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Interaction processes with the radiation for the KP equation, through symmetry transformations

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Abstract

Some new solutions of the Kadomtsev–Petviashvili equation are obtained by applying some of the elements of the symmetry group of this equation to its basic solutions, it is the coherent structures: the soliton solution, the two soliton solution and the resonant soliton.

We observe that in these and in many other cases the new solutions can be interpreted as interaction processes in which a certain structure interacts with the radiation and as a consequence of the interaction it transforms (at least asymptotically, in compact subsets of $\mathbb{R}^2$) in to the solution we have deformed.

1. Introduction

The Kadomtsev–Petviashvili equation is a very well-known integrable system

$$(u_t + 6uu_x + u_{xxx})_x + u_{yy} = 0. \tag{1}$$

Among its applications, this equation describes the evolution of quasi-one-dimensional shallow-water waves when effects of the surface tension and the viscosity are negligible [10].

Several methods in the context of integrable systems have been developed in order to construct large families of solutions [1,11]. A particular class of these solutions are the multisoliton solutions, describing interaction processes of solitary waves. It is found that in these processes, solitons emerge from interaction without changing their forms or their velocities.

On the other hand this equation has also been studied from the point of view of the theory of symmetry transformations on partial differential equations. In this sense the Lie algebra associated to the Lie group of symmetry transformations of (1) have been obtained [2,3,9]. It is found that the general element of the Lie algebra has the form

$$V = V_1(f) + V_2(g) + V_3(h) \tag{2}$$

with f, g, h arbitrary functions on t, and where $V_1(f)$, $V_2(g)$, $V_3(h)$ are given by

$$\begin{aligned} V_1(f) &= \left(\frac{1}{3}f'(t)x - \frac{1}{6}f''(t)y^2\right)\frac{\partial}{\partial x} + \frac{2}{3}f'(t)y\frac{\partial}{\partial y} + f(t)\frac{\partial}{\partial t} + \left(-\frac{1}{36}f'''(t)y^2 + \frac{1}{18}f''(t)x - \frac{2}{3}f'(t)u\right)\frac{\partial}{\partial u}, \\ V_2(g) &= g(t)\frac{\partial}{\partial x} + \frac{1}{6}g'(t)\frac{\partial}{\partial u}, \\ V_3(h) &= \frac{1}{2}h'(t)y\frac{\partial}{\partial x} - h(t)\frac{\partial}{\partial y} + \frac{1}{12}h''(t)y\frac{\partial}{\partial u}. \end{aligned} \tag{3}$$

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The symmetry group can be integrated from (2) and (3) [3], and in this way, given a solution of the KP equation, we can obtain a new solution by applying to the first one an element of the group. Due to the involved expression of the general element of the group (see [3]) and due also to the fact that we can use the composition operation in the group, we just provide here the expression of the new solution $U(X,Y,T)$ in terms of the known solution $u(x,y,t)$, in the cases $g \equiv h \equiv 0$ and $f \equiv 0$, respectively. In this cases we have

$$U(X,Y,T) = \left(\frac{f(t)}{f(T)}\right)^{2/3}\left[u(x,y,t) - \frac{y^2}{108f(t)^2}\Big(3f''(T)f(T) - f'(T)^2 - f'(T)f'(t) - 3f''(t)f(t) + 2f'(t)^2\Big) + \frac{x}{18f(t)}(f'(T) - f'(t))\right], \tag{4}$$

where

$$\begin{aligned} t &= \Phi^{-1}(\Phi(T) - s), \\ y &= \left(\frac{f(t)}{f(T)}\right)^{2/3} Y, \\ x &= \left(\frac{f(t)}{f(T)}\right)^{1/3}\left[X + \frac{f'(T) - f'(t)}{f(T)} Y^2\right] \end{aligned} \tag{5}$$

and

$$\Phi(T) = \int^T \frac{1}{f(\xi)}\,\mathrm{d}\xi,$$

$$U(X,Y,T) = \frac{sg'(T)}{6} - \frac{sYh''(T)}{12} + \frac{s^2h(T)h''(T)}{24} + u\left(X - sg(T) + \frac{sYh'(T)}{2} - \frac{s^2h(T)h'(T)}{4}, Y - sh(T), T\right). \tag{6}$$

In this work, we will use the symmetry transformations in order to construct new solutions of the KP equation starting with the basic coherent structures of this equation. From (3) it is clear that the choice $f \equiv g \equiv h \equiv 0$ corresponds to the trivial group, for which $U(X,Y,T) = u(X,Y,T)$, i.e. each solution is transformed into itself. However, if we take into account that the arbitrary functions f, g, h are functions on t we note that

- If we choose the arbitrary functions satisfying that $\lim_{t\to\infty} f(t) = 0$, $\lim_{t\to\infty} g(t) = 0$, $\lim_{t\to\infty} h(t) = 0$, then the new solution behaves asymptotically (as $t \to \infty$) as the starting solution. The same fact holds if we choose the arbitrary functions tending to zero as $t \to -\infty$.
- If we choose the arbitrary functions as $\mathscr{C}^\infty$ functions with compact support, we have that the new solution differs only from our starting solution when t is in the support of one of the arbitrary functions.

Thus, using appropriate choices for f, g, h, we can consider the new solutions as deformations of the starting ones, in such a way that the solution recover its form without deformation in the asymptotic limit (in the first case above) or the deformations occurs only in a finite interval of time (in the second case above).

