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# CYCLIC PROPERTIES OF VOLTERRA OPERATOR

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**A bounded linear operator $T$ defined on a Hilbert space $H$ is said to be supercyclic if there exists a vector $x \in H$ such that the set $\{\lambda T^n x \ : \ n \in \mathbb{N}, \lambda \in \mathbb{C}\}$ is dense in $H$. In the present work, two open questions posed by N. H. Salas and J. Zemánek respectively, are solved. Namely, we will exhibit that the classical Volterra operator $V$ and the identity plus Volterra operator $I + V$ are not supercyclic.**

## 1. Introduction.

This paper deals with the classical Volterra operator $V$ which was introduced in 1896. It is defined on the Hilbert space $L^2[0,1]$ by

$$Vf(x) = \int_0^x f(s)\, ds.$$

An operator $T$ on a Hilbert space $H$ is said to be supercyclic if there exists a vector $x \in H$ such that the projective orbit $\{\lambda T^n x \ : \lambda \in \mathbb{C}\ , n \in \mathbb{N}\}$ is dense in $H$. The concept of supercyclicity was introduced originally in [**HW**] by Hilden and Wallen. Supercyclicity stands in the midway between hypercyclicity and cyclicity. An operator is said to be hypercyclic if there exists a vector whose orbit under $T$ is dense. On the other hand, if the linear span of some orbit is dense, the operator is called cyclic.

We have two goals:

a) To show that $V$ cannot be supercyclic on $L^2[0,1]$, and

b) the identity plus Volterra operator $I + V$ is not supercyclic on $L^2[0,1]$.

The first question was posed by N. H. Salas in [**Sa**] the second one by J. Zemánek in personal communication. In Section 2 we will renew acquaintance with the Volterra operator by proving that $V$ and $I + V$ are not hypercyclic, however they are cyclic. Section 3 is devoted to prove our main result.

Volterra operator has been studied by several authors. The norm of Volterra operator is $2/\pi$ (see [**Ha**, Problem 149]). The problem's book of P. R. Halmos contains several nice results (some of them not so elementary) related with Volterra operator. The asymptotic behaviour of the norm $\|V^n\|$ is described in [**LR**]. The most interesting fact about the Volterra operator is the determination of its invariant subspace lattice (see [**Co**, Chapter 4], and [**Br**], [**Dix**], [**Don**], [**Ka**] and [**Sar**]). Although Volterra operator is more than a hundred years old however still there exist several open questions, for example, it is not known the exact norm $\|V^n\|$ (see [**LR**]); in [**Ts**] appear new results about Volterra operator.

## 2. Hypercyclicity and cyclicity. Elementary facts.

The Volterra operator is quasinilpotent. Thus the orbit of every vector converges to zero. Therefore $V$ cannot be hypercyclic.

For the identity plus Volterra case the argument is not so easy. The following result was pointed to the authors by J. Zemánek:

**Proposition 2.1.** *Identity plus Volterra operator is not Hypercyclic on* $L^2[0,1]$.

*Proof.* The proof is based in this fact: The inverse of $(I+V)$ is power bounded (see [**Ha**, Problem 150]). Thus the orbit of any vector under $(I+V)^{-1}$ is bounded, therefore $(I+V)^{-1}$ cannot be hypercyclic. The result follows from a result of Herrero and Kitai which asserts that an invertible operator is hypercyclic if and only if its inverse is hypercyclic (see [**HK**]). □

However both operators are cyclic. Basically this fact is consequence of Weierstrass's Theorem.

