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Boolean algebras and uniform convergence of series ☆

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Abstract

Several classical results on uniform convergence of unconditionally Cauchy series are generalized to weakly unconditionally Cauchy series. This uniform convergence is characterized through subalgebras and subfamilies of $\mathcal{P}(\mathbb{N})$. A generalization of the Orlicz–Pettis theorem is also proved by mean of subalgebras of $\mathcal{P}(\mathbb{N})$.

Keywords: Uniform convergence; Unconditionally Cauchy series; Weakly unconditionally Cauchy series; Boolean algebras

1. Introduction

Let X be a real Banach space. A series $\sum_{i=1}^{\infty} x_i$ in X is called weakly unconditionally Cauchy (wuC) if $\sum_{i=1}^{\infty} |f(x_i)| < +\infty$, for $f \in X^*$, and it is called unconditionally convergent (uc) if $\sum_{i=1}^{\infty} x_{\pi(i)}$ is convergent for every permutation π of $\mathbb{N}$. It is well known (cf. [7,8]) that the series $\sum_{i=1}^{\infty} x_i$ is uc if and only if the series $\sum_{i=1}^{\infty} a_i x_i$ is convergent for every $(a_i)_{i\in\mathbb{N}} \in \ell_\infty$. It is also well known (cf. [5,7,12]) that a Banach space has a copy of c_0 if and only if there exists a wuC series $\sum_{i=1}^{\infty} x_i$ in X which is not unconditionally convergent.

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Let $\sum_{i=1}^{\infty} x_i$ be a wuC series. For every $(a_i)_{i\in\mathbb{N}} \in c_0$, the series $\sum_{i=1}^{\infty} a_i x_i$ is convergent. Therefore, the map $\sigma : c_0 \to X$ defined, for $(a_i)_{i\in\mathbb{N}} \in c_0$, by $\sigma((a_i)_i) = \sum_{i=1}^{\infty} a_i x_i$ is linear and continuous. Conversely, if $\sigma : c_0 \to X$ is a continuous linear map then $\sum_{i=1}^{\infty} \sigma(e_i)$ is a wuC series and, for $(a_i) \in c_0$, we can write $\sigma((a_i)_i) = \sum_{i=1}^{\infty} a_i \sigma(e_i)$. In this case,

$$\begin{aligned}\|\sigma\| &= \sup\left\{\left\|\sum_{i=1}^{\infty} a_i x_i\right\|:\ (a_i)_i \in B_{c_0}\right\} = \sup\left\{\left\|\sum_{i=1}^{n} a_i x_i\right\|:\ (a_i)_i \in B_{c_0},\ n \in \mathbb{N}\right\} \\ &= \sup\left\{\left\|\sum_{i=1}^{n} \epsilon_i x_i\right\|:\ |\epsilon_i| = 1,\ n \in \mathbb{N}\right\} = \sup\left\{\sum_{i=1}^{n} \left\|f(x_i)\right\|:\ f \in B_{X^*},\ n \in \mathbb{N}\right\}.\end{aligned} \tag{1.1}$$

If $\sum_{i=1}^{\infty} x_i$ is a wuC series then, for $(a_i)_i \in \ell_\infty$, the series $\sum_{i=1}^{\infty} a_i x_i$ is convergent in X^{**}, with the $*$-weak topology. The map $f : \ell_\infty \to X^{**}$ defined by $f((a_i)_i) = *\text{-}w \sum_{i=1}^{\infty} a_i x_i$ is linear and continuous and verifies that $\|f\| = \|\sigma\|$ (cf. [1,3]).

Let S be a subspace of ℓ_∞ containing c_0. Let us denote

$$X(S) = \left\{\bar{x} = (x_i)_i \in X^{\mathbb{N}}:\ \sum_{i=1}^{\infty} a_i x_i \text{ converges, for all } (a_i)_i \in S\right\}.$$

The space $X(S)$, endowed with the norm $\|\bar{x}\| = \sup\{\|\sum_{i=1}^{\infty} a_i x_i\|:\ (a_i)_i \in B_S\}$, is a Banach space (cf. [2,3]). It can be proved that $\|\bar{x}\|$ verifies (1.1).

