asymptotic behavior of u*

Setting $\phi = \frac{1}{n}T_n(u_\delta)$ in (3.4), we have

$$\frac{1}{n}\int_{\{|u_\delta|\le n\}} A_\delta(u_\delta)\,\nabla u_\delta\,\nabla u_\delta = \frac{1}{n}\int_\Omega (f_\delta - g_\delta(u_\delta))\,T_n(u_\delta)\,.$$

As before, passing to the limit when $\delta \to 0$ we have

$$\frac{1}{n}\int_{\{|u|\le n\}} A(u)\,\nabla u\,\nabla u\chi_{\{u>s_0\}} = \frac{1}{n}\int_\Omega (f - g(u))T_n(u)\,.$$

Hence,

$$\lim_{n\to\infty}\frac{1}{n}\int_{\{|u|\le n\}} A(u)\,\nabla u\,\nabla u\,\chi_{\{u>s_0\}} = 0\,,$$

and so

$$\lim_{n\to\infty}\frac{1}{n}\int_{\{s_0<u\le n\}} A(u)\,\nabla u\,\nabla u = 0\,.$$

3.3.8. *Step 8*: *u satisfies* (2.3)

Let $h \in W^{1,\infty}_{\mathrm{c}}(\mathbb{R})$, such that $h(s_0) = 0$, and $v \in \mathcal{D}(\Omega)$. Choosing $\phi = h(u_\delta)\, v$ as a test function in (3.4) we obtain

$$\int_\Omega w_\delta \,\nabla u_\delta\, h(u_\delta)\, v + \int_\Omega A_\delta(u_\delta)\,\nabla u_\delta\,\nabla v\, h(u_\delta) + \int_\Omega A_\delta(u_\delta)\,\nabla u_\delta\,\nabla u_\delta\, v\, h'(u_\delta) + \int_\Omega g_\delta(u_\delta)\, v\, h(u_\delta) = \int_\Omega f_\delta\, v\, h(u_\delta)\,. \quad (3.34)$$

Let $M > 0$ such that $\operatorname{supp} h \subset [-M, M]$. In order to pass to the limit in (3.34), we will analyze again each integral separately. It is straightforward that

$$\lim_{\delta\to 0} \int_\Omega (f_\delta - g_\delta(u_\delta)) h(u_\delta)\, v = \int_\Omega (f - g(u))\, h(u)\, v\,,$$

$$\lim_{\delta\to 0} \int_\Omega w_\delta\,\nabla u_\delta\, h(u_\delta)\, v = \lim_{\delta\to 0} \int_\Omega w_\delta\,\nabla T_M(u_\delta)\, h(u_\delta) v = \int_\Omega w \nabla T_M(u)\, h(u)\, v = \int_\Omega w\,\nabla u\, h(u)\, v\,.$$

On the other hand, thanks to strong convergence deduced in step 5,

$$\lim_{\delta\to 0} \int_\Omega A_\delta(u_\delta)\,\nabla T_M(u_\delta)\,\nabla T_M(u_\delta)\, h'(u_\delta)\, v = \int_\Omega A(u) \nabla T_M(u)\,\nabla T_M(u)\, h'(u)\, v\,\chi_{\{u>s_0\}}\,.$$

It only remains to examine the term

$$\int_\Omega A_\delta(u_\delta)\,\nabla u_\delta\,\nabla v\, h(u_\delta)\,,$$

which according to (3.15) can be written as

$$\int_\Omega \frac{A_\delta(u_\delta)}{\beta(T^\delta(u_\delta))}\,\nabla z_\delta\,\nabla v h(u_\delta)\,.$$

Using that $h(s_0) = 0$, hypothesis $(H2)$, and the almost everywhere convergence in Ω of u_δ to u yield

$$\frac{A_\delta(u_\delta)}{\beta(T^\delta(u_\delta))}\, h(u_\delta) \to \frac{A(u)}{\beta(u)}\, h(u)\,, \quad \text{a.e. } x \in \Omega, \text{ and *-weakly in } L^\infty(\Omega)\,.$$

Then, since $\nabla z_\delta \rightharpoonup \nabla z$ in $L^q(\Omega)^N$-weakly, we can pass to the limit in that integral when $\delta \to 0$. In fact, it would be enough that $\nabla v \in L^{q'}(\Omega)$, for some $q' > N$.

$$\lim_{\delta\to 0} \int_\Omega A_\delta(u_\delta)\,\nabla u_\delta\,\nabla v\, h(u_\delta) = \int_\Omega \frac{A(u)}{\beta(u)}\,\nabla z\,\nabla v\, h(u)\,.$$

