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The role of the angle in supercyclic behavior ☆

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Abstract

A bounded operator T acting on a Hilbert space $\mathscr{H}$ is said to be supercyclic if there is a vector $f \in \mathscr{H}$ such that the *projective orbit* $\{\lambda T^n f:\ n \geqslant 0 \text{ and } \lambda \in \mathbb{C}\}$ is dense in $\mathscr{H}$. We use a new method based on a very simple geometric idea that allows us to decide whether an operator is supercyclic or not. The method is applied to obtain the following result: A composition operator acting on the Hardy space whose inducing symbol is a parabolic linear-fractional map of the disk onto a proper subdisk is not supercyclic. This result finishes the characterization of the supercyclic behavior of composition operators induced by linear fractional maps and, thus, completes previous work of Bourdon and Shapiro.

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1. Introduction

A bounded operator T acting on a Hilbert space $\mathscr{H}$ is said to be cyclic if there is a vector $f \in \mathscr{H}$ such that $\operatorname{span}\{T^n f : n \geqslant 0\}$ is dense in $\mathscr{H}$. In such a case the vector f is called cyclic. A very strong form of cyclicity is hypercyclicity. An operator T is said to be hypercyclic if there is a vector $f \in \mathscr{H}$ such that the orbit $\{T^n f\}_{n \geqslant 0}$ is dense in $\mathscr{H}$.

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In this case the vector f is called hypercyclic. The first example of a hypercyclic operator in the Hilbert space setting goes back to Rolewicz [19] in the late 1960s. He proved that if B is the backward shift defined on ℓ^2, then λB is hypercyclic whenever $|\lambda| > 1$. For more on hypercyclicity and related subjects see Grosse-Erdmann's survey [13].

A subtler property is supercyclicity. An operator T is said to be supercyclic if there is a vector f such that $\{\lambda T^n f : \lambda \in \mathbb{C}, n = 0, 1, \dots\}$ is dense in $\mathscr{H}$. Such a vector f is said to be supercyclic for T. For instance, the backward shift B acting on ℓ^2 is supercyclic because λB is hypercyclic for $|\lambda| > 1$. A nice feature is that the orbit of either a hypercyclic or a supercyclic vector projects densely onto the unit sphere of $\mathscr{H}$. The first authors that considered supercyclic operators were Hilden and Wallen [15]. They proved that every unilateral weighted shift is supercyclic. See also Salas's recent work [20] in which he completely characterized the supercyclic bilateral weighted shifts.

At first glance, one might think that every supercyclic operator has a hypercyclic scalar multiple. But this is not the case. For instance, there are compact supercyclic unilateral weighted shifts [15]. They cannot have scalar hypercyclic multiples because each component of the spectrum of a hypercyclic operator must meet the unit circle [16, Theorem 2.8]. Also, in [12, Theorem 5.2] it is shown that for $\alpha \neq 0$ the operator $S = \alpha I \oplus T : \mathbb{C} \oplus \mathscr{H} \to \mathbb{C} \oplus \mathscr{H}$ is supercyclic if and only if $(1/\alpha)T : \mathscr{H} \to \mathscr{H}$ is hypercyclic. On the other hand, since the adjoint of a hypercyclic operator has empty point spectrum [16, Corollary 2.4] and this is not the case for such an S, we find that no scalar multiple of S can be hypercyclic. But in this vein, the most interesting example is the one given by Héctor Salas [20]. He constructed an invertible supercyclic bilateral weighted shift such that any scalar multiple of which fails to be hypercyclic and has empty point spectrum.

Section 2 is devoted to an overview of the Angle Criterion of supercyclicity, which turns out to be very useful. This criterion has been previously used by Salas and the second author [17, Sections 5 and 6] to construct non-supercyclic vectors for supercyclic operators. In the present work, we will show that the criterion can also be used to show that certain operators, not just vectors, fail to be supercyclic. As a first application of the Angle Criterion, we will provide a new proof of a result due to Herrero that asserts that for any supercyclic operator T there is $\tau \geqslant 0$ such that each connected component of $\sigma(T)$ meets the circle $|z| = \tau$ [14, Proposition 3.1]. In Section 3 we will show that if we add an extra hypothesis to Clancey–Rogers' Theorem, a classical theorem that provides sufficient conditions for an operator to be cyclic (see [5, Theorem 3]), we obtain sufficient conditions for the operator to be supercyclic.

Sections 4 and 5 are the major part of this work and they are devoted to proving that if φ is a linear fractional map that takes the unit disk into itself and is a parabolic non-automorphism, then the composition operator C_φ acting on the Hardy space is not supercyclic. This result illustrates the use of the Angle Criterion in a less trivial situation and completes previous work on cyclicity of Bourdon and Shapiro [4] and Ansari and Bourdon [2, Proposition 4.2]. Consequently, the supercyclic behavior of composition operators that are induced by linear fractional maps is completely characterized. From Herrero's work [14], it is known that, in order to determine whether an operator is supercyclic or not, the first thing to do is to look at its spectrum. The spectrum of C_φ is known by Carl Cowen's work [6,7]. In fact, our proof of the non-supercyclicity of C_φ depends, in an essential way, on very nice orthogonality properties possessed by certain well-known inner functions that are eigenfunctions of C_φ. These properties allow a construction, based on Gerschgorin's theorem about approximation of eigenvalues, that transfers certain estimates from finite dimensional C_φ-invariant subspaces to the Hardy space. These ideas are intimately related to Ahern and Clark's work [1] about functions orthogonal to invariant subspaces.

2. The angle criterion

The known criteria that enable one to prove an operator is not hypercyclic do not help in disproving supercyclicity. One of the ideas in [17, Section 5] to construct a non-supercyclic vector in any infinite-dimensional closed subspace $\mathcal{M}$ for certain weighted shifts W consists of finding a vector $f \in \mathcal{M}$ such that

$$\sup_n \frac{|\langle W^n f, e_0 \rangle|}{\|W^n f\|} < 1,$$

where e_0 is the first vector of the canonical basis of ℓ^2. The point is that the whole orbit $\{W^n f\}$ lies outside a cone around e_0. Therefore, scalar multiples of the orbit cannot approximate e_0 and, consequently, f cannot be supercyclic. Now, the following supercyclicity criterion for vectors, and therefore, for operators follows immediately.