With the former notations, the space $X(c_0)$ can be considered as the space of wuC series in X (it can also be identified with the space $\ell_1^{\omega}(X)$ of the weakly 1-summing sequences). The space $X(\ell_\infty)$ can be identified with the space of uc series. Clearly $X(\ell_\infty) \subset X(S) \subset X(c_0)$. These spaces have been extensively studied (cf. [2,8,13–15]). In this paper we generalize some results, that appear in [2,6,15], in terms of the unconditionally Cauchy series in the weak topology instead of the norm topology.

2. Uniform convergence of wuC series

Let $(\bar{x}^n)_{n\in\mathbb{N}}$ be a sequence in $X(c_0)$ that converges to $\bar{x}^0 \in X(c_0)$. Then

1. $\lim_{n\to\infty}(*\text{-}w \sum_{i=1}^{\infty} a_i x_i^n) = *\text{-}w \sum_{i=1}^{\infty} a_i x_i^0$ uniformly in $(a_i)_i \in B_{\ell_\infty}$.
2. If $(a_j)_j \in \ell_\infty$ is such that, for $n \in \mathbb{N}$, $w \sum_{j=1}^{\infty} a_j x_j^n = z_n$, then there exist $z_0 \in X$ such that $w\text{-} \sum_{j=1}^{\infty} a_j x_j^0 = z_0$ and $\lim_{n\to\infty} z_n = z_0$.
3. If $(a_j)_j \in \ell_\infty$ is such that, for every $n \in \mathbb{N}$, $\sum_{j=1}^{\infty} a_j x_j^n$ converges to some $z_n \in X$ then there exists a $z_0 \in X$ such that $\sum_{j=1}^{\infty} a_j x_j^0 = z_0$ and $\lim_{n\to\infty} z_n = z_0$.

Let $(\bar{x}^n)_n$ be a sequence in $X(c_0)$ and let $\bar{x}^0 \in X(c_0)$. It can be checked that if $\lim_{n\to\infty}(*\text{-}w \sum_{i=1}^{\infty} a_i x_i^n) = *\text{-}w \sum_{i=1}^{\infty} a_i x_i^0$ uniformly in $(a_i)_i \in B_{\ell_\infty}$ then $\lim_{n\to\infty} \bar{x}^n = \bar{x}^0$ in $X(c_0)$. In the next theorem we study if the pointwise convergence of $*\text{-}w \sum_{i=1}^{\infty} a_i x_i^n$ to $*\text{-}w \sum_{i=1}^{\infty} a_i x_i^0$ is sufficient to ensure that $\bar{x}^0$ coincides with $\lim_{n\to\infty} \bar{x}^n$ so that it does not incorrectly presume the limit exists. To solve this question, let us recall some concepts (cf. [2]).

Let M be a subspace of X^{**} such that $X \subset M$. We will say that X is a M-Grothendieck space if every $\sigma(X^*, X)$-convergent sequence in X^* is $\sigma(X^*, M)$-convergent. It is clear that a Banach space X is a Grothendieck space (every $*$-w-convergent sequence in X^* is w-convergent) if and only if X is X^{**}-Grothendieck.

If S is a subspace of ℓ_∞ such that $c_0 \subset S$ then ℓ_∞ can be identified with a subspace of S^{**}. This can be checked in the following way: if I is the inclusion mapping $I : c_0 \to S$ then I^{**} is a linear isometry from $c_0^{**} \equiv \ell_\infty$ to S^{**}; if $(a_j)_j \in \ell_\infty$, we can identify $(a_j)_j$ with the map $h : S^* \to \mathbb{R}$ defined, for $g \in S^*$, by $h(g) = \sum_{j=1}^{\infty} a_j g(e_j)$.

Theorem 2.1. *Let S be a ℓ_∞-Grothendieck subspace of ℓ_∞ such that $c_0 \subset S$. Let $(\bar{x}^n)$ be a sequence in $X(c_0)$ such that*

$$\lim_{n\to\infty}\left(*\text{-}w\sum_{i=1}^{\infty} a_i x_i^n\right)$$

exists for all $(a_i)_i \in S$. Then, there exists $\bar{x}^0 \in X(c_0)$ such that $\lim_{n\to\infty} \bar{x}^n = \bar{x}^0$.

