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Symmetry classification and optimal systems of a non-linear wave equation

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Abstract

In this paper the complete Lie group classification of a non-linear wave equation is obtained. Optimal systems and reduced equations are achieved in the case of a hyperelastic homogeneous bar with variable cross section.

Keywords: Non-linear wave equations; Symmetries

1. Introduction

Starting from the well-known paper of Ames et al. [1], concerned with the group properties of the non-linear wave equation

$$u_{tt} = [f(u)u_x]_x, \tag{1.1}$$

in the last two decades, the search for symmetries of one-dimensional non-linear wave equations has been considered in many papers.

In this context, Torrisi and Valenti [2,3], generalizing equation (1.1), have investigated on symmetries of the following equations:

$$u_{tt} = [f(x,u)u_x]_x, \tag{1.2}$$

$$u_{tt} = [f(u)u_x + g(x,u)]_x, \tag{1.3}$$

where g and h are arbitrary functions of their argument, t is the time coordinate and x is the one-space coordinate.

Eqs. (1.1)–(1.3), after the introduction of the non-local transformation $u = v_x$, can be written as follows:

$$v_{tt} = f(v_x)v_{xx}, \tag{1.4}$$

$$v_{tt} = f(x,v_x)v_{xx}, \tag{1.5}$$

$$v_{tt} = f(v_x)v_{xx} + g(x,v_x). \tag{1.6}$$

In [4], an attempt was done in order to collect many types of equations previously discussed by considering a wide class of non-linear wave equations of the form

$$v_{tt} = f(x,v_x)v_{xx} + g(x,v_x). \tag{1.7}$$

In that paper, by using the *preliminary group classification* approach [5], a partial classification was performed, extending by one-dimension the *Principal Lie Algebra* for each classes of equivalent equations found there.

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Successively, Eq. (1.7) was studied in [6,7]. It was still considered in [8], for classification through contact group transformations.

Recently one of these classes,

$$u_{tt} = g(u_x)u_{xx} + h(u_x), \tag{1.8}$$

was investigated in [9] and explicit forms of g and h were obtained by extending the dimension of the admitted *Principal Lie Algebra* by adding one vector. The approach followed in [9] was based on the *low-dimensional algebras* method [10].

However, the problem to find the complete group classification of Eq. (1.8) is still open.

An efficient method, developed in the last decade, which can solve the group classification problem when arbitrary functions are present in the equations, is based on the moving frames theory (for a wide list of references on this field see [11]).

In this paper we get the complete classification of Eq. (1.8) by means of the classical Lie direct approach. Moreover, by focussing our attention on some forms of g and h suggested by a specific physical application, the corresponding Lie algebras are specialized, the optimal systems of subalgebras are constructed and the corresponding reduced forms of Eq. (1.8) are obtained.

The knowledge of the optimal system of subalgebras gives the possibility to construct the optimal system of solutions [12,13] and permits the generation of new solutions starting from invariant or non-invariant solutions (see e.g. [14] for some applications to the multidimensional hydrodynamics).

In fact, following [12], two solutions u_1 and u_2 are said to be *essentially different* with respect to a group of transformations G if u_2 does not belong to the orbit (u_1, G) (that is the set of all those solutions generated by transforming u_1 by G) and, of course, $u_1 \notin (u_2, G)$. Starting from the last definition it is possible to separate the whole collection of invariant solutions in classes of equivalence. This implies that each class is characterized by a group of transformations G and all elements of the class are the family of solutions generated only from G. The list of the disjoint classes is called the *optimal system of solutions*. So, by recalling that a non-invariant solution of a PDE, with respect to an admitted Lie group of transformations G, is mapped by G, in a family of solutions, it is easy to affirm that we can generate not only new solutions but essentially different solutions starting from the optimal system of subalgebras.

The plan of the paper is the following. In the next section we give the complete classification of Eqs. (1.8) (with $g_{u_x} \neq 0$). We construct, in Section 3, the optimal system of one-dimensional subalgebras of the Principal Lie Algebra and their associated reduced equations. In Section 4, the case of a bar with variable cross section varying by an exponential law is considered; for two well-known expressions of the tension the additional generators of the optimal systems are obtained and their associated reduced equations are constructed. We present our conclusions in Section 5.