By virtue of (3.16), we have

$$\int_\Omega \frac{A(u)}{\beta(u)}\,\nabla z\,\nabla v\, h(u) = \int_\Omega A(u)\,\nabla u\,\nabla v\, h(u)\,\chi_{\{u>s_0\}}\,.$$

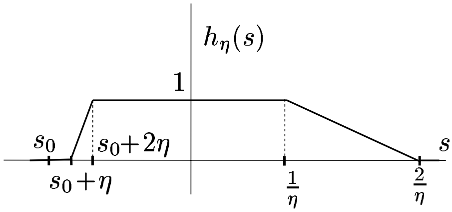

Fig. 5. Function $h_\eta(s)$ defined in (3.35).

Therefore, u satisfies the variational formulation (2.3). We have shown that u is a renormalized solution of problem (1.1) in the sense of Definition 1.

3.4. *Every renormalized solution u verifies* (3.1)

Let u be a renormalized solution of (1.1). We want to show that $u \in W_0^{1,q}(\Omega)$, with $1 \le q < \frac{N}{N-1}$. By Lemma 1, it is enough to show that $G_j(u) \in H_0^1(\Omega)$ and is bounded (not dependent on j) in this space, with $G_j(s)$ being given in (2.2). To do so, we choose $h = h_\eta$ (see Fig. 5) and $\phi = G_j(u)$ in (2.3), where $h_\eta(s)$ is defined by

$$h_\eta(s) = \begin{cases} 0\,, & \text{if } s < s_0 + \eta\,, \\ \dfrac{1}{\eta}(s - s_0 - \eta)\,, & \text{if } s_0 + \eta < s < s_0 + 2\eta\,, \\ 1\,, & \text{if } s_0 + 2\eta < s < \dfrac{1}{\eta}\,, \\ -\eta s + 2\,, & \text{if } \dfrac{1}{\eta} < s < \dfrac{2}{\eta} \\ 0\,, & \text{if } s > \dfrac{2}{\eta}\,. \end{cases} \tag{3.35}$$

We can choose these functions since $h_\eta \in W_c^{1,\infty}(\mathbb{R})$ and $\operatorname{supp} h \subset (s_0, +\infty)$. Also, $G_j(u) = T_{j+1}(u) - T_j(u) \in H_0^1(\Omega) \cap L^\infty(\Omega)$ (see last remark in Sec. 2). Then, we obtain

$$\int_\Omega A(u)\,\nabla u\,\nabla G_j(u)\,h_\eta(u) + \frac{1}{\eta}\int_{\{s_0+\eta<u<s_0+2\eta\}} A(u)\,\nabla u\,\nabla u\,G_j(u)$$
$$-\eta\int_{\{\frac{1}{\eta}<u<\frac{2}{\eta}\}} A(u)\,\nabla u\,\nabla u\,G_j(u) = \int_\Omega (f - g(u))\,h_\eta(u)\,G_j(u)\,.$$

Making $\eta \to 0$, and since u is a renormalized solution, it yields

$$\int_\Omega A(u)\,\nabla u\,\nabla G_j(u) = \int_\Omega (f - g(u))\,G_j(u)\,.$$

Consequently, $\int_\Omega |\nabla G_j(u)|^2 \le \|f - g(u)\|_{L^1(\Omega)}/C_1$, for all $j > 0$.

4. A Uniqueness Analysis of Renormalized Solutions

A uniqueness result of renormalized solutions may be deduced under more restrictive assumptions on data. We consider three new assumptions, namely

$(H6)$ $\displaystyle\int_0^{s_0} \beta(s)\,ds = -\infty$.

$(H7)$ $\frac{A(x,s)}{\beta(s)}$ is a locally Lipschitz function in the variable s, i.e.

$$\forall\, n \geq 1,\ \exists\, C_n > 0\,, \quad \text{such that } \left| \frac{A(x,s_1)}{\beta(s_1)} - \frac{A(x,s_2)}{\beta(s_2)} \right| \leq C_n |s_1 - s_2|\,,$$

$$\forall\, s_1, s_2 \text{ with } s_0 + \frac{1}{n} < s_i < 2n, i = 1,2, \text{ a.e. } x \in \Omega\,.$$

$(H8)$ Either $g(x,0) = 0$, or $g(\cdot,0) \in L^1(\Omega)$ and g verifies the condition

$$(g(x,t_1) - g(x,t_2))(t_1 - t_2) > 0\,, \quad \text{for all } t_1, t_2 > s_0\,.$$

$(H9)$ $\exists\,\sigma > N$ such that $w \in L^\sigma(\Omega)^N$ and $\operatorname{div} w = 0$, in Ω.

