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Ext groups of Janet systems

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Abstract

In this paper we develop the basic ideas of Janet by using the theory of $\mathscr{D}$-modules. For the so-called completely integrable systems (Janet Systems), we show that the higher Ext groups are null.

MSC: Primary: 16S32; secondary: 13P10; 13N10

1. Introduction

The aim of this paper is to give some contributions to the study of Completely Integrable System(C.I.S.) by using the theory of $\mathscr{D}$-modules. By considering a C.I.S. as a $\mathscr{D}$-module, we discuss its resolution and we prove that its Ext groups are null except for the first one.

As pioneer projects in the theory of $\mathscr{D}$-modules we consider the works of Quillen [12] and Sato et al. [16], who consider a system of homogeneous linear partial differential equations as a module of finite presentation over the ring $\mathscr{D}$ of linear differential operators. This work is based on the ideas developed by Janet in [7,8].

The structure of this paper is as follows. In Section 2, we introduce several notations and, following Janet, we consider systems in canonical form (Definition 1), called orthonomic systems by Riquier [13] and Ritt [14]. We also give the definitions of: multiplicative variables (Definition 2) and of complete systems (Definition 7); we also present some examples of these concepts. In Section 3, we study Janet systems

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(Definition 13) (i.e. completely integrable systems). These systems form a particular subclass of involutive systems. We have proved that, when the linear differential equations have their coefficients in a field, under certain conditions, every completely integrable system is a Gröbner basis and conversely. This is particularly useful in the case of rings of differential operators with constant coefficients. This fact was already observed by Pommaret [11] and was also proved by Gerdt in [5]. Here we adapt Buchberger's algorithm to give a criterion for deciding when a system of partial differential equations in canonical form is a completely integrable system. Using this criterion, we give an algorithm that, starting with a given system, constructs a completely integrable one (or, equivalently, given a generating system of a left ideal I of $\mathscr{D}$, constructs a Janet basis of I). In Section 4 we demonstrate that the elementary relations of a completely integrable system generate its module of relations; in particular, we get a resolution of length at most $n+1$ by induction. Section 5 is devoted to extension of the previous results to systems of linear differential equations with convergent coefficients.

Finally, in Section 6 we calculate the *Ext* groups of completely integrable system and we give a new proof of the equality $Ext^m_{\mathscr{D}}(M,\mathscr{O})=0$, for $m \geqslant 1$. In [7] Janet proved that each completely integrable system has an unique analytic solution, depending on certain initial conditions. In [10], we present an algorithm, following the one of Riquier [13] that compute a basis of the complex vector space $Ext^0_{\mathscr{D}}(M,\mathscr{O})$, where $\mathscr{O}$ is the ring of the convergent series in n variables, $\mathscr{D}$ is the ring of linear differential operators with coefficients in $\mathscr{O}$ and M is the finitely generated $\mathscr{D}$-module associated with a Janet system.

2. Definitions and notations

Let $\mathbf{k}$ be a field. We denote by $\mathbf{k}(X)$ (resp. $\mathbf{k}((X))$) the quotient field of the polynomial ring $\mathbf{k}[X]=\mathbf{k}[x_1,\dots,x_n]$ (resp. of the formal power series ring $\mathbf{k}[[X]]=\mathbf{k}[[x_1,\dots,x_n]]$). In this section we consider the following rings of linear differential operators: $\mathbf{k}[\partial]=\mathbf{k}[\partial_1,\dots,\partial_n]$, $Q_n(\mathbf{k})=\mathbf{k}(X)[\partial_1,\dots,\partial_n]$ and $\hat{Q}_n(\mathbf{k})=\mathbf{k}((X))[\partial_1,\dots,\partial_n]$, where $\partial_i=\partial/\partial x_i$. We denote by $\mathscr{D}$ any of these three rings and by $\mathscr{R}$ any of the corresponding fields $\mathbf{k}$, $\mathbf{k}(X)$, $\mathbf{k}((X))$. We will denote by $\mathscr{N}$ an arbitrary left $\mathscr{D}$-module.

Let us consider a system of homogeneous linear differential equations:

$$S: P_1(u)=0,\dots,P_r(u)=0, \tag{1}$$

where $P_i\in\mathscr{D}$, $1\leqslant i\leqslant s$, and the unknown u belongs to $\mathscr{N}$.