Proof. Let us suppose that $(\bar{x}^n)_n$ is not a Cauchy sequence in $X(c_0)$. There exists a $\delta > 0$ and a subsequence $(\bar{x}^{n_k})_{k\in\mathbb{N}}$ such that, for $k \in \mathbb{N}$, $\|\bar{x}^{n_k} - \bar{x}^{n_{k+1}}\| > \delta$.

For every $k \in \mathbb{N}$, we put $\bar{z}^k = (z_i^k)_i$, where $z_i^k = x_i^{n_k} - x_i^{n_{k+1}}$ for $i \in \mathbb{N}$. Clearly, $\|\bar{z}^k\| > \delta$, for $k \in \mathbb{N}$, and $\lim_{n\to\infty}(*\text{-}w\sum_{i=1}^{\infty} a_i z_i^n) = 0$, for $(a_i)_i \in S$. For every $k \in \mathbb{N}$, let $f_k \in B_{X^*}$ be such that

$$\sum_{j=1}^{\infty}\left|f_k\left(z_j^k\right)\right| > \delta, \tag{2.2}$$

let $\sigma_k : S \to X^{**}$ be the map defined by $\sigma_k((a_i)_i) = *\text{-}w\sum_{i=1}^{\infty} a_i z_i^k$ and let $\ell_k : X^{**} \to K$ be the map defined by $\ell_k(x^{**}) = x^{**}(f_k)$. If $(a_j)_{j\in\mathbb{N}} \in S$, we have that

$$\left|\ell_k\sigma_k\left((a_j)_j\right)\right| = \left|\left(*\text{-}w\sum_{j=1}^{\infty} a_j z_j^k\right)(f_k)\right| \leqslant \left\|*\text{-}w\sum_{j=1}^{\infty} a_j z_j^k\right\|.$$

Therefore, $\lim_{k\to\infty} \ell_k\sigma_k((a_j)_j) = 0$ and $(\ell_k\sigma_k)_k$ is $*$-w-convergent to 0 in S^*. If we identify ℓ_∞ with a subspace of S^{**}, we deduce that

$$(a_j)(\ell_k\sigma_k) = \sum_{j=1}^{\infty} a_j \ell_k\sigma_k(e_j) = \sum_{j=1}^{\infty} a_j f_k\left(z_j^k\right),$$

for $(a_j)_j \in \ell_\infty$. Therefore, $\lim_{k\to\infty}\sum_{j=1}^{\infty} a_j f_k(z_j^k) = 0$ and $((f_k(z_j^k))_j)_k$ is weakly convergent to 0 in ℓ_1, which implies that is norm-convergent to 0. This contradicts (2.2). □

Remark 2.2.

1. As a consequence of Theorem 2.1, we have the following result: *Let $(\bar{x}^n)_n$ be a sequence in $X(\ell_\infty)$ such that $\lim_{n\to\infty}\sum_{i=1}^{\infty} t_i x_i^n$ exists for all $(t_i)_i \in \ell_\infty$. If*

$\lim_{n\to\infty}\bar{x}_i^n = \bar{x}_i^0$, *for* $i \in \mathbb{N}$, *then* $\lim_{n\to\infty}\sum_{i=1}^{\infty} t_i x_i^n = \sum_{i=1}^{\infty} t_i x_i^0$ *uniformly in* $(t_i)_i \in B_{\ell_\infty}$. This result, when X is an F-space, was proved by Swartz [15]. Theorem 3.1 in [2] can also be obtained as a consequence of Theorem 2.1.

2. It is well known that if $\mathcal{F}$ is a Boolean algebra and T is the corresponding Stone space then T is a 0-dimensional compact space and the algebra of clopen sets in T is isomorphic to $\mathcal{F}$. If $\mathcal{C}(T)$ is the space of the real-valued continuous functions defined on T and $\mathcal{C}_0(T)$ is the subspace of $\mathcal{C}(T)$ of the finite-valued functions, then $\mathcal{C}_0(T)$ is dense in $\mathcal{C}(T)$.
   The Boolean algebra (cf. [16]) $\mathcal{F}$ has:
   (a) The Nikodym (N) property if and only if $\mathcal{C}_0(T)$ is a barrelled space.
   (b) The Grothendieck (G) property if $\mathcal{C}(T)$ is a Grothendieck space.
   (c) The Vitali–Hahn–Saks (VHS) property if and if $\mathcal{F}$ has properties (N) and (G).
   We will denote by $\mathcal{P}(\mathbb{N})$ (resp. $\phi(\mathbb{N})$) the Boolean algebra of the subsets of $\mathbb{N}$ (resp. finite or cofinite subsets of $\mathbb{N}$).