We recall the primitive of $\beta(s)$ which is null at 0,

$$B(s) = \int_0^s \beta(t)\,dt\,. \tag{4.1}$$

We will need the following result in the sequel.

Lemma 1. *Let u be a renormalized solution of problem* (1.1). *We will assume hypothesis* (H1)–(H6). *Then*

(i) $u > s_0$, *a.e.* $x \in \Omega$,
(ii) $B(u) \in W_0^{1,1}(\Omega)$ *and* $\nabla B(u) = \beta(u)\nabla u$,
(iii) $T_j(B(u)) \in H_0^1(\Omega)$, *for all* $j > 0$,
(iv) $\displaystyle\lim_{n\to\infty} \frac{1}{n} \int_{\{|B(u)|\leq n\}} A(u)\nabla u \nabla B(u) = 0.$

Proof of Lemma 1. Let $B_j(s) = \int_0^s T_j(\beta(t))\,dt$ for $j > 0$; then B_j is a globally Lipschitz function in $\mathbb{R}$. Since u is a renormalized solution of problem (1.1), u satisfies (3.1) and also $B_j(u) \in W_0^{1,q}(\Omega) \subset W_0^{1,1}(\Omega)$, for all $q < N/(N-1)$ with $\nabla B_j(u) = T_j(\beta(u))\nabla u$. The definition of a renormalized solution also yields $u \geq s_0$ almost everywhere in Ω, and therefore

$$B_j(u) \overset{j\to\infty}{\longrightarrow} B(u), \text{ a.e. } x \in \Omega\,. \quad \nabla B_j(u) \overset{j\to\infty}{\longrightarrow} \beta(u)\nabla u \chi_{\{u>s_0\}}\,, \text{ a.e. } x \in \Omega\,. \tag{4.2}$$

In fact, we have $(\nabla B_j(u))_j > 0$ bounded above by $\beta(u)\nabla u\chi_{\{u>s_0\}} \in L^1(\Omega)$ since u is a renormalized solution. Consequently, by Lebesgue's theorem,

$$\nabla B_j(u) \to \beta(u)\nabla u \chi_{\{u>s_0\}}\,, \quad \text{strongly in } L^1(\Omega)\,. \tag{4.3}$$

By virtue of (4.2), (4.3) we deduce that $B(u) \in W_0^{1,1}(\Omega)$ and we also have that $\nabla B(u) = \beta(u)\nabla u \chi_{\{u>s_0\}}$.

Using $(H6)$ and the fact that $B(u) \in L^1(\Omega)$, yield $\{u > s_0\}$ almost everywhere in Ω, and consequently $\nabla B(u) = \beta(u)\nabla u$. This proves (i) and (ii). Now we will show (iii). First of all, it is obvious that $T_j(B(u)) \in L^\infty(\Omega) \subset L^2(\Omega)$. On the other hand,

$$\begin{aligned}\nabla T_j(B(u))\nabla B(u)\,\chi_{\{|B(u)|\le j\}} &= \beta(u)\,\nabla u\,\chi_{\{|B(u)|\le j\}}\\ &= \beta(u)\,\nabla u\,\chi_{\{B^{-1}(-j)\le u\le B^{-1}(j)\}}\,.\end{aligned}$$

Assuming $(H6)$, the interval $[B^{-1}(-j), B^{-1}(j)]$ is a compact set of $(s_0, +\infty)$ and so

$$\beta(u)\chi_{\{B^{-1}(-j)\le u\le B^{-1}(j)\}} \in L^\infty(\Omega)\,.$$

Furthermore, if we denote $\rho(j) = \max\{B^{-1}(j), -B^{-1}(-j)\}$ we get

$$|\nabla u\chi_{\{B^{-1}(-j)\le u\le B^{-1}(j)\}}| \le \nabla u|\chi_{\{-\rho(j)\le u\le\rho(j)\}} = |\nabla T_{\rho(j)}(u)| \in L^2(\Omega)$$

thanks to condition $(R2)$. Hence, $T_j(B(u)) \in H_0^1(\Omega)$.