**Proposition 2.2.** *Volterra and identity plus Volterra operators are cyclic.*

*Proof.* Let us denote by $L^2_{\mathbb{R}}[0,1]$ the subspace $\{f \in L^2[0,1]$ : such that $f[0,1] \subset \mathbb{R}\}$. The orbit of the identity function 1 under $V$ is the set

$$\mathrm{Orb}(V,1) = \left\{1, x, \frac{x^2}{2}, \dots, \frac{x^n}{n!}, \dots\right\}.$$

By Weierstrass's Theorem, the linear span of $\mathrm{Orb}(V,1)$ is dense in $L^2_{\mathbb{R}}[0,1]$. That is, $V$ is cyclic on $L^2_{\mathbb{R}}[0,1]$. Pick $f \in L^2[0,1]$ and $\varepsilon > 0$. The function $f = u+iv$ with $u, v \in L^2_{\mathbb{R}}[0,1]$, therefore there exists polynomials $p_u, p_v$ such that $\|p_u(V)1 - u\|^2 < \varepsilon/2$ and $\|p_v(V)1 - v\|^2 < \varepsilon$. Thus

$$p_u(x) = u_0 + u_1 x + \dots + u_n x^m \quad p_v(x) = v_0 + v_1 x + \dots + v_m x^m$$

with $u_i, v_i \in \mathbb{R}$, let us consider $p(z) = \sum_{k=0}^m a_k z^k$ with $a_k = u_k + iv_k$, $k = 0, \dots, m$, and compute

$$\begin{aligned}\|f - p(V)(1)\|^2 &= \|u + iv - p_u(V)(1) - ip_v(V)(1)\|^2 \\ &= \|u - p_u(V)(1)\|^2 + \|v - p_v(V)(1)\|^2 < \varepsilon,\end{aligned}$$

therefore 1 is a cyclic vector for $V$. For the case of $I+V$ the proof is similar. □

## 3. (Non) Supercyclicity.

The adjoint of Volterra operator is defined by

$$V^\star f(x) = \int_x^1 f(s)\,ds,$$

that is, it is an integral operator. It easy to compute that $\sigma_p(V^\star) = \emptyset$. Observe that Volterra operator is defined on complex valued functions. The following result which appear in [**LM**] will reduce our problem to real functions.

**Theorem 3.1** (Positive-Supercyclicity's Theorem)**.** *Let $T$ be a bounded linear operator defined on a separable Banach space $\mathcal{B}$. If $\sigma_p(T^\star) = \emptyset$ then $T$ is supercyclic if and only if there exists a vector $x \in \mathcal{B}$ such that $\{rT^n x : r > 0\ , n \in \mathbb{N}\}$ is dense in $\mathcal{B}$.*

**Theorem 3.2.** *Volterra and the identity plus Volterra operators are not supercyclic on $L^2[0,1]$.*

*Proof.* Let us denote by $T = V$ or $I + V$. The proof will be done in several steps:

**(1)** If $T$ is supercyclic on $L^2[0,1]$ then $T$ is supercyclic on $L^2_{\mathbb{R}}[0,1]$.

*Proof.* Let us denote by $f = u + iv$ a supercyclic vector for $T$. Observe that $T(L^2_{\mathbb{R}}[0,1]) \subset L^2_{\mathbb{R}}[0,1]$ and $T^n f = T^n u + iT^n v$. It is easy to see (using the positive-supercyclicity's Theorem) that the function $u$ is supercyclic for $T$ on $L^2_{\mathbb{R}}[0,1]$.

**(2)** If $f \in L^2_{\mathbb{R}}[0,1]$ is a continuous function (more precisely, there exists a continuous function in the coset determined by $f$) and $f$ is a supercyclic vector for $T$ then the point 0 is an accumulation point of zeros of $f$.

*Proof.* Observe that if $f$ is a continuous function so that $f$ is positive (respectively negative) on $[0,\delta]$ then the function $Vf(x)$ is also positive (respectively negative) on $[0,\delta]$. Since $Tf$ is a continuous function we obtain that the orbit under $T$ of $f$ is positive (negative) a.e. $[0,\delta]$. By way of contradiction suppose that $\delta \in (0,1]$ is the smaller zero of $f$ and without loss of generality suppose that $f$ is positive on $(0,\delta)$. In this situation the function $-1$ is separated more than $\delta$ from the set

$$\{cT^n f\ :\ c > 0\,, n \in \mathbb{N}\}.$$

Therefore $f$ cannot be supercyclic for $T$.

**(3)** If $f \in L^2_{\mathbb{R}}[0,1]$ is a continuous function, and $f$ is a supercyclic vector for $T^\star$ then the point 1 is an accumulation point of zeros of $f$.