Let $<$ be a well ordering in $\mathbb{N}^n$, compatible with addition. Rewrite the equation $P_i(u)=0$ as follows:

$$a_{\alpha^i}\partial^{\alpha^i}(u)=\sum_{\beta<\alpha^i} a_\beta\partial^\beta(u) \tag{2}$$

with $a_{\alpha^i},a_\beta\in\mathscr{R}$ and $a_{\alpha^i}\neq 0$. Here ∂^γ stands for the monomial $\partial^\gamma=\partial_1^{\gamma_1}\dots\partial_n^{\gamma_n}$ for each $\gamma=(\gamma_1,\dots,\gamma_n)\in\mathbb{N}^n$. The element $a_{\alpha^i}\partial^{\alpha^i}(u)$ is called the leading derivative (see [9]) of this equation. We will identify $\partial^{\alpha^i}(u)$ with ∂^{α^i} and with α^i.

Definition 1. Let S be a system where each equation is written in the form (2) with respect to $<$. We say that S is in canonical form (with respect to $<$) if the following conditions hold:

(1) $a_{\alpha^i} = 1$ for $i = 1, \dots, r$.
(2) $\alpha^i \neq \alpha^j$ for $i \neq j$.

2.1. Multiplicative variables. Classes

We will use the notion of multiplicative variable and of class of a monomial given by Janet in [7, pp. 75–76]. These notions are different from those given by Gerdt in [4, p. 80], [6, p. 522]; and by Zharkov in [18, p. 390].

Definition 2. Let $\mathscr{F}$ be a finite set of monomials in the variables $\partial_1, \dots, \partial_n$ and let $\partial^\alpha \in \mathscr{F}$.

(1) We say that ∂_n is a multiplicative variable for ∂^α in $\mathscr{F}$, if $\beta_n \leqslant \alpha_n$ for each $\partial^\beta \in \mathscr{F}$.
(2) We say that ∂_j, $1 \leqslant j \leqslant n-1$, is a multiplicative variable for ∂^α in $\mathscr{F}$, if $\beta_j \leqslant \alpha_j$ for each $\partial^\beta \in \mathscr{F}$ such that $\beta_n = \alpha_n, \dots, \beta_{j+1} = \alpha_{j+1}$.

We denote by $mult(\partial^\alpha, \mathscr{F})$ the set of multiplicative variables for ∂^α in $\mathscr{F}$. The variables $\partial_i \notin mult(\partial^\alpha, \mathscr{F})$ are called non-multiplicative variables for ∂^α in $\mathscr{F}$.

Example 3. Let $\mathscr{F} = \{\partial_2^4, \partial_1\partial_2^4, \partial_1^2, \partial_1^3\} \subseteq \mathbf{k}[\partial_1, \partial_2]$, then

- $mult(\partial_2^4, \mathscr{F}) = \{\partial_2\}$.
- $mult(\partial_1\partial_2^4, \mathscr{F}) = \{\partial_1, \partial_2\}$.
- $mult(\partial_1^2, \mathscr{F}) = \emptyset$.
- $mult(\partial_1^3, \mathscr{F}) = \{\partial_1\}$.

Definition 4. Let ∂^α be a monomial in $\mathscr{F}$. Following Janet, we call the class of ∂^α in $\mathscr{F}$, denoted by $\mathscr{C}_{\alpha,\mathscr{F}}$, the set of monomials $\partial^{\alpha+\beta}$ such that each variable in ∂^β belongs to $mult(\partial^\alpha, \mathscr{F})$.

Example 5. We go back to Example 3,

- $\mathscr{C}_{(0,4),\mathscr{F}} = \{\partial_2^{4+b}\colon\ b \geqslant 0\}$.
- $\mathscr{C}_{(1,4),\mathscr{F}} = \{\partial_1^a\partial_2^{4+b}\colon\ a > 0, b \geqslant 0\}$.
- $\mathscr{C}_{(2,0),\mathscr{F}} = \{\partial_1^2\}$.
- $\mathscr{C}_{(3,0),\mathscr{F}} = \{\partial_1^{3+a}\colon\ b \geqslant 0\}$.

Remark 6. Classes corresponding to different monomials are disjoint (see [7, pp. 76–77]).

Definition 7. We say that $\mathscr{F}$ is complete if for each $\partial^\alpha \in \mathscr{F}$ and for each $\partial_i \notin mul(\partial^\alpha, \mathscr{F})$ there exists $\partial^\beta \in \mathscr{F}$ such that $\partial_i\partial^\alpha \in \mathscr{C}_{\beta,\mathscr{F}}$.