The Angle Criterion: Let T be a bounded operator on a separable Hilbert space $\mathcal{H}$. Then f is supercyclic for T if and only if for any non-zero vector $g \in \mathcal{H}$

$$\sup_n \frac{|\langle T^n f, g \rangle|}{\|T^n f\| \, \|g\|} = 1.$$

Remark. We note here that, in the above supremum, it is sufficient to consider all the positive integers larger than a fixed one.

The most interesting feature of the Angle Criterion is that the scalar multiples that appear in the definition of supercyclic operator have been absorbed by the quotient. In this way, supercyclicity becomes easier to handle. To prove that a given operator T is not supercyclic it is sufficient to show that for any vector $f \in \mathcal{H}$ there is g such that condition (2) fails to be true. The proof of the following proposition [14, Proposition 3.1] is an excellent example of the use of the Angle Criterion. Roughly speaking, one has to look for two subspaces: in one of them the operator is "small" and in the other one the operator is "big". We denote by $\sigma(T)$ the spectrum of T.

Proposition 2.1. *Suppose that* $T : \mathscr{H} \to \mathscr{H}$ *is supercyclic. Then there is* $\tau \geqslant 0$ *such that each connected component of* $\sigma(T)$ *meets the circle* $\{\lambda \in \mathbb{C} : |\lambda| = \tau\}$.

Proof. The result is trivial if $\sigma(T)$ has just one connected component. Thus suppose that $\sigma(T)$ has two or more connected components. If the conclusion of the proposition does not hold, then we can find $\tau > 0$ and non-void compact subsets K_i, $i = 1, 2$ with $K_1 \subset \mathbb{D}$ and $K_2 \subset \mathbb{C} \setminus \overline{\mathbb{D}}$, and a possibly void compact subset K_3 disjoint from $K_1 \cup K_2$ such that $\sigma(\tau T) = K_1 \cup K_2 \cup K_3$. The Riesz Decomposition Theorem (see [18, Theorem 2.10], for instance) implies that $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2 \oplus \mathscr{H}_3$ and $\tau T = T_1 \oplus T_2 \oplus T_3$ with $T_i : \mathscr{H}_i \to \mathscr{H}_i$ and $\sigma(T_i) = K_i$, $i = 1, 2, 3$. It is known and easy to show that $S = T_1 \oplus T_2$ is also supercyclic on $\mathscr{H}_1 \oplus \mathscr{H}_2$. Now, the spectral radius formula provides a constant c in $(0, 1)$ such that $\|T_1^n\| < c^n$ for n large enough. Analogously, for n large enough there is a constant $C > 1$ such that $\|T_2^n f\| > C^n \|f\|$ for any $f \in \mathscr{H}_2$.

Let f be a supercyclic vector for S; then $f = f_1 \oplus f_2$, with $f_i \in \mathscr{H}_i$, $i = 1, 2$. Since f_2 must be a supercyclic vector for T_2, we find that $\|f_2\| \neq 0$. Now, for any non-zero vector g in the orthogonal complement of $\mathscr{H}_2$ we have

$$\frac{|\langle S^n f, g \rangle|}{\|S^n f\| \, \|g\|} = \frac{|\langle T_1^n f_1 \oplus T_2^n f_2, g \rangle|}{\|T_1^n f_1 \oplus T_2^n f_2\| \, \|g\|} \leqslant \frac{|\langle T_1^n f_1, g \rangle|}{(\|T_2^n f_2\| - \|T_1^n f_1\|)\|g\|} \leqslant \frac{c^n \|f_1\|}{(C^n - c^n)\|f_2\|},$$

which goes to zero as $n \to \infty$. Therefore, upon applying the Angle Criterion we see that f is not supercyclic; a contradiction. Thus the result is proved. □

3. Supercyclicity in Clancey–Rogers' theorem

It is known [11] that when there is a rich supply of eigenvectors, splitting the spectrum of an operator in two can also be used to prove that certain operators are hypercyclic or supercyclic. This is now folklore for the specialists, see for instance Herrero's characterization of the norm closure of hypercyclic and supercyclic operators [14]. In this section we will see that the same method applies to show that many times there is supercyclicity in Clancey–Rogers' Theorem.

Let $\sigma_r(T)$ denote the right spectrum of T, that is, the set of complex numbers λ such that $T - \lambda$ is not right invertible. It is a compact set. Also we denote by $\rho_r(T) = \mathbb{C} \setminus \sigma_r(T)$ the right resolvent of T. Clancey–Rogers' Theorem [5, Theorem 3] asserts that if $\operatorname{span}\{\ker(T - \lambda) : \lambda \in \rho_r(T)\}$ is dense in $\mathscr{H}$, then T has a dense set of cyclic vectors. Observe the dualism between the proof below and that of Proposition 2.1.

Proposition 3.1. *Let* T *be a bounded linear operator on a separable Hilbert space* $\mathscr{H}$ *such that* $\operatorname{span}\{\ker(\lambda - T) : \lambda \in \rho_r(T)\}$ *is dense in* $\mathscr{H}$. *Suppose also that there is a positive number* $\tau > 0$ *such that each component of* $\rho_r(T)$ *meets the circle* $|z| = \tau$. *Then* T *is supercyclic. Furthermore,* $(1/\tau)T$ *is hypercyclic.*

Proof. Let C_n, $n = 1, 2, \ldots$, denote the (possibly finite) collection of components of $(1/\tau)\rho_r(T)$. Obviously, each of these components meets the unit circle. Since $(1/\tau)\rho_r(T)$ is an open set, we can take two sequences of non-empty open disks $\{D_n\}$ and $\{D'_n\}$ satisfying $D_n \subset C_n \cap \{z : |z| < 1\}$ and $D'_n \subset C_n \cap \{z : |z| > 1\}$. Consider the following sets:

$$X = \text{span}\left\{\ker(T - \lambda) : \lambda \in \bigcup D_n\right\}$$

$$\text{and} \quad Y = \text{span}\left\{\ker(T - \lambda) : \lambda \in \bigcup D'_n\right\}.$$

By Lemma 1 in [5] both sets X and Y are dense in $\mathscr{H}$. Since the eigenvalues corresponding to the eigenvectors in X are <1 in modulus and the eigenvalues corresponding to the eigenvectors in Y are >1 in modulus, it can be concluded as in the proof of Proposition 2.4 in [11] that $(1/\tau)T$ is hypercyclic.