2. Lie group classification

We consider the class of non-linear wave equations

$$u_{tt} = g(u_x)u_{xx} + h(u_x), \quad g_{u_x} \neq 0. \tag{2.1}$$

In order to give the group classification of this class, by means of the classical invariant Lie criterion, we consider the infinitesimal operator X of the Lie algebra

$$X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u}, \tag{2.2}$$

where ξ^1, ξ^2 and η are functions of t, x and u.

The prolongation of the operator X, keeping only the necessary terms, is

$$\tilde{X} = X + \zeta_2 \frac{\partial}{\partial u_x} + \zeta_{11} \frac{\partial}{\partial u_{tt}} + \zeta_{22} \frac{\partial}{\partial u_{xx}}, \tag{2.3}$$

where following the well-known monograph on these arguments (see e.g. [12,13,15,16]), we have set

$$\zeta_1 = D_t(\eta) - u_t D_t(\xi^1) - u_x D_t(\xi^2),$$

$$\zeta_2 = D_x(\eta) - u_t D_x(\xi^1) - u_x D_x(\xi^2),$$

$$\zeta_{11} = D_t(\zeta_1) - u_{tt} D_t(\xi^1) - u_{tx} D_t(\xi^2),$$

$$\zeta_{22} = D_x(\zeta_2) - u_{tx} D_x(\xi^1) - u_{xx} D_x(\xi^2), \tag{2.4}$$

and the operators D_t and D_x denote total derivatives with respect to t and x.

The determining system of Eq. (2.1) arises from the following invariance condition:

$$\tilde{X}(u_{tt} - g(u_x)u_{xx} - h(u_x))\big|_{u_{tt}=gu_{xx}+h} = 0. \tag{2.5}$$

After some calculations, the determining system arising from (2.5) leads to the following conditions:

$$\xi^1 = \frac{1}{2}a_2t^2 + a_1t + a_0,$$

$$\xi^2 = \psi(x),$$

$$\eta = \left(\frac{a_2t + a_1}{2} + c_5\right)u + c_4tx + \frac{1}{6}c_3t^3 + \frac{1}{2}c_2t^2 + c_1t + \beta(x),$$

$$\left(c_4 + \frac{1}{2}a_2u_x\right)g_{u_x} = -2a_2g,$$

$$\left[\beta_x + \left(\frac{1}{2}a_1 + c_5 - \psi_x\right)u_x\right]g_{u_x} = 2(\psi_x - a_1)g,$$

$$\left(c_4 + \frac{1}{2}a_2u_x\right)h_{u_x} = -\frac{3}{2}a_2h + c_3,$$

$$\left[\beta_x + (\frac{1}{2}a_1 + c_5 - \psi_x)u_x\right]h_{u_x} = (c_5 - \frac{3}{2}a_1)h + (\psi_{xx}u_x - \beta_{xx})g + c_2, \tag{2.6}$$

where $a_0, a_1, a_2, c_1, c_2, c_3, c_4$ and c_5 are constants, while ψ and β are functions of x.

Obviously, for arbitrary g and h, from (2.6) one obtains the four-dimensional *Principal Lie Algebra* $L_{\mathscr{P}}$ of (2.1), which is spanned by the operators [4,9]

$$X_1 = \frac{\partial}{\partial t},\ X_2 = \frac{\partial}{\partial x},\ X_3 = \frac{\partial}{\partial u},\ X_4 = t\frac{\partial}{\partial u}. \tag{2.7}$$

The complete Lie classification of the non-linear wave equation (2.1), arising from conditions (2.6), is showed in Tables 1 and 2.

By comparing these results with the six cases obtained in [9], it is possible to ascertain that the cases l, 2, 4, 5, 6 in [9] correspond, respectively, to cases IV, IX, VI, VII and III of Tables 1 and 2 in the present paper. The case 3 in [9] is not obtained in this paper because we assume a priori $g_{u_x} \neq 0$. Finally, the generator X_5 of case 6 in [9] is incorrect. It does not make Eq. (1.8) invariant when the corresponding constitutive forms of g and h are utilized.

3. Optimal system of the principal Lie algebra and reduced equations

When the dimension of the Lie algebra, associated to a group of invariant transformations of a PDE, is greater than one, there are often, an infinite number of subgroups. To each s-parameter subgroup corresponds a family of group-invariant solutions. So that, in general, it is quite impossible to determine all possible group-invariant solutions of a PDE. In order to minimize this search, it is useful to construct the optimal system of solutions.