Note also that solutions with similar properties for other integrable models have been interpreted as interaction processes in which the coherent structure is interacting with the radiation [5].

The work is organized as follows: in Sections 2–4 we analyze the properties of solutions obtained by applying some elements of the symmetry group to the basic coherent structures of the KP equation. More specifically in Section 2, we consider the solution obtained by applying the element of the group corresponding to $f(t) = \mathrm{e}^{-t}$, $g \equiv h \equiv 0$ to the one soliton solution of the KP equation; in Section 3 we choose $f \equiv g \equiv 0$ and h as a $\mathscr{C}^\infty$ compact support function, and we start with the two soliton solution of (1), and in Section 4 we take $f(t) = 1/t$, $g \equiv h \equiv 0$ and the resonant soliton of (1) [7]. Finally, we summarize in Section 5 the main results of the work.

2. Deformation of the one soliton solution

We start with the one soliton solution, i.e. the solution of the KP equation given by

$$u(x,y,t) = \frac{(p-q)^2}{2}\,\mathrm{sech}\left[\frac{1}{2}\Big((p-q)x + \sqrt{3}(p^2 - q^2)y - 4(p^3 - q^3)t\Big)\right] \tag{7}$$

with p and q being arbitrary constants. It is a coherent structure localized on a line, moving with constant velocity, and without deformation. The well known graphic of this solution is plotted in Fig. 1 for $p = 4$, $q = -4$ and $t = 0$.

Now, we apply the symmetry group element determined by the choice of the arbitrary functions

$$f(t) = \mathrm{e}^{-t}, \quad g \equiv 0, \quad h \equiv 0.$$

According to the preceding discussion, we will have that the new solution behaves as the one soliton solution for large values of t. The analytic expression of the new solution is easily obtained making use of (4), (5), (7). Thus, we find

$$U(x,y,t) = \frac{s\mathrm{e}^{-t}x}{18(1-s\mathrm{e}^{-t})} + \frac{s(3-s\mathrm{e}^{-t})\mathrm{e}^{-t}y^2}{108(1-s\mathrm{e}^{-t})^2} + \frac{(p-q)^2}{2(1-s\mathrm{e}^{-t})^{2/3}} \times \operatorname{sech}\left[\frac{1}{2}\left((p-q)\left(\frac{x}{(1-s\mathrm{e}^{-t})^{1/3}} + \frac{s\mathrm{e}^{-t}y^2}{6(1-s\mathrm{e}^{-t})^{4/3}}\right) + \frac{\sqrt{3}(p^2-q^2)y}{(1-s\mathrm{e}^{-t})^{2/3}} - 4(p^3-q^3)(t+\log(1-s\mathrm{e}^{-t}))\right)\right]^2. \tag{8}$$

In order to understand this new solution let us denote

$$R(x,y,t) = \frac{s\mathrm{e}^{-t}x}{18(1-s\mathrm{e}^{-t})} + \frac{s(3-s\mathrm{e}^{-t})\mathrm{e}^{-t}y^2}{108(1-s\mathrm{e}^{-t})^2} \tag{9}$$

and

$$S(x,y,t) = \frac{(p-q)^2}{2(1-s\mathrm{e}^{-t})^{2/3}} \times \operatorname{sech}\left[\frac{1}{2}\left((p-q)\left(\frac{x}{(1-s\mathrm{e}^{-t})^{1/3}} + \frac{s\mathrm{e}^{-t}y^2}{6(1-s\mathrm{e}^{-t})^{4/3}}\right) + \frac{\sqrt{3}(p^2-q^2)y}{(1-s\mathrm{e}^{-t})^{2/3}} - 4(p^3-q^3)(t+\log(1-s\mathrm{e}^{-t}))\right)\right]^2, \tag{10}$$

so that $U(x,y,t) = R(x,y,t) + S(x,y,t)$. Then, we observe the following facts

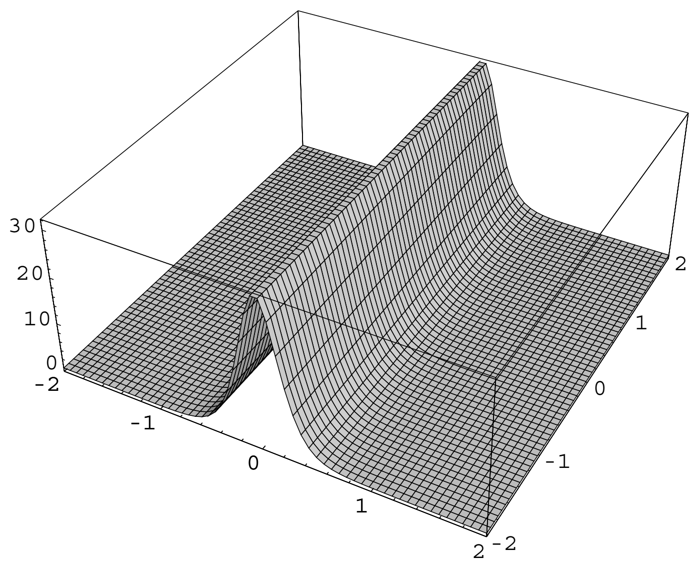

Fig. 1. Soliton solution.