Example 8. The set $\mathscr{F}=\{\partial_1\partial_2^2, \partial_1^2, \partial_1^3, \partial_1^2\partial_2\} \subseteq \mathbf{k}[\partial_1, \partial_2]$ is complete. However the set $\mathscr{F}=\{\partial_2^4, \partial_1\partial_2^4, \partial_1^2, \partial_1^3\} \subseteq \mathbf{k}[\partial_1, \partial_2]$ is not complete.

Let $S=\{E_1,\ldots,E_r\}$ be a system of equations in canonical form where

$$E_1 \equiv \partial^{\alpha^1}(u) = \sum_{\beta<\alpha^1} a_\beta^1 \partial^\beta(u),$$

$$\vdots$$

$$E_r \equiv \partial^{\alpha^r}(u) = \sum_{\beta<\alpha^r} a_\beta^r \partial^\beta(u).$$

Let $\mathscr{F}$ be the set of the leading derivatives of S, that is to say

$$\mathscr{F}=\{\partial^{\alpha^1},\ldots,\partial^{\alpha^r}\}.$$

Definition 9. With the above notations, let E_i be an equation of S.

- We will say that a variable is a multiplicative (resp. non-multiplicative) variable for E_i (in S) if it is so for ∂^{α^i} (in $\mathscr{F}$).
- The class of E_i in S, denoted by $\mathscr{C}_{E_i,S}$, will be the class of its leading derivative in $\mathscr{F}$ (i.e. $\mathscr{C}_{E_i,S}=\mathscr{C}_{\alpha^i,\mathscr{F}}$).

Remark 10. Classes corresponding to different equations are disjoint, that is to say $\mathscr{C}_{E_i,S}\cap\mathscr{C}_{E_j,S}=\emptyset$, for $i\neq j$.

Definition 11. The system S is complete if $\mathscr{F}$ is complete.

3. Completely integrable systems

In this section, we will study completely integrable systems and learn how to detect when a given system is a Completely Integrable System.

Definition 12. Let $E\equiv a_\alpha\partial^\alpha(u)=\sum_{\beta<\alpha} a_\beta\partial^\beta(u)$ be a linear differential equation with $a_\alpha, a_\beta\in\mathscr{R}$ and $a_\alpha\neq 0$. We call the set $\mathrm{supp}(E)=\{\gamma\in\mathbb{N}^n \mid a_\gamma\neq 0\}$ the support of E. Let $<$ be the well ordering introduced in Section 2. We call α the leading exponent of E (with respect to $<$) and we denote it by $\exp_<(E)$. We write $\exp(E)$ when no confusion is possible.

Let $S=\{E_1,\ldots,E_r\}$ be a complete system of homogeneous linear partial differential equations written in canonical form.

We suppose

$$E_i\equiv P_i(u)=0,\quad \forall i=1,\ldots,r.$$

If no confusion arises, we will identify the equation $P_i(u)=0$ (that is E_i) with the linear differential operator P_i.

We denote by $\Delta(S)=\bigcup_{i=1}^r(\exp(E_i)+\mathbb{N}^n)$ and let I be the left ideal (in $\mathscr{D}$) generated by $\{P_1,\ldots,P_r\}$.

Definition 13. Let S be a complete system. We say that S is a completely integrable system if for all $P \in I - 0$, $\operatorname{supp}(P)$ is not contained in $\mathbb{N}^n \setminus \Delta(S)$.

Definition 14. The system S is called a Janet basis (of I) if S is completely integrable.

The theory of Gröbner bases developed by Buchberger for commutative polynomial rings has been generalized to ideals in rings of differential operators and in particular to ideals in $\mathscr{D}$ (see [1,2]).

Let $<$ be a well ordering, compatible with the sum, in $\mathbb{N}^n$. Let $P = \sum_\alpha a_\alpha \partial^\alpha$ be an element of $\mathscr{D} \setminus \{0\}$. We call the element of $\mathbb{N}^n$, $\max_<\{\mathrm{N}\mathbf{w}(\mathrm{P})\}$ (where $\mathrm{N}\mathbf{w}(\mathrm{P})$ is the Newton diagram of P) the privileged exponent of P. We denote it by $\exp_<(P)$. We write $\exp(P)$ when no confusion is possible. If I is a left ideal of $\mathscr{D}$ we denote the set $\{\exp(P)\colon P \in I\}$ by Exp(I). A finite subset $\{P_1, \ldots, P_r\} \subset I$ is said to be a Gröbner basis of I if $\operatorname{Exp}(I) = \bigcup_{j=1}^r (\exp(P_j) + \mathbb{N}^n)$.