It is well known that the problem of the construction of the optimal system of solutions is equivalent to that of the construction of the optimal system of subalgebras [12,13]. Here, we will deal with the construction of the optimal system of subalgebras of $L_{\mathscr{P}}$.

Let G be a Lie group with L its Lie algebra. Each transformation $\tau \in G$ yields an inner automorphism $\tau_a \to \tau\tau_a\tau^{-1}$ of the group G. Every automorphism of the group G induces an automorphism of L. The set of all these automorphism is a Lie group called the *adjoint group* G^A. The Lie algebra of G^A is the *adjoint algebra* of L, defined as follows. Let two infinitesimal generators $X, Y \in L$. The linear mapping $\mathrm{Ad}\,X(Y) : Y \to [X, Y]$ is an automorphism of L, called *inner derivation* of L. The set of all inner derivations $\mathrm{ad}\,X(Y)$ $(X, Y \in L)$ together with the Lie bracket $[\mathrm{Ad}\,X, \mathrm{Ad}\,Y] = \mathrm{Ad}[X, Y]$ is a Lie algebra L^A called the *adjoint algebra of* L. Clearly L^A is the Lie algebra of G^A.

Two subalgebras in L are *conjugate* (or *similar*) if there is a transformation of G^A which takes one subalgebra into the other. The collection of pairwise non-conjugate s-dimensional subalgebras is the *optimal system of subalgebras of order s*.

The construction of the one-dimensional optimal system of subalgebras can be made by using a global matrix of the adjoint transformations as suggested by Ovsiannikov [12].

In this paper we follow, instead, the method suggested by Olver [13] which uses a slightly different technique. It consists in constructing a table showing the separate adjoint actions of each element of the Lie algebra on all other elements. This table is usually called the *adjoint table*.

Taking into account the commutator Table 3 and the adjoint Table 4 here we show the non-trivial operators

Table 1
Lie group classification. g_0, h_0, h_1, A, m and n are constitutive constants

Case	Forms of g and f	Extensions w.r.t $L_{\mathscr{P}}$
I	arbitrary g, $h=h_0$,	$X_5 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + \left(u+\frac{1}{2}h_0t^2\right)\frac{\partial}{\partial u}$
II	$g=g_0e^{u_x/m}$, $h=h_1u_x+h_0$, $m,h_1\neq 0$	$X_5 = t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + (2u+h_1mt^2+2mx)\frac{\partial}{\partial u}$
III	$g=g_0e^{u_x/m}$, $h=h_1e^{u_x/2m}+h_0$, $m,h_1\neq 0$	$X_5 = x\frac{\partial}{\partial x} + (u-\frac{1}{2}h_0t^2+2mx)\frac{\partial}{\partial u}$
IV	$g=g_0e^{u_x/m}$, $h=h_1e^{u_x/n}+h_0$, $m,h_1,n\neq 0$, $n\neq 2m$	$X_5 = t\frac{\partial}{\partial t} + \frac{2(m-n)}{2m-n}x\frac{\partial}{\partial x} + \left[\frac{2(m-n)}{2m-n}u + \frac{mh_0}{2m-n}t^2 - \frac{2mn}{2m-n}x\right]\frac{\partial}{\partial u}$
V	$g=g_0e^{u_x/m}$, $h=h_0$, $m\neq 0$	$X_5 = t\frac{\partial}{\partial t} + (h_0t^2-2mx)\frac{\partial}{\partial u}$, $X_6 = x\frac{\partial}{\partial x} + \left(u-\frac{1}{2}h_0t^2+2mx\right)\frac{\partial}{\partial u}$
VI	$g=g_0(A+u_x)^{-2}$, $h=\log(A+u_x)^n+h_0$, $n\neq 0$	$X_5 = x\frac{\partial}{\partial x} - \left(\frac{1}{2}nt^2+Ax\right)\frac{\partial}{\partial u}$
VII	$g=g_0(A+u_x)^m$, $h=\log(A+u_x)^n+h_0$, $m\neq 0,-2$, $n\neq 0$	$X_5 = t\frac{\partial}{\partial t} + \frac{2(m+1)}{m+2}x\frac{\partial}{\partial x} + \left(2u+\frac{n}{m+2}t^2+\frac{2A}{m+2}x\right)\frac{\partial}{\partial u}$

of the optimal system of the *Principal Lie Algebra*:

$$X_{o1} = aX_1 + X_2 + bX_4 + cX_3 = a\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + (bt+c)\frac{\partial}{\partial u}, \qquad (3.1)$$

$$X_{o2} = X_1 + aX_2 + bX_4 = \frac{\partial}{\partial t} + a\frac{\partial}{\partial x} + bt\frac{\partial}{\partial u}. \qquad (3.2)$$