- R has a polynomial dependence in the spatial variables (x, y). Consequently, it is not a bounded term, but it increase for large values of $|x|$ and $|y|$.
- For any fixed point (x, y); R tends to zero as $t \to \infty$. However, we have only uniform convergence for compact sets in the plane.
- In the other asymptotic limit we have:

$$\lim_{t\to-\infty} R(x,y,t) = -\frac{x}{18} - \frac{y^2}{108}.$$

- S describes a structure localized on a parabola of equation

$$(p-q)\left(\frac{x}{(1-se^{-t})^{1/3}} + \frac{se^{-t}y^2}{6(1-se^{-t})^{4/3}}\right) + \frac{\sqrt{3}(p^2-q^2)y}{(1-se^{-t})^{2/3}} - 4(p^3-q^3)(t+\log(1-se^{-t})) = 0. \tag{11}$$

- The amplitude of this structure depends on time satisfying that it tends to zero as $t \to -\infty$, and to the amplitude to the soliton (7) as $t \to +\infty$.
- The parabola of Eq. (11), in which our structure is localized depends on time, and it tends to the straight line of equation

$$\left((p-q)x + \sqrt{3}(p^2-q^2)y - 4(p^3-q^3)t\right) = 0,$$

  as $t \to +\infty$. This is the line in which the soliton (7) is localized.
- The vertex of the parabola moves according to the equations

$$\begin{aligned} x_V(t) &= \frac{9}{2s}(p+q)^2 e^t (1-se^{-t})^{1/3} + 4(p^2+pq+q^2)(1-se^{-t})(t-\log(1-se^{-t})), \\ y_V(t) &= \frac{3\sqrt{3}}{s}(p+q)e^t(1-se^{-t})^{2/3}. \end{aligned} \tag{12}$$

  That means that the velocity tends to infinity as $t \to \infty$. This is a consequence that for large values of t the parabola transforms into a straight line. We can avoid this effect by starting with the KdV soliton for the KP equation, i.e. by taking in (7) $p = -q$ (as we did in Fig. 1). In this case, all the preceding discussion holds, but we have that the vertex of the parabola moves along the straight line $y = 0$, according to

$$x = 4q^2(1-se^{-t})(t-\log(1-se^{-t})).$$

  Consequently, the vertex moves with a velocity

$$v = 4q^2 se^{-t}(t-\log(1-se^{-t})) + 4q^2(1-2se^{-t}).$$

  We note that this velocity tends to the one soliton transversal velocity as $t \to +\infty$.

Taking into account all these facts we interpret solution (8) as a process of interaction between the radiation (represented by R) and a soliton (represented by S). In the asymptotic limit $t \to -\infty$ the solution only contains radiation, then, as time increases a parabolic soliton emerges from the radiation. The amplitude of this soliton increases with t, while the radiation is decreasing and the soliton transforms into a line soliton. As $t \to +\infty$ the solution behaves as the one soliton solution, but far enough of the vertex of the parabola, the soliton is still a parabolic soliton and the radiation is still arbitrary large.

We represent solution (8) with $p = -q$ in Figs. 2–6. The properties of this solution previously discussed can be observed in these figures. We have choose the parameter $p = 4$ and the parameter of the group $s = -1$.

3. Deformation in finite time

In the next example we consider, we start with the two soliton solution of the KP equation, i.e.

$$u(x,y,t) = 2\partial_x^2 \log \tau(x,y,t)$$

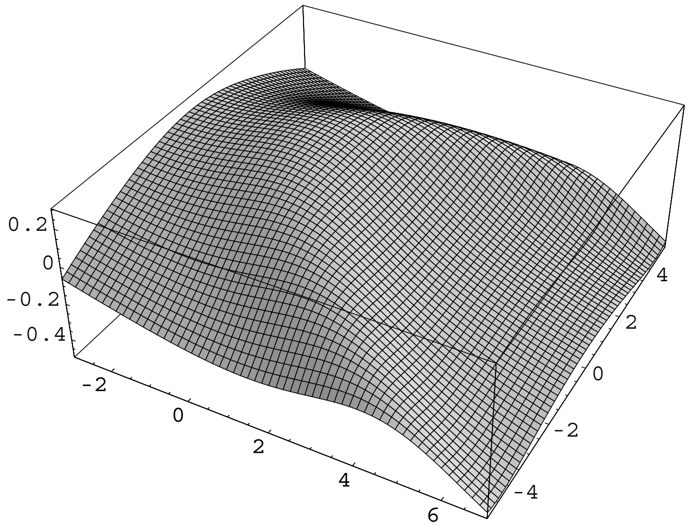

Fig. 2. (8) for $t = -7$.