An operator is said to be hyponormal if $T^\star T - TT^\star \geqslant 0$. Recently, Bourdon [3] has proved that no hyponormal operator is supercyclic. An operator is said to be cohyponormal if its adjoint is hyponormal. The following corollary establishes that there are many cohyponormal operators that are supercyclic. It is an immediate consequence of Theorems 1 and 2 in [5] and Proposition 3.1.

Corollary 3.2. *Suppose that T is a completely non-normal cohyponormal operator such that $\sigma_r(T)$ has planar Lebesgue measure zero and there is $\tau > 0$ such that each connected component of $\rho_r(T)$ meets the circle $|z| = \tau$. Then T is supercyclic.*

Remark. Proposition 3.1 as well as Corollary 3.2 have particular interest when $\rho_r(T)$ is connected; they can be applied to many classes of operators, but we will not pursue this direction here.

Most recently, Feldman, Miller and Miller independently obtained some results related to the work in this section.

4. Composition operators: background

If φ is a holomorphic self-map of the unit disk $\mathbb{D}$ into itself, the associated composition operator C_φ maps the holomorphic function f on $\mathbb{D}$ to the function $C_\varphi f = f \circ \varphi$. It is known that C_φ acts boundedly on the Hardy space $\mathscr{H}^2$ [8,21].

The simplest composition operators are those induced by linear fractional maps

$$\varphi(z) = \frac{az + b}{cz + d} \quad \text{with } ad - bc \neq 0$$

and $\varphi(\mathbb{D}) \subset \mathbb{D}$. The first results about cyclic linear fractional composition operators appeared in [24, Chapter 5]. The characterization of the linear fractional composition operators that induce cyclic or hypercyclic operators can be found in [4]. In [22] it is proved that, if φ is a parabolic non-automorphism, then scalar multiples of C_φ are never hypercyclic on the Hardy space. Ansari and Bourdon [2] have proved that composition operators whose inducing symbols have a fixed point in $\mathbb{D}$ are not supercyclic in the Hardy space. This latter result, along with the fact that supercyclicity is an intermediate property between cyclicity and hypercyclicity, allows us to determine which linear fractional composition operators are supercyclic (see [4]) except in one case. The exception is when φ is a parabolic non automorphism. Although in this case no scalar multiple is hypercyclic, there still is the possibility for C_φ to be supercyclic. In the next section, we will prove the following theorem, which completes the characterization of the supercyclic behavior of linear fractional composition operators on the Hardy space.

Main Theorem 4.1. *Let φ be a parabolic non-automorphism that takes the unit disk into itself. Then C_φ acting on the Hardy space is not supercyclic.*

We stress here that when considering the whole space of holomorphic functions $\mathscr{H}(\mathbb{D})$, the composition operator C_φ is not only supercyclic but also hypercyclic (see [21, p. 123]). On the other hand, as a consequence of the above theorem and Theorem 2.5 in [4] we see that if the underlying space is the Hardy space, then C_φ is merely cyclic.

The remainder of this section is devoted to developing those facts that are necessary to prove our Main Theorem. Recall that a function f in $\mathscr{H}^2$ has radial limits almost everywhere on the unit circle, and the norm of f is given by

$$\|f\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{i\theta})|^2\, d\theta.$$

Throughout this section φ will stand for a parabolic non-automorphism that takes $\mathbb{D}$ into itself. We will obtain a simple formula for φ that allows us to compute the iterates of C_φ. By definition, parabolic linear fractional maps have just one fixed point α. In addition, the fact that $\varphi(\mathbb{D})\subset\mathbb{D}$ implies that α is on $\partial\mathbb{D}$. Set $\psi(z)=\alpha z$. We have $C_\psi C_\varphi C_\psi^{-1} = C_{\psi^{-1}\circ\varphi\circ\psi}$. Therefore, since supercyclicity is invariant under similarity, we may assume that α is equal to 1. Using the map $z\to(1+z)/(1-z)$ that takes $\mathbb{D}$ onto the right half-plane one easily shows that there exists a complex number a with $\Re a>0$ such that φ is given by

$$\frac{1+\varphi(z)}{1-\varphi(z)} = \frac{1+z}{1-z} + a \quad \text{and} \quad \Re a > 0. \tag{1}$$

The condition $\Re a>0$ means that φ is not an automorphism of the unit disk. Now, from (1) we have the following expression:

$$\varphi(z) = \frac{(2-a)z+a}{-az+2+a}. \tag{2}$$

Let φ_0 denote the identity map and $\varphi_n = \varphi \circ \varphi_{n-1}$. The connection between the iterates of φ and the iterates of C_φ is given by $C_\varphi^n = C_{\varphi_n}$. It follows from (1) that the following expression for the nth iterate holds:

$$\varphi_n(z) = \frac{(2 - na)z + na}{-naz + 2 + na}. \tag{3}$$

The key point of the proof of our Main Theorem is to get norm estimates from above and from below of C_φ acting on certain subspaces. These subspaces are built from the eigenfunctions of C_φ. Cowen [6] proved that the spectrum of C_φ is $\{e^{-at} : t \geqslant 0\} \cup \{0\}$. But for our purposes, it is enough to know the eigenvalues and eigenfunctions of C_φ. For each $t \geqslant 0$ we consider the inner function

$$e_t(z) = \exp\left[t\frac{z+1}{z-1}\right].$$

An easy computation, using (1), shows that

$$C_\varphi e_t(z) = e^{-at} e_t(z).$$

Hence, e^{-at} is an eigenvalue corresponding to the eigenfunction $e_t(z)$. This fact will be used throughout the proof of the Main Theorem. It is not difficult to show that e_t, $t \geqslant 0$, are *all* the eigenfunctions of C_φ. In addition, the corresponding eigenvalues are simple, but we will not make use of this fact.