By applying the invariant surface condition, through (3.1), we obtain

$$u = \omega(\lambda) - \tfrac{1}{2}abx^2 + btx + cx, \qquad (3.3)$$

where the similarity variable λ has the following expression:

$$\lambda = t - ax \qquad (3.4)$$

and the function ω must satisfy the following ODE to which PDE (2.1) is reduced by (3.1)

$$(1-a^2g)\omega'' + abg - h = 0, \qquad (3.5)$$

with g and h arbitrary functions of $z = c + b\lambda - a\omega'$.

By applying the invariant surface condition, through (3.2), we obtain

$$u = \omega(\lambda) - \tfrac{1}{2}bt^2, \qquad (3.6)$$

Table 2
Lie group classification. g_0, h_0, h_1, A, m and n are constitutive constants

Case	Forms of g and f	Extensions w.r.t $L_{\mathscr{P}}$
VIII	$g = g_0(A+u_x)^m$, $h = h_1(A+u_x x)^{(m+2)/2} + h_0$, $m \neq 0, -2$	$X_5 = \frac{m}{m+2} x \frac{\partial}{\partial x} + \left(u - \frac{1}{2}h_0 t^2 + \frac{2A}{m+2} x\right) \frac{\partial}{\partial u}$
IX	$g = g_0(A+u_x)^m$, $h = h_1(A+u_x)^n + h_0$, $m, n \neq 0, \quad n \neq \frac{m+2}{2}$	$X_5 = t\frac{\partial}{\partial t} + \frac{2(m-n+1)}{m+2-2n} x \frac{\partial}{\partial x} + \left[\frac{2A}{m+2-2n} x + \frac{2(m+2-n)}{m+2-2n} u - \frac{nh_0}{m+2-2n} t^2\right] \frac{\partial}{\partial u}$
X	$g = g_0(A+u_x)^{-2}$, $h = h_1 g_0 (A+u_x)^{-1} + h_0$,	$X_5 = t\frac{\partial}{\partial t} + \left(u + \frac{1}{2}h_0 t^2 + Ax\right)\frac{\partial}{\partial u}$, $X_6 = \mathrm{e}^{h_1 x}\left(\frac{\partial}{\partial x} - A\right)\frac{\partial}{\partial u}$
XI	$g = g_0(A+u_x)^m$, $h = h_0$ $m \neq 0$	$X_5 = t\frac{\partial}{\partial t} + \left(-\frac{2}{m}u + \frac{m+1}{m}h_0 t^2 - \frac{2}{m}Ax\right)\frac{\partial}{\partial u}$, $X_6 = x\frac{\partial}{\partial x} + \left(\frac{m+2}{m}u - \frac{m+2}{2m}h_0 t^2 + \frac{2}{m}Ax\right)\frac{\partial}{\partial u}$
XII	$g = g_0(A+u_x)^{-4}$, $h = h_1(A+u_x)^{-3} + h_0$,	$X_5 = t\frac{\partial}{\partial t} + \left(\frac{1}{2}u + \frac{3}{4}h_0 t^2 + \frac{1}{2}Ax\right)\frac{\partial}{\partial u}$, $X_6 = t^2\frac{\partial}{\partial t} + \left(tu + Atx + \frac{1}{2}h_0 t^3\right)\frac{\partial}{\partial u}$
XIII	$g = g_0(A+u_x)^{-4}, \quad h = h_0$	$X_5 = t\frac{\partial}{\partial t} + \frac{1}{2}\left(u + \frac{3}{2}h_0 t^2 + Ax\right)\frac{\partial}{\partial u}$, $X_6 = t^2\frac{\partial}{\partial t} + \left(tu + Atx + \frac{1}{2}h_0 t^3\right)\frac{\partial}{\partial u}$, $X_7 = x\frac{\partial}{\partial x} + \frac{1}{2}\left(u - \frac{1}{2}h_0 t^2 - Ax\right)\frac{\partial}{\partial u}$