Theorem 2.3. *Let $\mathcal{F}$ be a Boolean algebra with the VHS property such that $\phi(\mathbb{N}) \subset \mathcal{F}$ and $\mathcal{F}$ is subalgebra of $\mathcal{P}(\mathbb{N})$. Let $(\bar{x}^n)_n$ be a sequence in $X(c_0)$. Let us suppose that there exists $\lim_{n\to\infty} *\text{-}w\sum_{i\in A} x_i^n$ in X^{**}, for every $A \in \mathcal{F}$. There exists $\bar{x}^0$ such that $x_i^0 = \lim_{n\to\infty} x_i^n$, $\bar{x}^0 \in X(c_0)$ and $\lim_{n\to\infty}\bar{x}^n = \bar{x}^0$ in $X(c_0)$. As a consequence,*

$$\lim_{n\to\infty} *\text{-}w\sum_{i\in A} x_i^n = *\text{-}w\sum_{i\in A} x_i^0$$

uniformly in $A \in \mathcal{F}$.

Proof. Let T be the Stone space of $\mathcal{F}$. Clearly, $\mathcal{C}(T)$ is linearly isometric to a closed subspace $S \subset \ell_\infty$ with the Grothendieck property. If S_0 is the subspace of S generated by the sequences with a finite range then S_0 is a barrelled space that is dense in S. For $n \in \mathbb{N}$, let us denote by $\sigma_n : S \to X^{**}$ the map defined by $\sigma_n((a_i)_i) = *\text{-}w\sum_{i=1}^{\infty} a_i x_i^n$. Let σ_{0n} be the restriction of σ_n to S_0. It is easy to check that σ_n is a continuous linear map and $\|\sigma_n\| = \|\sigma_{0n}\| = \|\bar{x}^n\|$, for $n \in \mathbb{N}$. It is also clear that, for all $(b_i)_i \in S_0$, $\lim_{n\to\infty} *\text{-}w\sum_{i=1}^{\infty} b_i x_i^n$ exists. Therefore, the sequence $(\sigma_{0n})_n$ is pointwise bounded in S_0. Since S_0 is a barrelled space, there exists $H > 0$ such that $\|\sigma_n\| = \|\sigma_{0n}\| < H$, for $n \in \mathbb{N}$. We also have that, for $(a_i)_i \in S$, there exists $\lim_{n\to\infty}\sigma_n((a_i)_i)$, because S_0 is dense in S. Since S is a Grothendieck space, we deduce (from Theorem 2.1) that there exists $\bar{x}^0 \in X(c_0)$ such that $\lim_{n\to\infty}\bar{x}^n = \bar{x}^0$. □

Remark 2.4. A Boolean algebra $\mathcal{F}$ is called subsequentially complete (SC) [7,11] if every disjoint sequence $(A_i)_i$ in $\mathcal{F}$ has a subsequence with a least upper bound. It can be checked that Boolean algebras with the property (SC) have the VHS property. In [11], Haydon constructed, by transfinite induction, a Boolean algebra $\mathcal{F}_H$ with the following characteristics:

(i) $\mathcal{F}_H$ is a subalgebra of $\mathcal{P}(\mathbb{N})$ such that $\phi(\mathbb{N}) \subset \mathcal{P}(\mathbb{N})$;
(ii) $\mathcal{F}_H$ is subsequentially complete;
(iii) If T_H is the Stone space of $\mathcal{F}_H$ then $\mathcal{C}(T_H)$ does not have a copy of ℓ_∞.

We have, therefore, that Theorem 2.3 can be applied to the Boolean algebra $\mathcal{F}_H$. The space $\mathcal{C}(T_H)$ can isometrically be identified with a closed subspace S_H of ℓ_∞ such that $c_0 \subset S_H$. Theorem 2.1 can be applied to S_H, but S_H does not have a copy of ℓ_∞.