The following result is well known for specialists and it is already contained in the proof of Theorem 6.1 in [1]. A different proof can be found in [24]. An alternative and elementary proof which is based on Laguerre polynomials can be found in [9]. The proof we include here was indicated by Donald Sarason.

Lemma 4.2. *The set of eigenfunctions of C_φ is an spanning set of $\mathscr{H}^2$. That is*

$$\overline{\text{span}}\{e_t : t \geqslant 0\} = \mathscr{H}^2.$$

Proof. Under the standard isometry of $\mathscr{H}^2$ onto $L^2(0,\infty)$, the function e_t maps to a multiple of the function that equals 0 on $(0,t)$ and equals to $(x-t)e^{-x}$ on (t,∞). The latter functions are easily seen to span $L^2(0,\infty)$. $\square$

5. Proof of the Main Theorem

In what follows, the unidimensional subspace generated by a function $f \in \mathscr{H}^2$ will be denoted by $[f]$. The orthogonal decompositions that precede Lemma 5.1 improve the orthogonal decompositions that were obtained in Ahern and Clark's work [1].

Proof of the Main Theorem. First of all, we will decompose $\mathscr{H}^2$ as the orthogonal sum of subspaces that are nearly invariant under C_φ. To this end, we observe that

$$\langle e_t, e_s \rangle = e^{s-t} \quad \text{whenever } t \geqslant s.$$

Now, suppose that $t \neq t'$. We orthogonalize e_t and $e_{t'}$ by the Gram–Schmidt method

$$f_{tt'} = e_t - \langle e_t, e_{t'} \rangle e_{t'} = e_t - e^{-|t-t'|} e_{t'}.$$

The point here is that if $t < t'$, then $f_{tt'}$ is orthogonal not only to $e_{t'}$ but also to e_s for any $s \geqslant t'$. Indeed, we have

$$\langle f_{tt'}, e_s \rangle = e^{t-s} - e^{t-t'} e^{t'-s} = 0 \quad \text{whenever } 0 \leqslant t < t' \leqslant s. \tag{1}$$

Consequently,

$$\langle f_{tt'}, f_{st'} \rangle = 0 \quad \text{whenever } 0 \leqslant t < t' < s. \tag{2}$$

Let τ_1 be any positive real number. We define the subspaces

$$X_{\tau_1} = \overline{\text{span}}\{ f_{t\tau_1} : 0 \leqslant t < \tau_1 \} \quad \text{and} \quad Y_{\tau_1} = \overline{\text{span}}\{ f_{t\tau_1} : t > \tau_1 \}.$$

Therefore, Lemma 4.2 along with (2) allows us to obtain the following orthogonal decomposition:

$$\mathscr{H}^2 = X_{\tau_1} \oplus [e_{\tau_1}] \oplus Y_{\tau_1}. \tag{3}$$

Next, we take an arbitrary real number $\tau_2 > \tau_1$. We will show that we can decompose Y_{τ_1} in a similar way as $\mathscr{H}^2$. As before, for each $t > \tau_1$ and $t \neq \tau_2$ we orthogonalize $f_{t\tau_1}$ and $f_{\tau_2\tau_1}$ by the Gram–Schmidt method

$$g_t = f_{t\tau_1} - \frac{\langle f_{t\tau_1}, f_{\tau_2\tau_1} \rangle}{\| f_{\tau_2\tau_1} \|^2} f_{\tau_2\tau_1}.$$

Again g_t is not only orthogonal to $f_{\tau_2\tau_1}$ but also for $\tau_1 < t < \tau_2 < s$ we have

$$\begin{aligned}
\langle g_t, f_{s\tau_1} \rangle &= \langle f_{t\tau_1}, f_{s\tau_1} \rangle - \frac{\langle f_{t\tau_1}, f_{\tau_2\tau_1} \rangle}{\| f_{\tau_2\tau_1} \|^2} \langle f_{\tau_2\tau_1}, f_{s\tau_1} \rangle \\
&= e^{t-s} - e^{2\tau_1 - t - s} - \frac{(e^{t-\tau_2} - e^{2\tau_1 - t - \tau_2})(e^{\tau_2 - s} - e^{2\tau_1 - \tau_2 - s})}{1 - e^{2\tau_1 - 2\tau_2}} \\
&= e^{t-s} - e^{2\tau_1 - t - s} - \frac{e^{t-s} - e^{2\tau_1 - 2\tau_2 + t - s} - e^{2\tau_1 - s - t} + e^{4\tau_1 - 2\tau_2 - s - t}}{1 - e^{2\tau_1 - 2\tau_2}} \\
&= 0.
\end{aligned}$$

Consequently,

$$\langle g_t, g_s \rangle = 0 \quad \text{whenever } \tau_1 < t < \tau_2 < s. \tag{4}$$

Now, we consider the subspaces,

$$Z_{\tau_1,\tau_2} = \overline{\text{span}}\{g_t : \tau_1 < t < \tau_2\} \quad \text{and} \quad Z_{\tau_2} = \overline{\text{span}}\{g_t : t > \tau_2\}.$$

The definitions of Y_{τ_1} and g_t along with (4) yields the following orthogonal decomposition:

$$Y_{\tau_1} = Z_{\tau_1\tau_2} \oplus [f_{\tau_2\tau_1}] \oplus Z_{\tau_2}.$$

Hence, the following decomposition of $\mathscr{H}^2$ holds:

$$\mathscr{H}^2 = X_{\tau_1} \oplus [e_{\tau_1}] \oplus Z_{\tau_1\tau_2} \oplus [f_{\tau_2\tau_1}] \oplus Z_{\tau_2}.$$

Replacing τ_1 by τ_2 in the decomposition in (3) we have

$$\mathscr{H}^2 = X_{\tau_2} \oplus [e_{\tau_2}] \oplus Y_{\tau_2},$$

where X_{τ_2} and Y_{τ_2} are defined in the obvious way. On the other hand, one easily checks that the following identity holds:

$$X_{\tau_1} \oplus [e_{\tau_1}] \oplus Z_{\tau_1\tau_2} \oplus [f_{\tau_2\tau_1}] = X_{\tau_2} \oplus [e_{\tau_2}].$$