In order to show (iv), we choose $h = h_\eta$ and $v = T_n(B(u))/n$ in the variational formulation (2.3), with $h_\eta(s)$ the function defined in (3.35). Then (2.3) becomes

$$\begin{aligned}&\frac{1}{n}\int_{\{|B(u)|\le n\}} A(u)\nabla u\nabla B(u)h_\eta(u) + \frac{1}{\eta}\int_{\{s_0+\eta<u<s_0+2\eta\}} A(u)\nabla u\nabla u\frac{1}{n}\,T_n(B(u))\\ &\quad -\eta\int_{\{\frac{1}{\eta}<u<\frac{2}{\eta}\}} A(u)\nabla u\nabla u\frac{1}{n}T_n(B(u)) = \int_\Omega (f-g(u))h_\eta(u)\frac{1}{n}\,T_n(B(u))\,.\end{aligned} \tag{4.4}$$

It is easy to check that

$$\lim_{\eta\to 0}\frac{1}{n}\int_{\{|B(u)|\le n\}} A(u)\nabla u\nabla B(u)h_\eta(u) = \frac{1}{n}\int_{\{|B(u)|\le n\}} A(u)\nabla u\nabla B(u)\,.$$

On the other hand, by virtue of $(R3)$

$$\lim_{\eta\to 0}\frac{1}{\eta}\int_{\{s_0+\eta<u<s_0+2\eta\}} A(u)\nabla u\nabla u\frac{1}{n}\,T_n(B(u)) = 0\,.$$

Also, by $(R4)$

$$\left|\eta\int_{\{\frac{1}{\eta}<u<\frac{2}{\eta}\}} A(u)\nabla u\nabla u\frac{1}{n}\,T_n(B(u))\right| \le \left|\eta\int_{\{\frac{1}{\eta}<u<\frac{2}{\eta}\}} A(u)\nabla u\nabla u\right| \overset{\eta\to 0}{\longrightarrow} 0\,.$$

Finally,

$$\lim_{\eta\to 0}\int_\Omega (f-g(u))h_\eta(u)\frac{1}{n}\,T_n(B(u)) = \int_\Omega (f-g(u))\frac{1}{n}\,T_n(B(u))\,.$$

In brief, we have

$$\frac{1}{n}\int_{\{|B(u)|\le n\}} A(u)\nabla u\nabla B(u) = \int_\Omega (f-g(u))\frac{1}{n}\,T_n(B(u))\,,$$

and making n goes to infinity, we deduce

$$\lim_{n\to\infty}\frac{1}{n}\int_{\{|B(u)|\le n\}} A(u)\nabla u\nabla B(u) = 0\,.$$

This result ends the proof of Lemma 1. □

We have established some properties about the regularity of $B(u)$. They will be used along the proof of the following uniqueness result.

Theorem 1. *Under assumptions* (H1)–(H3), (H5)–(H9), *problem* (1.1) *admits at most one renormalized solution.*

It will be enough to state the following comparison principle.

Lemma 2. *Under the hypotheses of Theorem* 1, *let* f_1, $f_2 \in L^1(\Omega)$ *and* u_1, u_2 *be renormalized solutions of* (1.1), *related to data* f_1, f_2, *respectively. Then,*

$$\int_\Omega |g(u_1) - g(u_2)| \le \int_\Omega |f_1 - f_2|\,. \tag{4.5}$$

In particular, if $f_1 = f_2$ then $g(u_1) = g(u_2)$ and the equality $u_1 = u_2$ is readily deduced from $(H8)$.

Proof of Lemma 2. We denote $z_i = B(u_i)$, $i = 1, 2$, B defined in (4.1). Since Lemma 1 holds, it is $u_i > s_0$ almost everywhere in Ω, $i = 1, 2$ and also

$$z_i \in W_0^{1,1}(\Omega)\,, \quad \nabla z_i = \beta(u_i)\nabla u_i\,, \quad T_j(z_i) \in H_0^1(\Omega)\,, \quad \text{for all } j > 0\,,$$
$$\lim_{n\to\infty}\frac{1}{n}\int_{\{|z_i|\le n\}} A(u_i)\nabla u_i\nabla z_i = 0\,.$$

On the other hand, for $z = z_1, z_2$, (2.3) becomes

$$\begin{aligned}
&\int_\Omega w\nabla z\,\frac{1}{\beta(B^{-1}(z))}\,h(B^{-1}(z))v + \int_\Omega \frac{A(B^{-1}(z))}{\beta(B^{-1}(z))}\nabla z\nabla v\,h(B^{-1}(z))\\
&\qquad + \int_\Omega \frac{A(B^{-1}(z))}{\beta(B^{-1}(z))}\,\nabla z\nabla z\frac{1}{\beta(B^{-1}(z))}\,h'(B^{-1}(z))v\\
&\quad = \int_\Omega [f - g(B^{-1}(z))]\,h(B^{-1}(z))v\,,\\
&\qquad\qquad \text{for all } v \in \mathcal{D}(\Omega)\,, \quad \text{for all } h \in W_c^{1,\infty}(s_0, +\infty)\,.
\end{aligned} \tag{4.6}$$