In Theorem 2.3 we have supposed that $\mathcal{F}$ has the Vitali–Hahn–Saks property. This property does not give information on the supremum and separation characteristics of $\mathcal{F}$. It has been obtained interesting results in measure theory by means of this information on the supremum and separation characteristics [16]. Also, we can mentioned a generalization of the Orlicz–Pettis theorem in terms of a Boolean algebra with the separation property S_1 [4] which is defined in the next paragraph.

We will assume that $\mathcal{F}$ is a subfamily of $\mathcal{P}(\mathbb{N})$ such that $\phi_0(\mathbb{N}) \subseteq \mathcal{F}$, where $\phi_0(\mathbb{N})$ is the family of finite subsets of $\mathbb{N}$, and $\mathcal{F}$ verifies the following separation property [4]: "For any pair $((A_i)_i, (B_i)_i)$ of disjoint sequences of mutually disjoint elements of $\phi_0(\mathbb{N})$, there exist an infinite set $M \subseteq \mathbb{N}$ and a $B \in \mathcal{F}$ such that $A_i \subseteq B$ and $B_i \subseteq \mathbb{N} \setminus B$, for $i \in M$." Then, we will say that $\mathcal{F}$ has the property S_1.

In our next result we prove that Theorem 2.3 is valid when $\mathcal{F}$ verifies, instead of Vitali–Hahn–Saks property, the property S_1. Let us first give an example of family $\mathcal{F}$ in $\mathcal{P}(\mathbb{N})$ with the property S_1. Let $\mathcal{L}$ the family of subsets $A \subseteq \mathbb{N}$ such that A and A^c have infinite even numbers and infinite odd numbers. It can be checked that $\mathcal{F} = \mathcal{L} \cup \phi_0(\mathbb{N})$ has the former property. Let us observe that $\mathcal{F}$ is not subsequentially complete: the union of the members of any subsequence of $(\{2n\})_{n\in\mathbb{N}}$ is not an element of $\mathcal{F}$.

Theorem 2.5. *Let X be a Banach space and let $\mathcal{F}$ be a subfamily of $\mathcal{P}(\mathbb{N})$ such that $\phi_o(\mathbb{N}) \subseteq \mathcal{F}$ and has the property S_1. Let $(\bar{x}^n)_{n\in\mathbb{N}}$ be a sequence in $X(c_0)$ such that $\lim_n(*\text{-}w \sum_{j\in A} x_j^n)$ exists for all $A \in \mathcal{F}$. Then, there exists $\bar{x}^0 \in X(c_0)$ such that $\lim_n \bar{x}^n = \bar{x}^0$.*

Proof. Let us suppose that $(\bar{x}^n)_n$ is not a Cauchy sequence in $X(c_0)$. There exists an $\epsilon > 0$ and a subsequence $(\bar{x}^{n_k})$ such that, for $k \in \mathbb{N}$, $\|\bar{x}^{n_k} - \bar{x}^{n_{k+1}}\| > \epsilon$. For every $k \in \mathbb{N}$, we put $\bar{z}^k = (z_i^k)_i$, where $z_i^k = x_i^{n_k} - x_i^{n_{k+1}}$ for $i \in \mathbb{N}$. For every $k \in \mathbb{N}$, let $f_k \in B_{X^*}$ be such that

$$\sum_{j=1}^{\infty} \left| f_k\left(z_j^k\right)\right| > \epsilon. \tag{2.3}$$

For every $i, j \in \mathbb{N}$, we set $a_{ij} = f_i(z_j^i)$. It is easy to check that the matrix $(a_{ij})_{ij}$ is such that:

(i) $\lim_i a_{ij} = 0$ if $j \in \mathbb{N}$.
(ii) The sequence $(\sum_{j\in A} a_{ij})_i$ is convergent for every $A \in \mathcal{F}$.
(iii) $\sum_{j=1}^{\infty} a_{ij}$ is unconditionally convergent for every $i \in \mathbb{N}$.

We will prove that the sequence

$$\left(\sum_{j\in P} a_{ij}\right)_i \tag{2.4}$$

is convergent, for every $P \subseteq \mathbb{N}$.