Hence, it follows from the uniqueness of the orthogonal complement of a subspace that $Z_{\tau_2} = Y_{\tau_2}$. Now, we set $F = \text{span}\{e_{\tau_1}, f_{\tau_2\tau_1}\} = \text{span}\{e_{\tau_1}, e_{\tau_2}\}$. Obviously, F is a two-dimensional invariant subspace of C_φ. Let $\hat{\mathscr{H}}^2$ denote the quotient space $\mathscr{H}^2/F$. Given a function $f \in \mathscr{H}^2$ we denote by $\hat{f}$ its projection onto $\hat{\mathscr{H}}^2$. Also, given any subspace Z of $\mathscr{H}^2$, its projection onto $\hat{\mathscr{H}}^2$ will be denoted by $\hat{Z}$. Clearly, the following orthogonal decomposition holds:

$$\hat{\mathscr{H}}^2 = \hat{X}_{\tau_1} \oplus \hat{Z}_{\tau_1,\tau_2} \oplus \hat{Y}_{\tau_2}. \tag{5}$$

If $f \in \mathscr{H}^2$ is any representative of $\hat{f} \in \hat{\mathscr{H}}^2$, then we define $\hat{C}_\varphi \hat{f} = \widehat{C_\varphi f}$. As F is invariant under C_φ, the operator $\hat{C}_\varphi$ is well defined. Now we have the advantage that all the spaces that appear in (5) are invariant under $\hat{C}_\varphi$. In the remainder of the proof and in the proof of Lemma 5.1 below it is convenient to have in mind the natural identification of $\hat{\mathscr{H}}^2$ as the orthogonal complement of F in the Hardy space. In particular, as X_{τ_1} is already orthogonal to $f_{\tau_2\tau_1}$, we can identify $\hat{X}_{\tau_1} = X_{\tau_1}/F$ with $X_{\tau_1}/[e_{\tau_1}]$ and this space can, in turn, be identified with X_{τ_1}. Of course, the corresponding identifications can also be made for $\hat{Y}_{\tau_2}$. In addition, there is also an obvious identification between the operator $\hat{C}_\varphi$ acting on $\hat{\mathscr{H}}^2$ and the compression of C_φ to the orthogonal complement of F. In particular, it is possible to identify $\hat{C}_\varphi|_{\hat{X}_{\tau_1}}$ and $C_\varphi|_{X_{\tau_1}}$. To show the non supercyclicity of C_φ we need an estimate of the norm of the restriction of $\hat{C}_\varphi^n$ to $\hat{Y}_{\tau_2}$. To do this, we set $Y = \overline{\text{span}}\{e_t : t \geqslant \tau_2\}$ and observe that the following equalities hold:

$$e_{\tau_2}\mathscr{H}^2 = e_{\tau_2}\,\overline{\text{span}}\{e_t : t \geqslant 0\} = \overline{\text{span}}\{e_{t+\tau_2} : t \geqslant 0\} = Y.$$

In fact, since $e_{\tau_2}(z)$ is an inner function, the multiplication operator $M_{e_{\tau_2}}$ defined by the pointwise multiplication $(M_{e_{\tau_2}}f)(z) = e_{\tau_2}(z)f(z)$ is an isometric isomorphism from $\mathscr{H}^2$ onto Y. On the other hand, for any $f \in \mathscr{H}^2$ we have

$$C_\varphi(e_{\tau_2}(z)f(z)) = e_{\tau_2}(\varphi(z))f(\varphi(z)) = e^{-\tau_2 a}e_{\tau_2}(z)C_\varphi f(z). \tag{6}$$

Therefore, we find that $C_\varphi^n|_Y$ is similar under $M_{e_{\tau_2}}$ to $e^{-n\tau_2 a}C_\varphi^n$ acting on the Hardy space. Since $M_{e_{\tau_2}}$ is an isometry, it follows that

$$\|C_\varphi^n|_Y\| = e^{-n\tau_2\Re a}\|C_\varphi^n\|.$$

Since Y_{τ_2} is contained in Y, we conclude that

$$\|\hat{C}_\varphi^n|_{\hat{Y}_{\tau_2}}\| = \|C_\varphi^n|_{Y_{\tau_2}}\| \leqslant e^{-n\tau_2\Re a}\|C_\varphi^n\|. \tag{7}$$

We also need the following lemma whose proof is delayed.

Lemma 5.1. *Suppose that* $\tau_1 > 0$ *and let* $X_{\tau_1} = \overline{\text{span}}\{f_{t\tau_1} : 0 \leqslant t \leqslant \tau_1\}$. *Then* $\hat{C}_\varphi$ *is invertible on* $\hat{X}_{\tau_1} = X_{\tau_1}/F$.

An immediate consequence of Lemma 5.1 is that the restriction of $\hat{C}_\varphi$ to $\hat{X}_{\tau_1}$ is bounded from below, that is, there is a constant $C > 0$ such that

$$\|\hat{C}_\varphi \hat{f}\| \geqslant C\|\hat{f}\| \quad (\hat{f} \in \hat{X}_{\tau_1}). \tag{8}$$

We have all the necessary ingredients to apply the Angle Criterion. Suppose that C_φ acting on $\mathscr{H}^2$ is supercyclic. Therefore, since F is invariant under C_φ, it is easy to see that $\hat{C}_\varphi$ acting on $\hat{\mathscr{H}}^2$ is also supercyclic (see the proof of Proposition 2.2 in [14]). Thus assume that $\hat{f}$ is a supercyclic vector for $\hat{C}_\varphi$ and let $\hat{f}_{\tau_1}$ and $\hat{f}_{\tau_2}$ be its orthogonal projections onto $\hat{X}_{\tau_1}$ and $\hat{Y}_{\tau_2}$, respectively. As the set of supercyclic vectors is dense, we may suppose that $\hat{f}_{\tau_1}$ is different from zero. Let $\hat{g} \neq 0$ be any function in $\hat{Y}_{\tau_2}$. The first inequality below is due to the orthogonal decomposition in (5) and the fact that $\hat{X}_{\tau_1}$ and $\hat{Y}_{\tau_2}$ are invariant under $\hat{C}_\varphi$; the second is due to the Schwarz inequality and (8); and the third inequality is due to (7)