Table 3
Commutator table of the Principal Lie algebra

	X_1	X_2	X_3	X_4
X_1	0	0	0	$-X_3$
X_2	0	0	0	0
X_3	0	0	0	0
X_4	$-X_3$	0	0	0

Table 4
Adjoint table of the Principal Lie algebra

Ad	X_1	X_2	X_3	X_4
X_1	X_1	X_2	X_3	$X_4 - \varepsilon X_3$
X_2	X_1	X_2	X_3	X_4
X_3	X_1	X_2	X_3	X_4
X_4	$X_1 + \varepsilon X_3$	X_2	X_3	X_4

where the similarity variable λ has the following expression:

$$\lambda = x - at \tag{3.7}$$

and the function ω must satisfy the following ODE to which PDE (2.1) is reduced by (3.2)

$$(a^2 - g)\omega'' - h + b = 0, \tag{3.8}$$

with g and h arbitrary functions of $z = \omega'$.

4. A physical case

In this section we restrict ourselves to the physical case of a bar with variable cross section [17], in order to find the extensions of the optimal system of the principal Lie algebra and the corresponding reduced equations for special forms of the constitutive functions. However, it is possible to reach an equation belonging to the class (2.1) in some engineering applications in which two parts of the same material or two parts of different materials are attached together by a third material known as adhesive (see e.g. [18]).

The equation of motion of a hyperelastic homogeneous bar, whose cross-sectional area is variable along the bar, reads [17,19]

$$\rho\, u_{tt} = T_x + \frac{S'(x)}{S(x)}\, T, \tag{4.1}$$

where p is the constant mass density, $u = y - x$ is the displacement, y is the coordinate of the point P in the *present reference frame*, x represents the coordinate of the corresponding point P_0 of P in the reference frame, $T(u_x)$ is the tension and $S(x)$ is the cross-sectional area.

Moreover, Eq. (4.1), assuming that the cross-sectional area is varying with an exponential law

$$S = S_0\, e^{\mu x}, \tag{4.2}$$

with S_0 and μ constants, takes the following form:

$$u_{tt} = \frac{1}{\rho}\frac{dT}{du_x}\, u_{xx} + \frac{\mu}{\rho}\, T. \tag{4.3}$$

Eq. (4.3) by setting

$$\frac{1}{\rho}\frac{dT}{du_x} = g, \qquad \frac{\mu}{\rho}\, T = h \tag{4.4}$$

is included in (2.1).

Taking (4.4) into account, it is a simple matter to verify that only the following cases:

- Case IV, with $m = n$;
- Case VII, with $m = -1$;
- Case IX, with $m = n + 1$;
- Case X;
- Case XII;

can give an equation of the type (4.3).

In the framework of the previous cases we take into consideration the following forms for the tension function T:

1.

$$T = T_0 \log(1 + u_x), \tag{4.5}$$

suggested by Capriz [20,21]. Then

$$h = \frac{\mu T_0}{\rho}\log(1 + u_x), \qquad g = \frac{\mu T_0}{\rho(1 + u_x)}. \tag{4.6}$$

So we fall in the Case VII with the following identifications:

$$h_0 = 0, \quad A = 1, \quad g_0 = \frac{T_0}{\rho}, \quad m = -1,$$

$$n = \frac{\mu T_0}{\rho}. \tag{4.7}$$

and Eq. (2.1) reads

$$u_{tt}=\frac{\mu T_0}{\rho(1+u_x)}\,u_{xx}+\frac{\mu T_0}{\rho}\log(1+u_x). \qquad (4.8)$$

The additional operator X_5 assumes the following form:

$$X_5=t\frac{\partial}{\partial t}+\left(2u+\frac{\mu T_0}{\rho}t^2+2x\right)\frac{\partial}{\partial u}. \qquad (4.9)$$

2.

$$T=-T_0\left(\frac{3T_0}{\rho V_0^2}\right)^3\left(\frac{3T_0}{\rho V_0^2}+u_x\right)^{-3}+T_0, \quad (4.10)$$

which models the *the ideal soft materials* whose main feature is the lagrangian speed of sound decreases monotonically to zero as u_x increases without bound [22,23]. Then

$$h=-\frac{T_0\mu}{\rho}\left(\frac{3T_0}{\rho V_0^2}\right)^3\left(\frac{3T_0}{\rho V_0^2}+u_x\right)^{-3}+\frac{T_0\mu}{\rho}, \qquad (4.11)$$

$$g=\frac{1}{V_0^2}\left(\frac{3T_0}{\rho V_0}\right)^4\left(\frac{3T_0}{\rho V_0^2}+u_x\right)^{-4}. \qquad (4.12)$$