Let us suppose that there exists $P \subseteq \mathbb{N}$ such that $(\sum_{j\in P} a_{ij})_i$ is not a Cauchy sequence; let $\alpha > 0$ be such that, for every $k \in \mathbb{N}$, there exists $n > k$ such that $|\sum_{j\in P}(a_{kj} - a_{nj})| > \alpha$.

Because of the properties (i), (ii), and (iii) of $(a_{ij})_{ij}$, we can obtain, inductively, three increasing sequences $(k_i)_i$, $(n_i)_i$, and $(m_i)_i$ of natural numbers with the following properties:

(a) $k_1 < n_1 < k_2 < n_2 < \cdots$.
(b) $|\sum_{j\in C}(a_{k_i j} - a_{n_i j})| < \alpha/8$, with $C \subseteq \{1, 2, \ldots, m_{i-1}\}$ and $i > 1$.
(c) $|\sum_{j\in F_i}(a_{k_i j} - a_{n_i j})| > 3\alpha/4$ if $F_i = P \cap \{m_{i-1} + 1, \ldots, m_i\}$, for $i > 1$.
(d) $|\sum_{j\in B}(a_{k_i j} - a_{n_i j})| < \alpha/8$, with $B \subseteq \{m_i + 1, m_i + 2, \ldots\}$ and $i > 1$.

Let $B_i = \{m_{i-1} + 1, \ldots, m_i\} \setminus F_i$, for $i > 1$. For the pair $((F_i)_i, (B_i)_i)$ we can find $B \subseteq \mathbb{N}$ and an infinite subset $M \subseteq \mathbb{N}$ such that $(\sum_{j\in B} a_{ij})_i$ is convergent, $F_i \subseteq B$ and $B_i \subseteq B^c$ for $i \in M$. However, for $i \in M$, $i > 1$,

$$\left|\sum_{j\in B}(a_{k_i j} - a_{n_i j})\right| \geqslant \left|\sum_{j\in F_i}(a_{k_i j} - a_{n_i j})\right| - \left|\sum_{\substack{j\in B\\ j\leqslant m_{i-1}}}(a_{k_i j} - a_{n_i j})\right| - \left|\sum_{\substack{j\in B\\ j> m_i}}(a_{k_i j} - a_{n_i j})\right| > \frac{\alpha}{2}.$$

This contradicts the Cauchy condition for $(\sum_{j\in B} a_{ij})_i$.

From (i) and (iii), it can inductively be deduced that there exists the increasing sequences $(i_r)_r$ and $(m_r)_r$ of natural numbers with the following properties:

(i) $m_r + 1 < m_{r+1}$ if $r \in \mathbb{N}$.
(ii) $\sum_{j\in C} |a_{i_r j}| < \epsilon/4$ if $C \subseteq \{1, \ldots, m_{r-1}\}$.
(iii) $\sum_{j\in B} |a_{i_r j}| < \epsilon/4$ if $B \subseteq \{m_r, \ldots\}$.

Hence, by (2.3) for every $r \in \mathbb{N}$ is $\sum_{j\in(m_{r-1},m_r)} |a_{i_r j}| > \epsilon/2$. For every $r > 1$, let $A_r = \{j \in (m_{r-1}, m_r)\colon\ a_{i_r j} \geqslant 0\}$ and $B_r = \{j \in (m_{r-1}, m_r)\colon\ a_{i_r j} < 0\}$ be; then, either $|\sum_{j\in A_r} a_{i_r j}| > \epsilon/5$ for an infinite number of indices r or $|\sum_{j\in B_r} a_{i_r j}| > \epsilon/5$ for an infinite number of indices r.

Let us suppose that $M = \{r \in \mathbb{N}\colon\ r > 1 \text{ and } |\sum_{j\in A_r} a_{i_r j}| > \epsilon/5\}$ is an infinite set. For each $i \in \mathbb{N}$ we can consider the measure $\mu_i : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ defined by $\mu_i(P) = \sum_{j\in P} a_{ij}$. It follows, from the Vitali–Hahn–Saks Theorem and (2.4), that the sequence $(\mu_i)_i$ is uniformly strongly additive; i.e., $(\mu_i(A_j))_j$ converges to zero uniformly in $i \in \mathbb{N}$, where $(A_j)_j$ is a disjoint sequence of $\mathcal{P}(\mathbb{N})$. However, $|\mu_i(A_r)| > \epsilon/5$ for every $r \in M$. This contradiction proves the theorem. □

3. Boolean algebras and unconditionally convergence of series

In this section we obtain generalization of the Orlicz–Pettis theorem on unconditional convergence of series.