$$\frac{|\langle \hat{C}_\varphi^n \hat{f}, \hat{g}\rangle|}{\|\hat{C}_\varphi^n \hat{f}\|\,\|\hat{g}\|} \leqslant \frac{|\langle \hat{C}_\varphi^n \hat{f}_{\tau_2}, \hat{g}\rangle|}{\|\hat{C}_\varphi^n \hat{f}_{\tau_1}\|\,\|\hat{g}\|} \leqslant \frac{\|\hat{C}_\varphi^n|_{\hat{Y}_{\tau_2}}\|\,\|\hat{f}_{\tau_2}\|}{C^n\|\hat{f}_{\tau_1}\|} \leqslant \frac{\|C_\varphi\|^n e^{-\tau_2 n\Re a}\|\hat{f}_{\tau_2}\|}{C^n\|\hat{f}_{\tau_1}\|}.$$

Since $\Re a > 0$, we can choose τ_2 satisfying $C^{-1}\|C_\varphi\| e^{-\tau_2 \Re a} < 1$ and, then, the last quantity above tends to zero as $n \to \infty$. Hence, $\hat{f}$ cannot be supercyclic; a contradiction. Therefore, C_φ is not supercyclic.

Remark. The similarity in (6) was used in [22] to prove the non-hypercyclicity of λC_φ. Alternatively, to obtain the required upper estimate of the norm of $\hat{C}_\varphi^n$ on $\hat{Y}_{\tau_2}$ we could use the techniques of the proof of Lemma 5.1. The orthogonal decompositions that appeared in Ahern's and Clark's work [1] are of the form

$$\mathscr{H}^2 = \overline{\operatorname{span}}\{e_t - e^{t-\tau} e_\tau : 0 \leqslant t < \tau\} \oplus \overline{\operatorname{span}}\{e_t : t \geqslant \tau\},$$

and they are what is needed to prove that λC_φ is not hypercyclic.

Proof of Lemma 5.1. To begin with, we observe that $\hat{X}_{\tau_1} = \overline{\operatorname{span}}\{\hat{e}_t : 0 \leqslant t < \tau_1\}$ and that $\hat{C}_\varphi \hat{e}_t = e^{-at}\hat{e}_t$ for $0 \leqslant t < \tau_1$. The proof will be done by constructing the inverse S of $\hat{C}_\varphi|_{\hat{X}_{\tau_1}}$. First, S will be defined on $\operatorname{span}\{\hat{e}_t : 0 \leqslant t < \tau_1\}$. If $P = \{0 \leqslant t_1 < \cdots < t_m < \tau_1\}$ is any partition of $[0, \tau_1)$ and c_i, $i = 1, \ldots, m$ are complex numbers, then we can define

$$S \sum_{k=1}^{m} c_i \hat{e}_{t_i} = \sum_{i=1}^{m} e^{at_i} c_i \hat{e}_{t_i}.$$

As the collection of functions $\{\hat{e}_t : 0 \leqslant t < \tau_1\}$ is linearly independent, we see that S is well defined on $\operatorname{span}\{\hat{e}_t : 0 \leqslant t < \tau_1\}$. Also, $S\hat{C}_\varphi|_{\hat{X}_{\tau_1}} = \hat{C}_\varphi|_{\hat{X}_{\tau_1}} S$ is the identity map on $\operatorname{span}\{\hat{e}_t : 0 \leqslant t < \tau_1\}$. Next, we will show that S can be extended to a bounded operator on $\hat{X}_{\tau_1}$. Clearly, it is sufficient to prove that there is a constant C such that

$$\|S\hat{f}\| \leqslant C\|\hat{f}\| \quad (\hat{f} \in \operatorname{span}\{\hat{e}_t : 0 \leqslant t < \tau_1\}).$$

Toward this end, we consider again any partition P as above and observe that $\{\hat{e}_{t_1}, \ldots, \hat{e}_{t_m}\}$ is a basis of an m-dimensional subspace $\hat{G}_m$ that is invariant under S. This basis will be replaced by an appropriate orthonormal basis. We set $t_{m+1} = \tau_1$ and consider the functions

$$\hat{g}_i = \hat{f}_{t_i, t_{i+1}} = \hat{e}_{t_i} - e^{t_i - t_{i+1}} \hat{e}_{t_{i+1}} \quad (1 \leqslant i \leqslant m).$$

Observe that $\hat{e}_{t_{m+1}} = \hat{e}_{\tau_1}$ is the zero vector in $\hat{\mathscr{H}}^2$ and, consequently, $\hat{f}_{t_m, t_{m+1}} = \hat{e}_{t_m}$. Obviously, $\{\hat{g}_1, \ldots, \hat{g}_m\}$ spans the same space as $\{\hat{e}_{t_1}, \ldots, \hat{e}_{t_m}\}$. As a consequence of (1), we have the following orthogonal relations in the Hardy space $\mathscr{H}^2$:

$$\langle f_{tt'}, f_{ss'} \rangle = 0 \quad \text{whenever } 0 \leqslant t < t' \leqslant s < s'.$$

Now, for $0 \leqslant t < t' < \tau_1$ the function $f_{tt'}$ is orthogonal to e_{τ_1} and e_{τ_2}. Therefore, the above orthogonality relations are transferred to $\hat{\mathscr{H}}^2$ and we have

$$\langle \hat{f}_{tt'}, \hat{f}_{ss'} \rangle = 0 \quad \text{whenever } 0 \leqslant t < t' \leqslant s < s' < \tau_1.$$