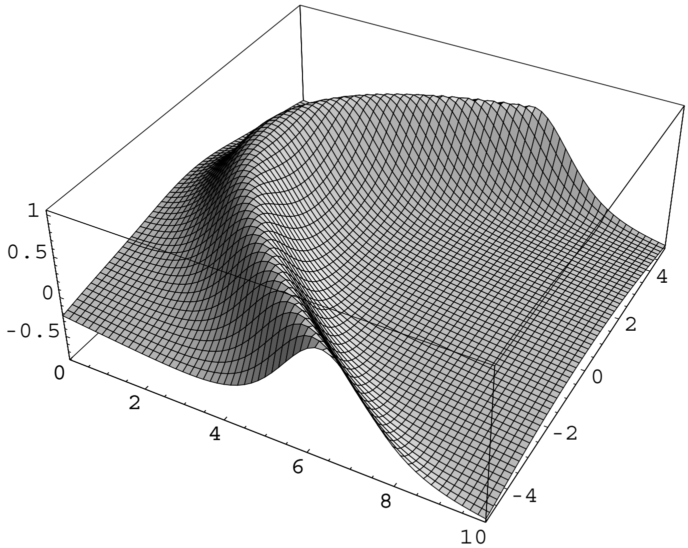

Fig. 3. (8) for $t = -5$.

with τ being

$$\tau(x,y,t) = 1 + \frac{a_1 q_1}{p_1 - q_1} \mathrm{e}^{(p_1-q_1)x+\sqrt{3}(p_1^2-q_1^2)y+4(p_1^3-q_1^3)t} + \frac{a_2 q_2}{p_2 - q_2} \mathrm{e}^{(p_2-q_2)x+\sqrt{3}(p_2^2-q_2^2)y+4(p_2^3-q_2^3)t} + \frac{a_1 a_2 (p_1 - p_2) q_1 (q_1 - q_2) q_2}{(p_1 - q_1)(q_1 - p_2)(p_1 - q_2)(p_2 - q_2)} \mathrm{e}^{(p_1-q_1+p_2-q_2)x+\sqrt{3}\left(p_1^2-q_1^2+p_2^2-q_2^2\right)y-4\left(p_1^3-q_1^3+p_2^3-q_2^3\right)t}. \quad (13)$$

This is again a coherent structure. Its graphic can be observed for the choices of the parameters $p_1 = 2/5$, $q_1 = 1/5$, $p_2 = 1/10$, $q_2 = -3/10$, $a_1 = 1$, $a_2 = -1$, and for $t = 0$ in Fig. 7.

We choose now the arbitrary functions $f \equiv 0$, $g \equiv 0$ and h as a $\mathscr{C}^\infty$ compact support function. According to the comments in the introduction, the new solution only differs from the two soliton solution when t is in the support of h. In this sense we say that the deformation is a deformation in finite time. The new solution is given by

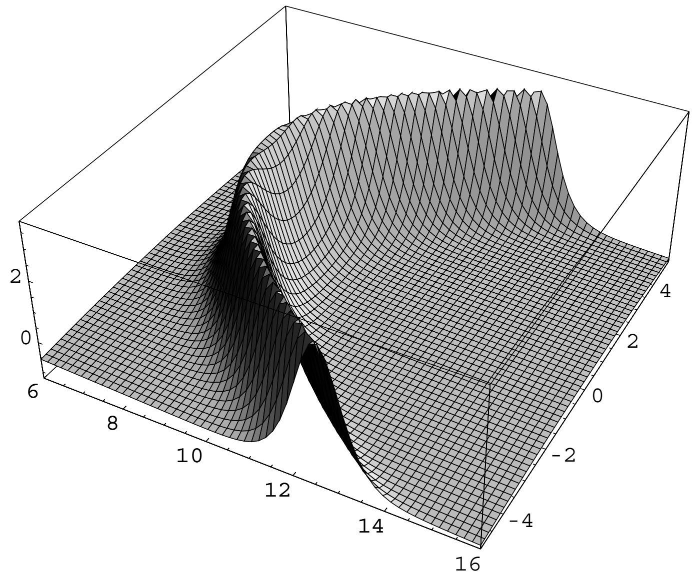

Fig. 4. (8) for $t = -3$.

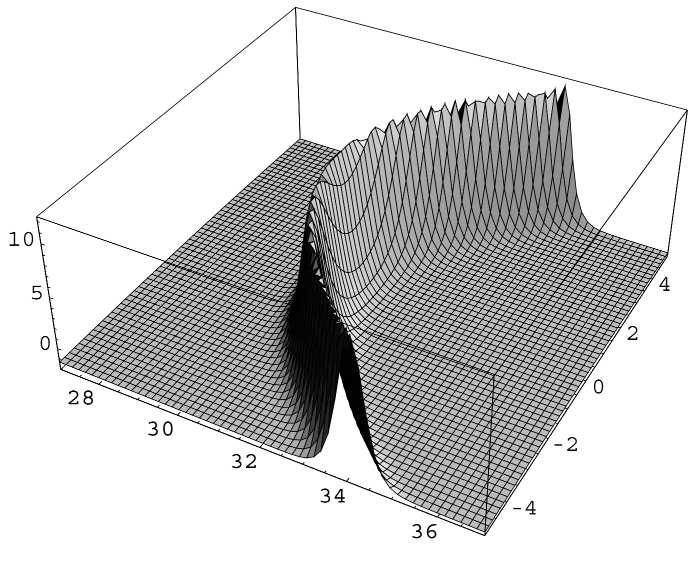

Fig. 5. (8) for $t = -1$.

$$U(x,y,t) = -\frac{syh''(t)}{12} + \frac{s^2h(t)h''(t)}{24} + 2\partial_x^2 \log \tilde{\tau}(x,y,t) \tag{14}$$

with $\tilde{\tau}$ being

$$\tilde{\tau}(x,y,t) = \tau\left(x + \frac{syh'(t)}{2} - \frac{s^2h(t)h'(t)}{4}, y - sh(t), t\right). \tag{15}$$