Now, we introduce the following notation

$$\tilde{h} = h \circ B^{-1}\,, \quad C(x,s) = \frac{A(x, B^{-1}(s))}{\beta(B^{-1}(s))}\,, \quad \tilde{g}(x,s) = g(x, B^{-1}(s)) \tag{4.7}$$

and (4.6) becomes

$$\int_\Omega w\nabla z \frac{\tilde{h}(z)}{\beta(B^{-1}(z))}\, v + \int_\Omega C(z)\nabla z \nabla v\, \tilde{h}(z) + \int_\Omega C(z)\nabla z \nabla z\, \tilde{h}'(z) v$$
$$= \int_\Omega (f - \tilde{g}(z))\tilde{h}(z) v\,, \quad \text{for all } v \in \mathcal{D}(\Omega)\,, \quad \text{for all } \tilde{h} \in W^{1,\infty}_{\mathrm{c}}(\mathbb{R})\,. \tag{4.8}$$

Problem (4.8) is a particular case (with diffusion matrix in $L^\infty(\Omega \times \mathbb{R})^{N\times N}$) to one studied by Gómez and Ortegón;[20] a similar comparison principle was stated in that reference. As a matter of fact, if we could make use of this result, we would obtain

$$\int_\Omega |g(B^{-1}(z_1)) - g(B^{-1}(z_2))| \le \int_\Omega |f_1 - f_2|$$

and Lemma 2 would already be proved, since $u_i = B^{-1}(z_i)$. However, it is not a straightforward application since our problem is not exactly the same because the convection term has a different form, namely

$$\int_\Omega w\nabla z \frac{\tilde{h}(z)}{\beta(B^{-1}(z))}\, v\,,$$

which contains the factor $\frac{1}{\beta(B^{-1}(z))}$. We remark that this term may also be written as

$$-\left[\int_\Omega w\, u\, \nabla z\, \tilde{h}'(z)\, v + \int_\Omega w\, u\, \nabla v\, \tilde{h}(z)\right].$$

Now, we try to remake the same arguments of this work. To do so, let the functions $S_\varepsilon(s) = \frac{1}{\varepsilon}T_\varepsilon(s)$ and $H_l \in W^{1,\infty}_{\mathrm{c}}(\mathbb{R})$ as in (3.21) be given.

We fix $\tilde{h} = \tilde{H}_l$ and $v = \tilde{H}_l(z_2)S_\varepsilon(z_1 - z_2)$ in the formulation (4.8) of the problem for z_1, and $\tilde{h} = \tilde{H}_l$ and $v = \tilde{H}_l(z_1)S_\varepsilon(z_1 - z_2)$ in the same problem for z_2. By subtracting both expressions, the contribution of the convection terms give

$$\int_\Omega (u_1 - u_2) w \nabla S_\varepsilon(z_1 - z_2)\tilde{H}_l(z_1)\tilde{H}_l(z_2)$$
$$+ \int_\Omega (u_1 - u_2) w \nabla z_1 \tilde{H}_l'(z_1) S_\varepsilon(z_1 - z_2)\tilde{H}_l(z_2)$$
$$+ \int_\Omega (u_1 - u_2) w \nabla z_2 \tilde{H}_l'(z_2) S_\varepsilon(z_1 - z_2)\tilde{H}_l(z_1)\,.$$

Since $B^{-1}(s)$ is globally Lipschitz, we have $|u_1 - u_2| \le C|z_1 - z_2|$, a.e. in Ω, and then the three integrals above can be bounded (up to a multiplicative constant) with $u_1 - u_2$ changed to $z_1 - z_2$, i.e.

$$C\left[\int_\Omega |z_1 - z_2|\,|w\nabla S_\varepsilon(z_1 - z_2)\tilde{H}_l(z_1)\tilde{H}_l(z_2)| + \int_\Omega |z_1 - z_2|\,|w\nabla z_1\,\tilde{H}_l'(z_1)S_\varepsilon(z_1 - z_2)\tilde{H}_l(z_2)| + \int_\Omega |z_1 - z_2|\,|w\nabla z_2\,\tilde{H}_l'(z_2)S_\varepsilon(z_1 - z_2)\tilde{H}_l(z_1)|\right],$$

which means that we have suppressed the factor $\frac{1}{\beta(B^{-1}(z_i))}$. From this point on, we are able to conclude the desired result. □

Remark. The results presented here are still valid in the case $N = 1$; moreover, since $W^{1,q}(\Omega) \subset L^\infty(\Omega)$, then $u \in H_0^1(\Omega)$ and also $\beta(u)\nabla u\chi_{\{u>s_0\}} \in L^2(\Omega)$.

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