Then the following proposition is obvious.

Proposition 15. *Let $\mathscr{B} = \{E_1, \ldots, E_r\}$ be a Gröbner basis, with respect to $<$, of a left ideal I of $\mathscr{D}$. Suppose $\exp(E_i) \neq \exp(E_j)$, for $i \neq j$, E_i is monic for all i and $\mathscr{B}$ is complete. Then $\mathscr{B}$ is a Janet basis of I.*

The converse is also clear (see [3]).

Proposition 16. *Let $S = \{E_1, E_2, \ldots, E_r\}$ be a homogeneous system of partial differential equations. Let us suppose that for all i and for all non-multiplicative variable ∂_k for E_i (in S) we have*

$$\partial_k E_i = \sum_{j=1}^{r} A_{ki}^{j} E_j \tag{3}$$

where the only variables in each monomial (in the variables $\partial_1, \ldots, \partial_n$) of A_{ki}^j are multiplicative variables for E_j, for $j = 1, 2, \ldots, r$. Let I be the ideal (of $\mathscr{D}$) generated by $P_1, \ldots, P_r$. Then for any $H \in \mathscr{D}$, $H \in I$ if and only if

$$H = \sum_{i=1}^{r} Q_i E_i$$

where the only variables of each monomial in Q_i are multiplicative variables for E_i in S.

Proof. We can suppose $\exp(E_r) < \exp(E_{r-1}) < \cdots < \exp(E_1)$. The hypothesis implies that S is complete, $\partial_k E_i = \sum_{j=1}^r A_{ki}^{c(k,i)} E_{c(k,i)}$ for an unique integer $c(k,i) < i$ (see 2.1) and $\exp(A_{ki}^j E_j) < \exp(\partial_k E_i)$ for $j \neq c(k,i)$.

If $H \in \langle E_1, E_2, \ldots, E_r \rangle$, then we have

$$H = \sum_{i=1}^{r} G_i E_i.$$

Each $G_i,\ i=1,2,\dots,r$ can be written as

$$G_i = G_i^{(1)} + H_i,$$

where $G_i^{(1)}$ is the sum of the monomials of G_i with only multiplicative variables for E_i in S. In particular $H_1 = 0$.

We have

$$H = \sum_{i=1}^{r} G_i E_i = \sum_{i=1}^{r} G_i^{(1)} E_i + \sum_{i=2}^{r} H_i E_i.$$

Let us denote

$$\delta = (\delta_1, \delta_2, \dots, \delta_n) = \max\{\exp(H_i, E_i),\ i = 2, \dots, r\}$$

and

$$i_0 = \max\{i \mid \exp(H_i E_i) = \delta\}.$$

We call (δ, i_0) the characteristic exponent of $\sum_{j=1}^{r} H_j E_j$.

We will consider on $\mathbb{N}^n \times \{1, \dots, r\}$ the well ordering defined as follows:

$$(\delta, i_0) \lhd (\delta', i_0') \quad \Leftrightarrow \quad \begin{cases} \delta < \delta' \\ \text{or} \\ \delta = \delta' \text{ and } i_0 < i_0'. \end{cases}$$

Then we can write

$$H_{i_0} E_{i_0} = a \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n} E_{i_0} + \hat{H}_{i_0} E_{i_0},$$

where $a \in \mathscr{R},\ \exp(a \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n} E_{i_0}) = \delta$ and $\exp(\hat{H}_{i_0} E_{i_0}) < \delta$.