Definition 3.1. Let $\mathcal{F}$ be a subalgebra of $\mathcal{P}(\mathbb{N})$ such that $\phi(\mathbb{N}) \subset \mathcal{F}$. Let $\sum_{i=1}^{\infty} x_i$ be a series in the space X. We will say that $\sum_{i=1}^{\infty} x_i$ is $\mathcal{F}$-convergent (resp. weakly $\mathcal{F}$-convergent) if $\sum_{i\in A} x_i$ is convergent (resp. weakly convergent), for every $A \in \mathcal{F}$.

Theorem 3.2. *Let $\mathcal{F}$ be a subalgebra of $\mathcal{P}(\mathbb{N})$ such that $\phi(\mathbb{N}) \subset \mathcal{F}$. Let $\sum_{i=1}^{\infty} x_i$ be a series in X.*

1. *If $\mathcal{F}$ has the VHS property and $\sum_{i=1}^{\infty} x_i$ is weakly $\mathcal{F}$-convergent, then $\sum_{i=1}^{\infty} x_i$ is uc.*
2. *If $\mathcal{F}$ has the G property and $\sum_{i=1}^{\infty} x_i$ is wuC and weakly $\mathcal{F}$-convergent then $\sum_{i=1}^{\infty} x_i$ is uc.*
3. *If $\mathcal{F}$ has the N property and $\sum_{i=1}^{\infty} f(x_i)$ is $\mathcal{F}$-convergent for every $f \in X^*$, then $\sum_{i=1}^{\infty} x_i$ is wuC.*

Proof.

1. Let $\sum_{i=1}^{\infty} x_i$ be a weakly $\mathcal{F}$-convergent series. For simplicity of notation we write x_A instead of $w\text{-}\sum_{i\in A} x_i$, if $A \in \mathcal{F}$. We can assume that X is a separable space.
   Let us suppose that $\sum_{i=1}^{\infty} x_i$ is not unconditionally convergent. Let $\epsilon > 0$ be such that for every $n \in \mathbb{N}$ there exists a finite set $F \subset \mathbb{N}$ such that $\inf F > n$ and $\|\sum_{i\in F} x_i\| > \epsilon$. Inductively, it can be obtained a disjoint sequence $(F_n)_{n\in\mathbb{N}}$ of finite subsets of $\mathbb{N}$ such that, for $n \in \mathbb{N}$,
   $$\sup F_n < \inf F_{n+1}$$
   and $\|\sum_{i\in F_n} x_i\| > \epsilon$. For each $n \in \mathbb{N}$ let $f_n \in S_{X^*}$ be such that
   $$f_n\left(\sum_{i\in F_n} x_i\right) > \epsilon.$$
   There exists a subsequence $(f_{n_j})_j$ of $(f_n)_n$ and $f_0 \in X^*$ such that
   $$*\text{-}w \lim_j f_{n_j} = f_0.$$
   For every $j \in \mathbb{N}$ let us consider the finitely additive measure $\mu_j : \mathcal{F} \to \mathbb{R}$ defined, for $A \in \mathcal{F}$, by $\mu_j(A) = f_{n_j}(x_A)$. For $A \in \mathcal{F}$, we have that $\lim_{j\to\infty} \mu_j(A) = f_0(x_A)$. Since $\mathcal{F}$ has the VHS property, (μ_j) is uniformly strongly additive (i.e., if $(A_i)_{i\in\mathbb{N}}$ is a sequence of disjoint elements of $\mathcal{F}$ then $\lim_{i\to\infty} \mu_j(A_i) = 0$ uniformly in $j \in \mathbb{N}$). This contradicts that
   $$\mu_j(F_{n_j}) = f_{n_j}\left(\sum_{i\in F_{n_j}} x_i\right) > \epsilon$$
   for $j \in \mathbb{N}$.
2. Since $\sum_{i=1}^{\infty} x_i$ is wuC, it is easy to check that the preceding sequence $(\mu_n)_{n\in\mathbb{N}}$ is uniformly bounded in $\mathcal{F}$.
3. If $\mathcal{F}$ is (N) and $\sum_{i=1}^{\infty} \alpha_i$ is a $\mathcal{F}$-convergent series in $\mathbb{R}$, then it can be checked that $\sum_{i=1}^{\infty} |\alpha_i| < \infty$ [9,10]. □

Remark 3.3.