So we fall in the Case XII with the following identifications:

$$h_0=\frac{T_0\mu}{\rho},\ h_1=-\frac{T_0\mu}{\rho}\left(\frac{3T_0}{\rho V_0^2}\right)^3,\ A=\frac{3T_0}{\rho V_0^2},$$

$$g_0=\frac{1}{V_0^2}\left(\frac{3T_0}{\rho V_0}\right)^4, \qquad (4.13)$$

and Eq. (2.1) reads

$$u_{tt}=\frac{1}{V_0^2}\left(\frac{3T_0}{\rho V_0}\right)^4\left(\frac{3T_0}{\rho V_0^2}+u_x\right)^{-4}u_{xx}-\frac{T_0\mu}{\rho}\times\left(\frac{3T_0}{\rho V_0^2}\right)^3\left(\frac{3T_0}{\rho V_0^2}+u_x\right)^{-3}+\frac{T_0\mu}{\rho}. \qquad (4.14)$$

The additional operators X_5 and X_6 assume the following forms:

$$X_5=t\frac{\partial}{\partial t}+\left(\frac{1}{2}u+\frac{3T_0\mu}{4\rho}t^2+\frac{3T_0}{2\rho V_0^2}x\right)\frac{\partial}{\partial u}, \qquad (4.15)$$

$$X_6=t^2\frac{\partial}{\partial t}+\left(tu+\frac{3T_0}{\rho V_0^2}tx+\frac{T_0\mu}{2\rho}t^3\right)\frac{\partial}{\partial u}. \qquad (4.16)$$

4.1. Optimal systems and reduced equations

Since the optimal system of each case is an extension of the optimal system of the *Principal Lie Algebra*, we take into consideration only the extensions with respect to the results obtained in Section 3.

1. By using law (4.5), taking identifications (4.7) into account, the Lie algebra can be represented by the set of all the generators $\{X_i\}_{i=1}^5$ given by (2.7) and (4.9) while the commutator and adjoint tables are Tables 5 and 6, respectively.
   In this case, the optimal system of the *Principal Lie Algebra* has an extension by one, whose operator is

$$X_{o3}=aX_2+X_5=t\frac{\partial}{\partial t}+a\frac{\partial}{\partial x}+\left(2u+\frac{\mu T_0}{\rho}t^2+2x\right)\frac{\partial}{\partial u}. \qquad (4.17)$$

By applying the invariant surface condition, through (4.17), we obtain

$$u=t^2\left[\omega(\lambda)+\frac{\mu T_0}{\rho}\log t\right]-x, \qquad (4.18)$$

where the similarity variable λ has the following expression:

$$\lambda=x-a\log t \qquad (4.19)$$

and the function ω must satisfy the following ODE to which the PDE (4.8), after (4.6) and (4.7), is

Table 5
Commutator table of the Lie algebra with $g = \mu T_0/\rho(1 + u_x)$ and $h = (\mu T_0/\rho)\log(1 + u_x)$

	X_1	X_2	X_3	X_4	X_5
X_1	0	0	0	X_3	$X_1 + \frac{2\mu T_0}{\rho} X_4$
X_2	0	0	0	0	$2X_3$
X_3	0	0	0	0	$2X_3$
X_4	$-X_3$	0	0	0	X_4
X_5	$-X_1 - \frac{2\mu T_0}{\rho} X_4$	$-2X_3$	$-2X_3$	$-X_4$	0

Table 6
Adjoint table of the Lie algebra with $g = \mu T_0/\rho(1 + u_x)$ and $h = (\mu T_0/\rho)\log(1 + u_x)$

Ad	X_1	X_2	X_3	X_4	X_5
X_1	X_1	X_2	X_3	$X_4 - \varepsilon X_3$	$X_5 - \varepsilon(X_1 + \frac{2\mu T_0}{\rho} X_4)$
X_2	X_1	X_2	X_3	X_4	$X_5 - \varepsilon 2X_3$
X_3	X_1	X_2	X_3	X_4	$X_5 - 2\varepsilon X_3$
X_4	$X_1 + \varepsilon X_3$	X_2	X_3	X_4	$X_5 - \varepsilon X_4$
X_5	S_1	S_2	$e^{2\varepsilon} X_3$	$e^{\varepsilon} X_4$	X_5