From this expression the following facts are easily noted

- The radiation term is linear in y and vanishes for $t \notin \text{supp}(h)$.
- The structure interacting with the radiation is the two soliton solution for $t \notin \text{supp}(h)$.
- For $t \in \text{supp}(h)$ the structure is localized in two lines of equations

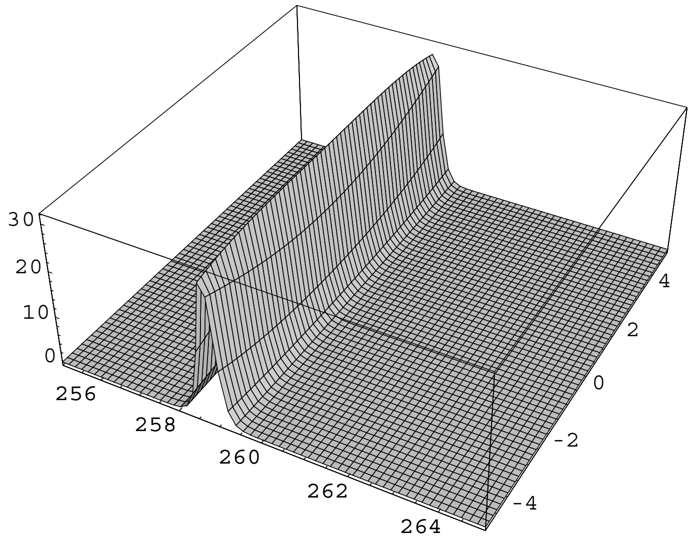

Fig. 6. (8) for $t = 4$.

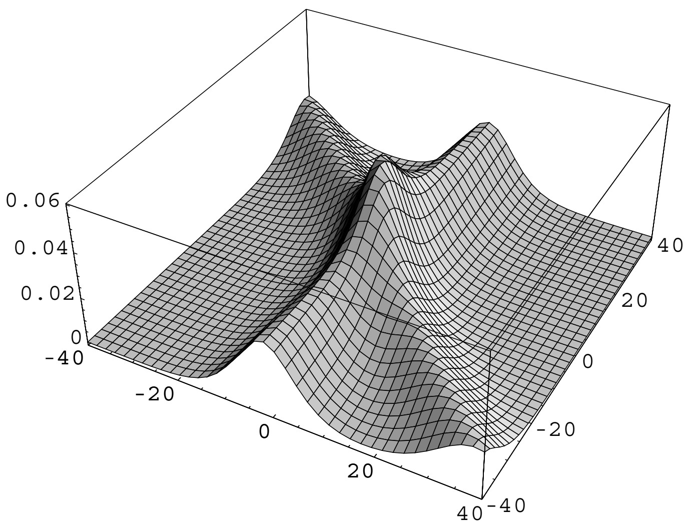

Fig. 7. Two soliton solution.

$$(p_1 - q_1)\left(x - 2sg(t) + \frac{syh'(t)}{2} - \frac{s^2h(t)h'(t)}{4}\right) + \sqrt{3}(p_1^2 - q_1^2)(y - sh(t)) - 4(p_1^3 - q_1^3)t = 0 \tag{16}$$

and

$$(p_2 - q_2)\left(x - 2sg(t) + \frac{syh'(t)}{2} - \frac{s^2h(t)h'(t)}{4}\right) + \sqrt{3}(p_2^2 - q_2^2)(y - sh(t)) - 4(p_2^3 - q_2^3)t = 0, \tag{17}$$

respectively. Consequently, it is not a coherent structure: the angle formed by the lines is not constant.

- The position of the center, the intersection point of the two lines, moves according to the equations

$$x_C(t) = 4(p_2^2 + p_2q_2 + q_2^2)t4\frac{(p_1^2 + p_1q_1 + q_1^2) - (p_2^2 + p_2q_2 + q_2^2)}{(p_1 + q_1) - (p_2 + q_2)}(p_2 + q_2)t$$
$$- \frac{2}{\sqrt{3}}\frac{(p_1^2 + p_1q_1 + q_1^2) - (p_2^2 + p_2q_2 + q_2^2)}{(p_1 + q_1) - (p_2 + q_2)}sh'(t) - \frac{s^2h(t)h'(t)}{4}, \tag{18}$$

$$y_C(t) = \frac{4(p_1^2 + p_1q_1 + q_1^2)t - 4(p_2^2 + p_2q_2 + q_2^2)t}{\sqrt{3}(p_1 + q_1) - \sqrt{3}(p_2 + q_2).} + sh'(t). \tag{19}$$

Thus, we have that the center moves on a straight line, at constant velocity for $t \notin \mathrm{supp}(h)$, while it describes a more complicated curve for $t \in \mathrm{supp}(h)$, i.e. when radiation is acting.

In order to illustrate the properties, we have commented above for this type of solution, we choose h as

$$h(t) = \begin{cases} 0.49\exp\left(-\frac{1}{(t-5)^2}\right)\exp\left(-\frac{1}{(t+5)^2}\right) & \text{if } t \in (-5, 5), \\ 0 & \text{otherwise,} \end{cases}$$

i.e. the $\mathscr{C}^\infty$ compact support function whose graphic is provided in Fig. 8. For this function, the same parameters than in Fig. 7, and the group parameter $s = 1$, we plot in Fig. 9, the trajectory of the center for $t \in (-8, 8)$, and in Fig. 10, the lines in which the structure is localized for $t = -5$, -4, -3.5 and -3.