1. Let us assume that X does not have a copy of c_0. If $\mathcal{F}$ is a subalgebra of $\mathcal{P}(\mathbb{N})$ with the property (N) and is such that $\phi(\mathbb{N}) \subset \mathcal{P}(\mathbb{N})$ then every $\mathcal{F}$-convergent series in X is unconditionally convergent.
2. If S_H is the space of sequences we have considered in Remark 2.4 then $X(S_H) = X(\ell_\infty)$ and S_H does not have a copy of ℓ_∞.
3. It can be proved [4] that the former theorem remains valid if $\mathcal{F}$ is a subfamily of $\mathcal{P}(\mathbb{N})$ such that $\phi_0(\mathbb{N}) \subseteq \mathcal{F}$ and $\mathcal{F}$ has the property S_1.
4. Some open problems on the subject we are studying are the following:
   (i) To characterize the subalgebras $\mathcal{F}$ of $\mathcal{P}(\mathbb{N})$ such that every weakly $\mathcal{F}$-convergent series is unconditionally convergent.
   (ii) To determine the subalgebras $\mathcal{F}$ of $\mathcal{P}(\mathbb{N})$ for which Theorem 2.3 is valid.
   (iii) To characterize the subspaces S of ℓ_∞ for which Theorem 2.1 is valid.

References

[1] A. Aizpuru, F.J. Pérez–Fernández, Characterizations of series in Banach spaces, Acta Math. Univ. Comenian. (N.S.) 68 (2) (1999) 337–344.

[2] A. Aizpuru, F.J. Pérez–Fernández, Spaces of $\mathcal{S}$-bounded multiplier convergent series, Acta Math. Hungar. 87 (1–2) (2000) 135–146.

[3] A. Aizpuru, F.J. Pérez-Fernández, Sequence spaces associated to a series in a Banach space, Indian, J. Pure Appl. Math. 33 (2002) 1317–1329.

[4] A. Aizpuru, A. Gutiérrez-Dávila, Summing Boolean algebras, Acta Math. Sinica, to appear.

[5] C. Bessaga, A. Pelczynski, On bases and unconditional convergence of series in Banach spaces, Stud. Math. 17 (1958) 151–164.

[6] J.K. Brooks, Sur les suites uniformément convergentes dans un espace de Banach, C. R. Acad. Sci. Paris 274 (1972) A1037–A1040.

[7] J. Diestel, Sequences and Series in Banach Spaces, Springer-Verlag, New York, 1984.

[8] J. Diestel, H. Jarchow, A. Tonge, Absolutely Summing Operators, Cambridge Univ. Press, Cambridge, UK, 1995.

[9] L. Drewnowski, M. Florencio, P.J. Paul, Barrelled subspaces of spaces with subseries decompositions of Boolean rings of projections, Glasgow Math. J. 36 (1994) 57–69.

[10] L. Drewnowski, P.J. Paul, The Nikodym property for ideals of sets defined by matrix summability methods, Rev. Real Acad. Cienc. Exact. Fís. Natur. Madrid, to appear.

[11] R. Haydon, A non-reflexive Grothendieck space that does not contain ℓ_∞, Israel J. Math. 40 (1981) 65–73.

[12] C.W. McArthur, On relationships amongst certain spaces of sequences in an arbitrary Banach space, Canad. J. Math. 8 (1956) 192–197.

[13] Q. Bu, C. Wu, Unconditionally convergent series of operators on Banach spaces, J. Math. Anal. Appl. 207 (1997) 291–299.

[14] R. Li, Q. Bu, Locally convex spaces containing no copy of c_0, J. Math. Anal. Appl. 172 (1993) 205–211.

[15] C. Swartz, The Schur lemma for bounded multiplier convergent series, Math. Ann. 263 (1983) 283–288.

[16] W. Schachermayer, On some classical measure theoretic theorems for non-σ-complete Boolean algebras, Technical report, Linz University, Linz, 1978.