Where $S_1 = X_1 e^{\varepsilon} - \frac{2\mu T_0}{\rho} X_4 \sum_{p=0}^{\infty} \frac{\varepsilon^{2p+1}}{(2p+1)!}$ and $S_2 = X_2 - 2X_3 \sum_{p=1}^{\infty} \frac{(-\varepsilon)^p}{p!}(-2)^{p-1}$.

reduced by (4.17)

$$\left(a^2 - \frac{T_0}{\rho\omega'}\right)\omega'' - 3a\omega' - \frac{\mu T_0}{\rho}\log\omega' + 2\omega + 3\frac{\mu T_0}{\rho} = 0. \tag{4.20}$$

2. By using law (4.10), taking identifications (4.13) into account, the Lie algebra can be represented by the set of all the generators $\{X_i\}_{i=1}^{6}$ given by (2.7), (4.15) and (4.16) while commutator and adjoint tables are Tables 7 and 8, respectively.
   In this case besides the operators X_{o1} and X_{o2}, the generators of the optimal system are

$$X_{o3} = aX_1 + bX_2 + X_6 = (t^2 + a)\frac{\partial}{\partial t} + b\frac{\partial}{\partial x} + \left(tu + \frac{3T_0}{\rho V_0^2} tx + \frac{T_0\mu}{2\rho} t^3\right)\frac{\partial}{\partial u}, \tag{4.21}$$

$$X_{o4} = bX_2 + X_6 = t^2\frac{\partial}{\partial t} + b\frac{\partial}{\partial x} + \left(tu + \frac{3T_0}{\rho V_0^2} tx + \frac{T_0\mu}{2\rho} t^3\right)\frac{\partial}{\partial u}, \tag{4.22}$$

$$X_{o5} = bX_2 + X_5 = t\frac{\partial}{\partial t} + b\frac{\partial}{\partial x} + \left(\frac{1}{2}u + \frac{3T_0\mu}{4\rho} t^2 + \frac{3T_0}{2\rho V_0^2} x\right)\frac{\partial}{\partial u}, \tag{4.23}$$

where a is a non-zero constant.
By applying the invariant surface condition, through (4.21), we obtain

$$u = \omega(\lambda)\sqrt{t^2 + a} - \frac{3T_0}{\rho V_0^2} x + \frac{T_0\mu}{2\rho} t^2 + \frac{3bT_0}{\rho a V_0^2} t + \frac{aT_0\mu}{\rho}, \tag{4.24}$$

where the similarity variable λ has the following expressions:

$$\lambda = x - \frac{b}{a}\arctan\left(\frac{t}{a}\right), \quad a > 0, \tag{4.25}$$

$$\lambda = x - \frac{b}{2a}\log\left(\frac{t-a}{t+a}\right), \quad a < 0 \tag{4.26}$$

and the function ω must satisfy the following ODE to which the PDE (4.14), after (4.11)–(4.13), is reduced by (4.21)

$$\left[\frac{1}{V_0^2}\left(\frac{3T_0}{\rho V_0}\right)^4 - b^2(\omega')^4\right]\omega'' - a\omega(\omega')^4 + \frac{T_0\mu}{\rho}\left(\frac{3T_0}{\rho V_0^2}\right)^3\omega' = 0 \tag{4.27}$$

By applying the invariant surface condition, through (4.22), we obtain

$$u = \omega(\lambda)t - \frac{3T_0}{\rho V_0^2} x - \frac{3T_0 b}{2\rho V_0^2 t} + \frac{T_0\mu}{2\rho} t^2, \tag{4.28}$$

where the similarity variable λ has the following expression:

$$\lambda = x - \frac{b}{t}, \tag{4.29}$$

Table 7
Commutator table of the Lie algebra with $g=(1/V_0^2)(3T_0/\rho V_0)^4(3T_0/\rho V_0^2+u_x)^{-4}$ and $h=-(T_0\mu/\rho)(3T_0/\rho V_0^2)^3(3T_0/\rho V_0^2+u_x)^{-3}+T_0\mu/\rho$