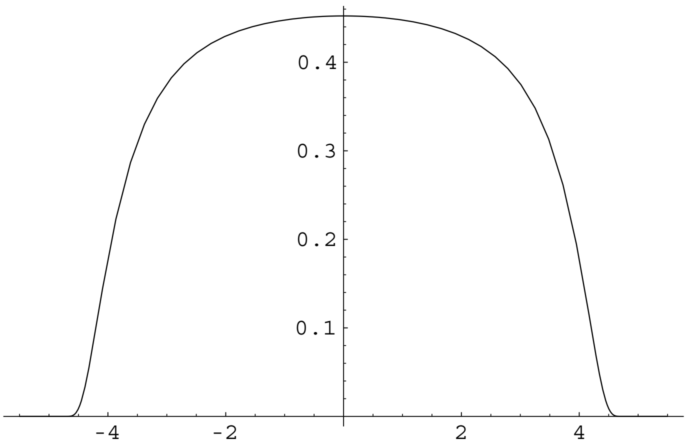

Fig. 8. $h(t)$.

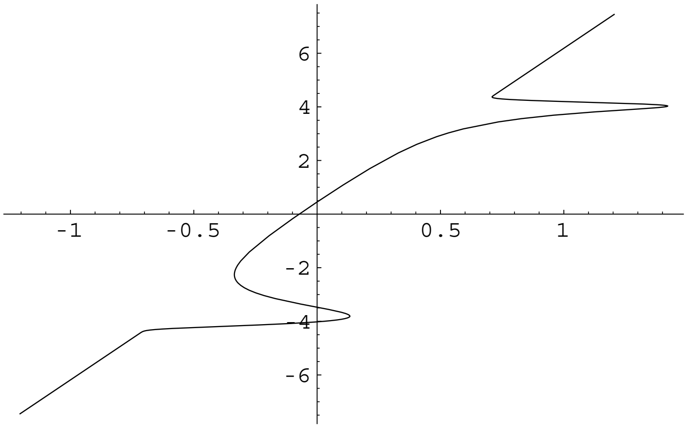

Fig. 9. Trajectory of the center.

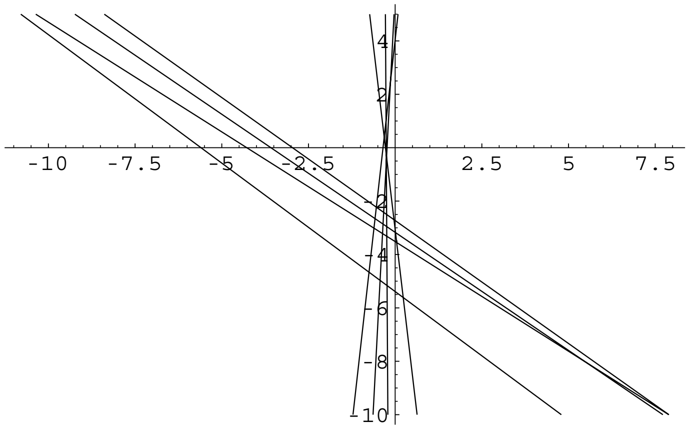

Fig. 10. Lines in which the soliton is localized for different values of t.

4. Deformation of the resonant soliton

As a last example, we consider the solution obtained by applying the group corresponding to $f(t) = 1/t$, $g \equiv h \equiv 0$, to the resonant soliton obtained by Miles [7]. That is the solution given by

$$\begin{aligned} &u(x,y,t) = 2\partial_x^2 \log \tau(x,y,t) \quad \text{with} \\ &\tau(x,y,t) = a_1 \exp\left(p_1 x + \sqrt{3}p_1^2 y - 4p_1^3 t\right) + a_2 \exp\left(p_2 x + \sqrt{3}p_2^2 y - 4p_2^3 t\right) + a_3 \exp\left(p_3 x + \sqrt{3}p_3^2 y - 4p_3^3 t\right). \end{aligned} \tag{20}$$

This solution can be obtained as a degenerate case of the two soliton solution [1,7] and describes a coherent structure moving at constant velocity

$$v = \left(-4(p_1p_2 + p_1p_3 + p_2p_3), \frac{4}{\sqrt{3}}(p_1 + p_2 + p_3) \right).$$

This structure is asymptotically localized on three rays of directions

$$\begin{aligned} &\left(\sqrt{3}(p_1 + p_2), -1\right) \quad y \to -\infty, \\ &\left(\sqrt{3}(p_1 + p_3), -1\right) \quad y \to +\infty, \\ &\left(\sqrt{3}(p_2 + p_3), -1\right) \quad y \to -\infty. \end{aligned}$$

These three rays intersect to a point, the center of the structure. Besides, as an important property we remark that in solutions describing interaction processes among these structures [8], it is found that, after interaction, they change both their forms and velocities [6]. We plot in Fig. 11, the contour lines corresponding to solution (20), for the values of the parameters $p_1 = -3$, $p_2 = 1$, $p_3 = 5$, $a_1 = 1$, $a_2 = 1$, $a_3 = 5$ and for $t = 0$.