*Proof.* The proof of (3) is analogous. It is sufficient to observe that if $f$ is a continuous function on $[0, 1]$ and $f$ is positive on $[\delta, 1]$ with $\delta \in [0, 1)$ then the orbit under $T^\star$ of $f$ is positive a.e. $[\delta, 1]$.

**(4)** The operator $T$ is supercyclic if and only if $T^\star$ is supercyclic.

*Proof.* Let us consider the isomorphism $R : L^2[0, 1] \to L^2[0, 1]$ defined by $Rf(x) = f(1 - x)$. Observe that $T = RT^\star R^{-1}$. Since Supercyclicity is invariant under similarity we obtain (4).

**(5)** Suppose that $V$ is supercyclic. Then there exists a supercyclic vector $f$ for $V$ which is so that the point 1 is an accumulation point of zeros of $V^n f$ for each integer $n$. Analagously, if $I + V$ is supercyclic then there exists a supercyclic vector $f$ for $(I + V)$ such that the point 1 is an accumulation point of zeros of the function $V(I + V)^n f$ for each integer $n$.

*Proof.* Let us suppose that $V$ is supercyclic, let us denote by $G$ the set of supercyclic vectors for $V$. It is well-known that the set of supercyclic vectors for a supercyclic bounded linear operator is a G-$\delta$ dense subset. By (4) let us denote by $G_\star$ the set of supercyclic vectors for $V^\star$. Since $V$ is continuous the set $V^{-n}(G_\star)$ is also a G-$\delta$ dense subset. Therefore the intersection $H = \bigcap_{n=1}^{\infty} V^{-n}(G_\star) \bigcap G$ contains a dense subset. Pick $f \in H$. Clearly $f$ is supercyclic for $V$, on the other hand if $n \geq 1$, $V^n f \in G_\star$ and $V^n f$ is a continuous function. Therefore by (3) the point 1 is an accumulation point of zeros of $V^n f$.

For the second part let us consider the set $\bigcap_{n=1}^{\infty} (I + V)^{-n} V^{-1} G_\star \bigcap G$ where $G$ and $G_\star$ denote now the sets of supercyclic vectors for $(I + V)$ and $(I + V)^\star$ respectively. The rest of the proof runs as before.

**(6)** The Volterra and the identity plus Volterra operators are not supercyclic on $L^2_{\mathbb{R}}[0, 1]$.

*Proof.* We first prove that Volterra operator is not supercyclic. It is sufficient to show that the orbit $V^n f$ of a possible supercyclic vector $f$ is orthogonal to the constants, that is, $\langle V^n f, 1\rangle = 0$ for all $n$. Fix $\epsilon > 0$. If $V$ is supercyclic let us consider the supercyclic function $f$ which guarantee (5). For $n \geq 1$ let us denote by $c_n$ a zero of $V^{n+1} f$ with $c_n \geq 1 - \epsilon$. Since $V^{n+1} f$ is a primitive function of $V^n f$ by applying Barrow's formula we have:

$$\begin{aligned} |\langle V^n f, 1\rangle|^2 &= \left( \left| \int_0^{c_n} V^n f(s)\, ds \right| + \left| \int_{c_n}^1 V^n f(s)\, ds \right| \right)^2 \\ &= \left| \int_{c_n}^1 V^n f(s)\, ds \right|^2 \\ &\leq (1 - c_n) \int_{c_n}^1 |V^n f(s)|^2 \, ds \\ &\leq (1 - c_n) \|V^n f\|^2 \leq \varepsilon \|V^n f\|^2. \end{aligned}$$

Since $\epsilon > 0$ is arbitrarily small (and independent of $n$) we obtain $\langle V^n f, 1\rangle = 0$ for all $n$, that is $f$ is not cyclic, a contradiction. For the case of $I + V$ the proof is similar.

Thus, by (1) and (6) the proof of Theorem 3.2 is established. □

Observe that although the results are stated in the space $L^2[0,1]$ the proofs runs as well for the spaces $L^p[0,1]$, $1 \leq p < \infty$.

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