Consequently, $\langle \hat{g}_i, \hat{g}_j \rangle = 0$ for $i \neq j$. Since the norm of $f_{tt'}$ in $\mathscr{H}^2$ is easily computed

$$\begin{aligned} \|f_{tt'}\|^2 &= \|e_t - e^{-|t-t'|} e_{t'}\|^2 \\ &= \|e_t\|^2 - 2e^{-2|t-t'|} + e^{-2|t-t'|}\|e_{t'}\|^2 \\ &= 1 - e^{-2|t-t'|}, \end{aligned}$$

the norm of $\hat{g}_i$ is easily obtained

$$\|\hat{g}_i\|_{\hat{\mathscr{H}}^2} = \|\hat{f}_{t_i t_{i+1}}\|_{\hat{\mathscr{H}}^2} = \|f_{t_i t_{i+1}}\|_{\mathscr{H}^2} = \sqrt{1 - e^{2(t_i - t_{i+1})}} \quad (1 \leqslant i \leqslant m).$$

Therefore, an orthonormal basis of $\hat{G}_m$ is formed by the functions

$$\hat{h}_i = \frac{1}{\|\hat{g}_i\|}(\hat{e}_{t_i} - e^{t_i - t_{i+1}} \hat{e}_{t_{i+1}}) \quad (1 \leqslant i \leqslant m). \tag{9}$$

Now, we can look for the matrix representation of the restriction of S to the invariant subspace $\hat{G}_m$. For $1 \leqslant j \leqslant m$ we have

$$\begin{aligned} S\hat{h}_j &= \frac{1}{\|\hat{g}_j\|}(e^{at_j}\hat{e}_{t_j} - e^{t_j - t_{j+1}} e^{at_{j+1}} \hat{e}_{t_{j+1}}) \\ &= \frac{e^{at_j}}{\|\hat{g}_j\|}(\hat{e}_{t_j} - e^{(a-1)(t_{j+1} - t_j)} \hat{e}_{t_{j+1}}) \\ &= \frac{e^{at_j}}{\|\hat{g}_j\|}(\hat{e}_{t_j} - e^{t_j - t_{j+1}} \hat{e}_{t_{j+1}} + (e^{t_j - t_{j+1}} - e^{(a-1)(t_{j+1} - t_j)})\hat{e}_{t_{j+1}}) \\ &= e^{at_j}\hat{h}_j + \frac{e^{t_j - t_{j+1}}(1 - e^{a(t_{j+1} - t_j)})}{\|\hat{g}_j\|} \hat{e}_{t_{j+1}}. \end{aligned} \tag{10}$$

From (9) we have

$$\hat{e}_{t_{j+1}} = \|\hat{g}_{j+1}\| \hat{h}_{j+1} + e^{t_{j+1} - t_{j+2}} \hat{e}_{t_{j+2}}. \tag{11}$$

Upon substituting (11) in (10) we obtain

$$\begin{aligned} S\hat{h}_j = e^{at_j}\hat{h}_j &+ \frac{\|\hat{g}_{j+1}\| e^{t_j - t_{j+1}}(1 - e^{a(t_{j+1} - t_j)})}{\|\hat{g}_j\|} \hat{h}_{j+1} \\ &+ \frac{e^{t_j - t_{j+2}}(1 - e^{a(t_{j+1} - t_j)})}{\|\hat{g}_j\|} \hat{e}_{t_{j+2}}. \end{aligned}$$

By iterating this procedure we arrive at

$$S\hat{h}_j = e^{at_j}\hat{h}_j + \sum_{i=j+1}^{m} \frac{\|\hat{g}_i\| e^{t_j - t_i}(1 - e^{a(t_{j+1}-t_j)})}{\|\hat{g}_j\|}\hat{h}_i.$$

Therefore, the matrix representation of the restriction of S to $\hat{G}_m$ is a lower triangular matrix $A_m = (a_{ij})$, where

$$a_{ij} = \begin{cases} 0 & \text{if } i<j, \\ e^{at_j} & \text{if } i=j, \\ \dfrac{\|\hat{g}_i\| e^{t_j - t_i}(1 - e^{a(t_{j+1}-t_j)})}{\|\hat{g}_j\|} & \text{if } i>j. \end{cases}$$

Since $\{\hat{h}_i, \dots, \hat{h}_m\}$ is an orthonormal basis of $\hat{G}_m$, we have

$$\|S\hat{f}\| = \|A_m\hat{f}\| \leqslant \|A_m\| \, \|\hat{f}\| \quad (\hat{f} \in \hat{G}_m),$$

where $\|A_m\|$ is the norm of the matrix A_m, that is, the square root of the maximum of the eigenvalues of $A_m^\star A_m$. Therefore, the invertibility of $C_\varphi|_{\hat{X}_{\tau_1}}$ will be established once we have shown that $\|A_m\| \leqslant C$, where C is a constant independent of m and of the partition P. But it is not an easy task to compute the eigenvalues of an $m \times m$ matrix. Fortunately, it will be sufficient to estimate $\|A_m\|$ for m large enough and only for special choices of the partition P. For each positive integer m we consider the partition

$$P_m = \{t_j = j\tau_1/m : j = 0, 1, \dots, m-1\}.$$

We claim that the set

$$\bigcup_{m=1}^{\infty} \operatorname{span}\{\hat{e}_t : t \in P_m\}$$

is dense in $\hat{X}_{\tau_1}$. Let $E(s)$ denote the integer valued function defined on the real numbers by $E(s) = j$ if $s \in [j, j+1)$. For each positive integer m we consider the function

$$k_m(s) = \frac{\tau_1 E(ms/\tau_1)}{m} \quad \text{with } s \in [0, \tau_1).$$

Now, consider an arbitrary linear combination $f = \sum_{j=1}^{m'} c_j \hat{e}_{s_j}$, where $0 \leqslant s_1 < \cdots < s_{m'} < \tau_1$. For each positive integer m such that

$$\frac{1}{m} \leqslant \min\{(s_{j+1} - s_j)/\tau_1 : j = 1, \dots, m'\}$$

(in particular, this implies that $m \geqslant m'$) we take the function $\hat{f}_m = \sum_{j=1}^{m'} c_j \hat{e}_{k_m(s_j)}$ which belongs to $\mathrm{span}\{\hat{e}_t : t \in P_m\}$. We have