Now, according to (4) and (5), if we act on (20) with the group corresponding to $f(t) = 1/t$, $g \equiv h \equiv 0$ we obtain the solution

$$\begin{aligned} U(x,y,t) = {} & \frac{st^{-3}x}{9(1-2st^{-2})} + \frac{s(9-8st^{-2})t^{-4}y^2}{54(1-2st^{-2})^2} + \frac{1}{(1-2st^{-2})^{1/3}} \\ & \times u\left(\frac{x}{(1-2st^{-2})^{1/6}} + \frac{st^{-3}y^2}{3(1-2st^{-2})^{7/6}}, \frac{y}{(1-2st^{-2})^{1/3}}, \sqrt{1-2st^{-2}}t \right), \end{aligned} \tag{21}$$

where $u(x,y,t)$ is given by (20). As in the previous cases we can appreciate in (21) a term of radiation and a term corresponding to the resonant soliton. The process described by (21) is again a process of a structure absorbing the radiation, and as a consequence of this absorbtion becoming, asymptotically in t, as a coherent structure. Indeed, as expected, the asymptotic behavior of the solution corresponds to the resonant soliton (20). The main properties of the structure are:

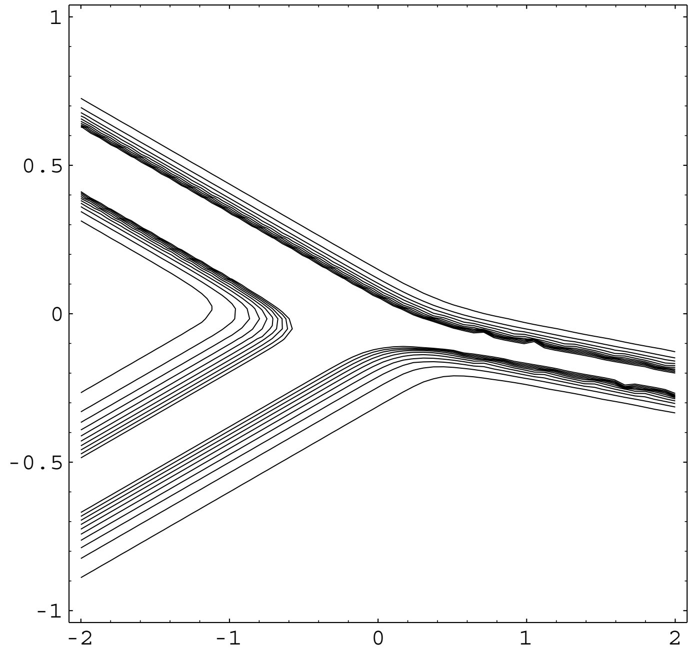

Fig. 11. Resonant soliton (20).

- It is asymptotically localized on three parabolas which intersect into a point. This point can be considered as the center of the structure.
- The center is moving according to the equations

$$x_C(t) = -\frac{4}{9t}(1-2st^{-2})^{2/3}\big(4(p_1^2+p_2^2+p_3^2)s + (p_1p_2+p_1p_3+p_2p_3)(8s+9t^2)\big),$$
$$y_C(t) = \frac{4}{\sqrt{3}}(p_1+p_2+p_3)\left(1-\frac{2s}{t^2}\right)^{5/6} t.$$

Thus, it describes an algebraic curve. We plot this curve in Fig. 12. We choose the same parameters than in Fig. 11 and the parameter of the group $s = -1/2$. In the first case we take $t \in (0, 6)$. We can see that as t increase the trajectory tends to a straight line. In order to have a better sight of the "angle" we plot also the same trajectory for $t \in (0.5, 1.5)$

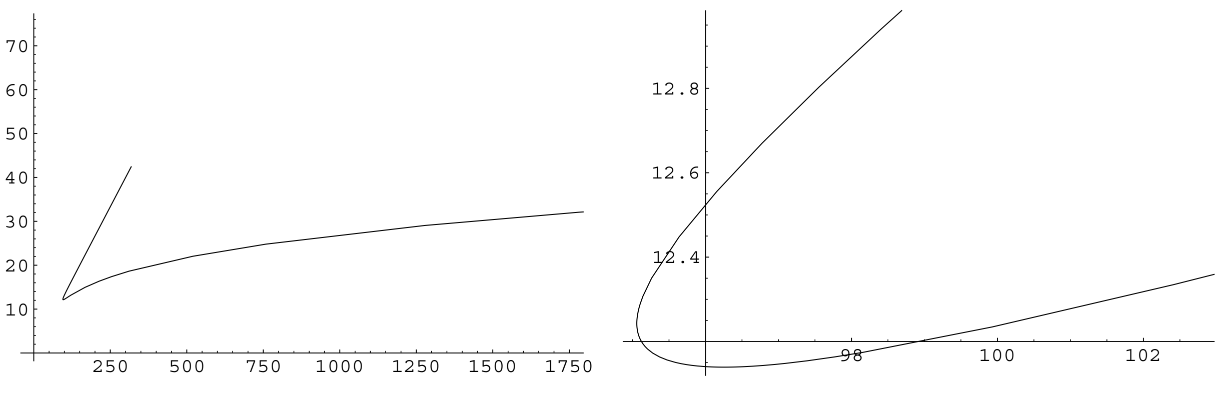

Fig. 12. Trajectory described by the center of (21) for $t \in (0, 6)$ and $t \in (0.5, 1.5)$, respectively.