	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	0	0	X_3	$X_1+\frac{3T_0\mu}{2\rho}X_4$	$2X_5$
X_2	0	0	0	0	$\frac{3T_0}{2\rho V_0^2}X_3$	$\frac{3T_0}{\rho V_0^2}X_4$
X_3	0	0	0	0	$\frac{1}{2}X_3$	X_4
X_4	$-X_3$	0	0	0	$-\frac{1}{2}X_4$	0
X_5	$-X_1-\frac{3T_0\mu}{2\rho}X_4$	$-\frac{3T_0}{2\rho V_0^2}X_3$	$-\frac{1}{2}X_3$	$\frac{1}{2}X_4$	0	X_6
X_6	$-2X_5$	$-\frac{3T_0}{\rho V_0^2}X_4$	$-X_4$	0	$-X_6$	0

Table 8
Adjoint table of the Lie algebra with $g=(1/V_0^2)(3T_0/\rho V_0)^4(3T_0/\rho V_0^2+u_x)^{-4}$ and $h=-(T_0\mu/\rho)(3T_0/\rho V_0^2)^3(3T_0/\rho V_0^2+u_x)^{-3}+T_0\mu/\rho$

Ad	X_1	X_2	X_3	X_4	X_5	X_6
X_1	X_1	X_2	X_3	$X_4-\varepsilon X_3$	$X_5-\varepsilon(X_1+\frac{3T_0\mu}{2\rho}X_4)$	$X_6-2\varepsilon X_5$
X_2	X_1	X_2	X_3	X_4	$X_5-\varepsilon\frac{3T_0}{2\rho V_0^2}X_3$	$X_6-\varepsilon\frac{3T_0}{\rho V_0^2}X_4$
X_3	X_1	X_2	X_3	X_4	$X_5-\varepsilon X_3$	X_6
X_4	$X_1+\varepsilon X_3$	X_2	X_3	X_4	$X_5+\frac{1}{2}\varepsilon X_4$	X_6
X_5	S_1	$X_2+\frac{3T_0}{2\rho V_0^2}\varepsilon X_3$	$e^{\varepsilon/2}X_3$	$e^{\varepsilon/2}X_4$	X_5	$e^{-\varepsilon}X_6$
X_6	$X_1+2\varepsilon X_5$	$X_2+\frac{3T_0}{\rho V_0^2}\varepsilon X_4$	$X_3+\varepsilon X_4$	X_4	$X_5+\varepsilon X_6$	X_6

Where $S_1=e^{\varepsilon}X_1+\frac{3T_0\mu}{2\rho}X_4\left[1+\sum_{p=1}^{\infty}\frac{\varepsilon^p}{(p+1)!}(1-\frac{1}{2^p})\right]$.

and the function ω must satisfy the following ODE to which the PDE (2.1), after (4.11)–(4.13), is reduced by (4.22)

$$\left[\frac{1}{V_0^2}\left(\frac{3T_0}{\rho V_0}\right)^4-b^2(\omega')^4\right]\omega''+\frac{3T_0 b}{\rho V_0^2}(\omega')^4 - \frac{T_0\mu}{\rho}\left(\frac{3T_0}{\rho V_0^2}\right)^3\omega'=0. \tag{4.30}$$

By applying the invariant surface condition, through (4.23), we obtain

$$u=\omega(\lambda)\sqrt{t}-\frac{3T_0}{\rho V_0^2}x+\frac{T_0\mu}{2\rho}t^2-\frac{6T_0 b}{\rho V_0^2}, \tag{4.31}$$

where the similarity variable λ has the following expression:

$$\lambda=x-b\log t, \tag{4.32}$$

and the function ω must satisfy the following ODE to which the PDE (2.1), after (4.11)–(4.13), is reduced by (4.23)

$$\left[\frac{1}{V_0^2}\left(\frac{3T_0}{\rho V_0}\right)^4-b^2(\omega')^4\right]\omega''+\frac{\omega(\omega')^4}{4} - \frac{T_0\mu}{\rho}\left(\frac{3T_0}{\rho V_0^2}\right)^3\omega'=0. \tag{4.33}$$

5. Conclusions

In this paper, following the classical Lie method, the complete Lie group classification for the class of non-linear wave equations (2.1) is obtained.

We show that the equation, which models the behavior of a hyperelastic homogeneous bar whose cross-sectional area is variable along the bar, falls in this class when the expression of the varying section is given by an exponential law. In this physical case, by considering two special forms of the stress function, the optimal systems are constructed and the corresponding reduced equations are obtained.

Of course it is also possible to construct the optimal systems and to obtain the corresponding reduced equations for all the cases in the classification reported in Tables 1 and 2. We omitted them for sake of brevity.

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