$$\left\| \sum_{j=1}^{m'} c_j \hat{e}_{k_m(s_j)} - \sum_{j=1}^{m'} c_j \hat{e}_{s_j} \right\|_{\hat{\mathscr{H}}^2} \leqslant \sum_{j=1}^{m'} |c_j| \, \| \hat{e}_{k_m(s_j)} - \hat{e}_{s_j} \|_{\hat{\mathscr{H}}^2}$$
$$\leqslant \sum_{j=1}^{m'} |c_j| \, \| e_{k_m(s_j)} - e_{s_j} \|_{\mathscr{H}^2}$$
$$= \sum_{j=1}^{m'} |c_j| \sqrt{2 - 2e^{k_m(s_j) - s_j}}$$
$$\leqslant \sqrt{2 - 2e^{-\tau_1/m}} \sum_{j=1}^{m'} |c_j|,$$

which goes to zero as $m \to \infty$ and our claim follows. Therefore, we may suppose that the original partition $P = \{0 \leqslant t_1 < \cdots < t_m < \tau_1\}$ is P_m for some positive integer m. In this way, the entries of the matrix A_m have the simpler form

$$a_{ij} = \begin{cases} 0 & \text{if } i < j, \\ e^{at_j} & \text{if } i = j, \\ e^{t_j - t_i}(1 - e^{a\tau_1/m}) & \text{if } i > j. \end{cases}$$

In addition, since $\mathrm{span}\{\hat{e}_t : t \in P_m\}$ is contained in $\mathrm{span}\{\hat{e}_t : t \in P_{m'}\}$ whenever P_m is contained in $P_{m'}$, we may consider just partitions P_m where m is as large as desired. The last ingredient we need to find a common bound for the norm of the matrices (a_{ij}) is the celebrated theorem of Gerschgorin [10] which states that the spectrum of an $m \times m$ matrix $B = (b_{ij})$ is contained in the union of the disks

$$\bigcup_{i=1}^{m} \left\{ \lambda \in \mathbb{C} : |b_{ii} - \lambda| \leqslant \sum_{j \neq i} |b_{ij}| \right\}.$$

See [23] for an interesting account on this and some related theorems. With Gerschgorin's Theorem at hand we can estimate the maximum of the eigenvalues of $A_m^\star A_m$. Since A_m is a lower triangular matrix, the entries of $A_m^\star A_m$ are

$$b_{ij} = \sum_{k=j}^{m} \bar{a}_{ki} a_{kj} \quad (i,j = 1, \ldots, m).$$

First, we estimate the entries on the diagonal. We have

$$b_{ii} = \sum_{k=i}^{m} |a_{ki}|^2 = |a_{ii}|^2 + \sum_{k=i+1}^{m} |a_{ki}|^2.$$

Therefore, substituting the values of a_{ij} we obtain

$$\begin{aligned}|b_{ii}| &= e^{2t_i \Re a} + |1 - e^{a\tau_1/m}|^2 \sum_{k=i+1}^{m} e^{2t_i - 2t_k} \\ &< e^{2t_i \Re a} + m|1 - e^{a\tau_1/m}|^2 \\ &\leqslant e^{2\tau_1 \Re a} + 1\end{aligned}$$

for m large enough. Second, we estimate

$$\sum_{\substack{j=1 \\ j \neq i}}^{m} |b_{ij}| = \sum_{\substack{j=1 \\ j \neq i}}^{m} \left| \sum_{k=j}^{m} \bar{a}_{ki}\, a_{kj} \right|. \tag{12}$$

Now, for $j \neq i$ we have

$$\begin{aligned}\left| \sum_{k=j}^{m} \bar{a}_{ki} a_{kj} \right| &\leqslant |a_{ji} a_{jj}| + \sum_{k=j+1}^{m} |a_{ki} a_{kj}| \\ &= e^{t_j \Re a} e^{t_i - t_j} |1 - e^{a\tau_1/m}| + |1 - e^{a\tau_1/m}|^2 \sum_{k=j+1}^{m} e^{t_i + t_j - 2t_k} \\ &\leqslant e^{\tau_1 \Re a} e^{\tau_1} |1 - e^{a\tau_1/m}| + e^{\tau_1} m |1 - e^{a\tau_1/m}|^2.\end{aligned}$$

Thus (12) is less than or equal to

$$e^{\tau_1 \Re a + \tau_1} m |1 - e^{a\tau_1/m}| + e^{\tau_1} m^2 |1 - e^{a\tau_1/m}|^2$$

and, for m large enough the above quantity is less than

$$|a|\tau_1 e^{\tau_1 \Re a + \tau_1} + |a|^2 \tau_1^2 e^{\tau_1}$$

which only depends on a and τ_1. Upon applying Gerschgorin's Theorem we see that if λ is an eigenvalue, then

$$|\lambda| \leqslant |\lambda - b_{ii}| + |b_{ii}| \leqslant M,$$

where M is a constant independent of m. It follows that $\|A_m\| \leqslant C$ for some constant C independent of m. Therefore, we may conclude that S can be extended to a bounded linear operator on $\hat{X}_{\tau_1} = \overline{\text{span}}\{\hat{e}_t : 0 \leqslant t < \tau_1\}$ that is the inverse of $\hat{C}_\varphi|_{\hat{X}_{\tau_1}}$. The proof of Lemma 5.1 and, therefore, that of the Main Theorem is complete. □

Remark. The constant M in the proof of Lemma 5.1 only depends on a and τ_1. By replacing a by na one can get the following estimate for the norm of S^n

$$\|S^n\| \leqslant (e^{2\tau_1 n \Re a} + n|a|\tau_1 e^{\tau_1 n \Re a + \tau_1} + n^2|a|^2\tau_1^2 e^{\tau_1} + 1)^{1/2}.$$

Therefore, the spectral radius of the restriction of S to $\hat{X}_{\tau_1}$ is $e^{\tau_1 \Re a}$.

6. Concluding remarks

We note here that, with easy modifications, the Angle Criterion also works for Banach spaces or even for Fréchet spaces. It can be concluded that the Angle Criterion makes the supercyclicity concept much clearer and the methods in this work give a transparent idea of how to use it to prove that a given operator is not supercyclic. We think that it will be very useful in future work on supercyclicity and, in particular, to solve the following questions posed by Héctor Salas [20].

Question 1. *Is the Volterra operator supercyclic?*

Question 2. *Can a finite rank perturbation of a hyponormal operator be supercyclic?*

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