Finally, we plot the contour lines of the solution (Fig. 13) we are studying for the same values of the parameters than in the previous figures in this section and for $t = 0.7$, 0.8, 0.9, 1, 1.1 and 1.6. It is clear that our solution it is not a coherent structure, but it evolves asymptotically to the coherent structure (20).

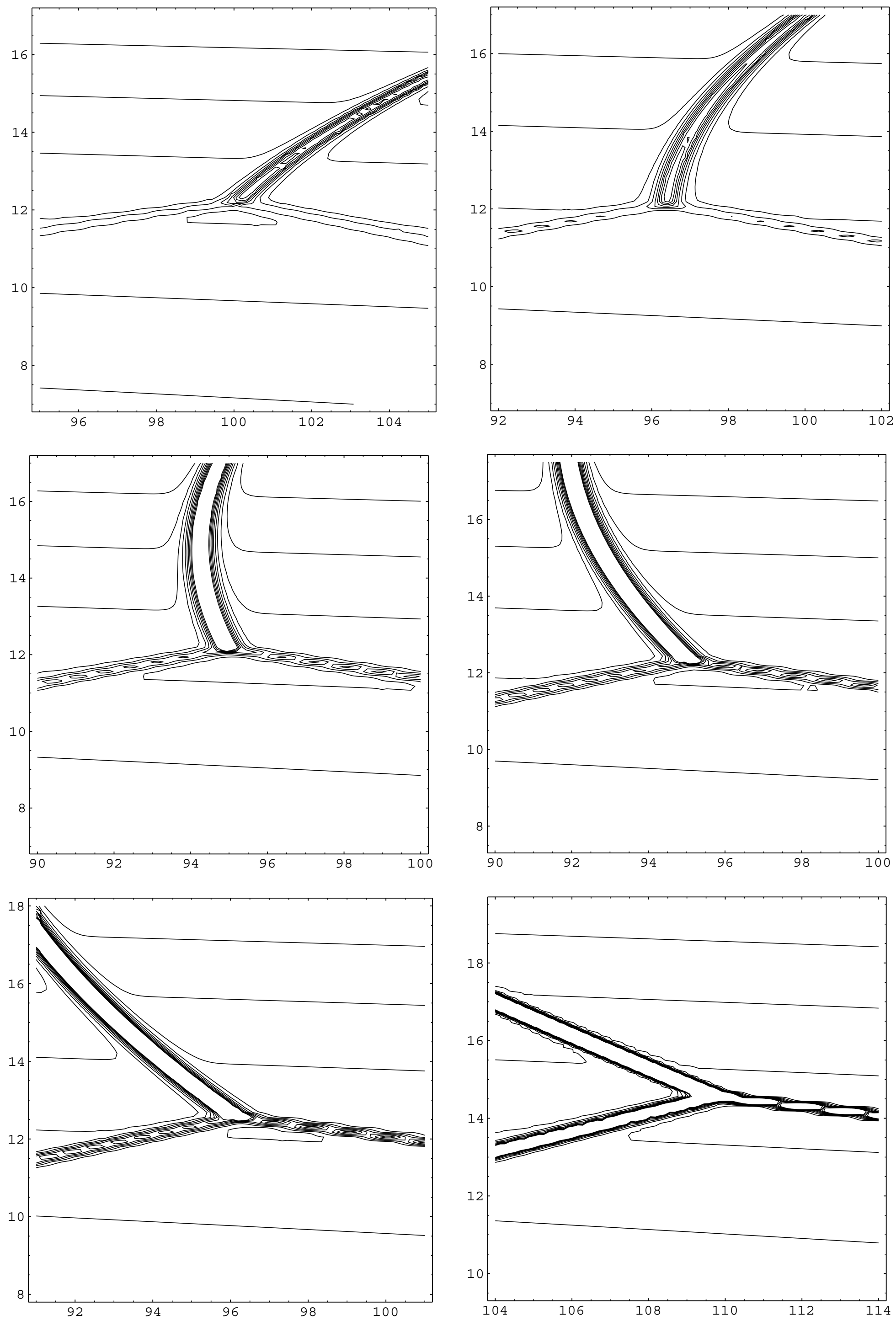

Fig. 13. Contour lines of solution (21).

5. Conclusions

In this work, we construct new solutions of the KP equation by applying some elements of its symmetry group to some of its most basic solutions. Most specifically, we have make use of the fact that the symmetry group depends on three arbitrary functions on the time variable. Thus, it is easy to see that if we choose the arbitrary functions vanishing in a certain interval for t, or vanishing asymptotically in one (or both) limits $t \to \pm\infty$, then, the new solution behaves as our starting solution, in the corresponding interval of time or asymptotically. In this sense, the solutions constructed in this way can be seen as deformations of the starting solutions.

In particular, we have worked with three concreted examples in which we have started with the basic coherent structures of the KP equation. In the three cases, the analytic expressions of the solutions are manageable enough to allow us to study some of their properties. Thus, we see that these solutions can be understood as interaction processes of the coherent structures with a non bounded term that we interpreted as the radiation.

Finally, note that more complicated solutions can be constructed by starting with solutions describing interaction processes of solitons [1,11], interaction processes of resonant solitons [6,8] or quasi-periodic solutions [4]. However, the analysis of the properties of the new solutions will be much more complicated.

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