Suppose ∂_k is a non-multiplicative variable for E_{i_0}, then by hypothesis we can write

$$\begin{aligned} H_{i_0} E_{i_0} &= a \partial_1^{\alpha_1} \cdots \partial_k^{\alpha_k - 1} \cdots \partial_n^{\alpha_n} (\partial_k E_{i0}) + \hat{H}_{i_0} E_{i_0} \\ &= a \partial_1^{\alpha_1} \cdots \partial_k^{\alpha_k - 1} \cdots \partial_n^{\alpha_n} \left(\sum_{j=1}^{r} A_{ki_0}^{j} E_j \right) + \hat{H}_{i_0} E_{i_0} \\ &= \sum_{j=1}^{r} a \partial_1^{\alpha_1} \cdots \partial_k^{\alpha_k - 1} \cdots \partial_n^{\alpha_n} A_{ki_0}^{j} E_j + \hat{H}_{i_0} E_{i_0}, \end{aligned}$$

where the only variables in each monomial of $A_{ki_0}^{j}$ are multiplicative variables for E_j. Then write

$$\begin{aligned} \sum_{i=2}^{r} H_i E_i &= H_{i_0} E_{i_0} + \sum_{j \neq i_0} H_j E_j \\ &= \sum_{j=1}^{r} a \partial_1^{\alpha_1} \cdots \partial_k^{\alpha_k - 1} \cdots \partial_n^{\alpha_n} A_{ki_0}^{j} E_j + \hat{H}_{i_0} E_{i_0} + \sum_{j \neq i_0} H_j E_j = \sum_{j=1}^{r} H_j' E_j, \end{aligned}$$

where

$$H'_j = a\partial_1^{\alpha_1}\cdots\partial_k^{\alpha_k-1}\cdots\partial_n^{\alpha_n}A^j_{ki_0} + H_j \quad \text{for } j \neq i_0,$$

$$H'_{i_0} = a\partial_1^{\alpha_1}\cdots\partial_k^{\alpha_k-1}\cdots\partial_n^{\alpha_n}A^{i_0}_{ki_0} + \hat{H}_{i_0}.$$

Now we will compute the characteristic exponent of this new expression:

(1) For $i_0+1 \leqslant j \leqslant r$ we have $\exp(H'_jE_j) = \exp(a\partial_1^{\alpha_1}\cdots\partial_k^{\alpha_k-1}\cdots\partial_n^{\alpha_n}A^j_{ki_0}E_j + H_jE_j) \leqslant \max\{\exp(a\partial_1^{\alpha_1}\cdots\partial_k^{\alpha_k-1}\cdots\partial_n^{\alpha_n}A^j_{ki_0}E_j), \exp(H_jE_j)\}$.

We have first $\exp(H_jE_j) < \delta$, because the definition of i_0, and then

$$\begin{aligned}&\exp(a\partial_1^{\alpha_1}\cdots\partial_k^{\alpha_k-1}\cdots\partial_n^{\alpha_n}A^j_{ki_0}E_j\\ &\quad=(\alpha_1,\ldots,\alpha_k-1,\ldots,\alpha_n) + \exp(A^j_{ki_0}E_j) < (\alpha_1,\ldots,\alpha_k-1,\ldots,\alpha_n)\\ &\qquad+\exp(\partial_kE_{i_0}) = \delta.\end{aligned}$$

So, $\exp(H'_jE_j) < \delta$ for $i_0+1 \leqslant j \leqslant r$.

(2) $$\begin{aligned}\exp(H'_{i_0}E_{i_0}) &= \exp(a\partial_1^{\alpha_1}\cdots\partial_k^{\alpha_k-1}\cdots\partial_n^{\alpha_n}A^{i_0}_{ki_0}E_{i_0} + \hat{H}_{i_0}E_{i_0})\\ &\leqslant \max\{\exp(a\partial_1^{\alpha_1}\cdots\partial_k^{\alpha_k-1}\cdots\partial_n^{\alpha_n}A^{i_0}_{ki_0}E_{i_0}), \exp(\hat{H}_{i_0}E_{i_0})\},\end{aligned}$$

and then $\exp(H'_{i_0}E_{i_0}) < \delta$.

(3) For $1 \leqslant j \leqslant i_0-1$ we have

$$\begin{aligned}\exp(H'_jE_j) &= \exp(a\partial_1^{\alpha_1}\cdots\partial_k^{\alpha_k-1}\cdots\partial_n^{\alpha_n}A^j_{ki_0}E_j + H_jE_j)\\ &< \max\{\exp(a\partial_1^{\alpha_1}\cdots\partial_k^{\alpha_k-1}\cdots\partial_n^{\alpha_n}A^j_{ki_0}E_j), \exp(H_jE_j)\}.\end{aligned}$$

The choice of j implies that $\exp(H_jE_j) \leqslant \delta$ and, on the other hand, we have

$$\begin{aligned}\exp(a\partial_1^{\alpha_1}\cdots\partial_k^{\alpha_k-1}\cdots\partial_n^{\alpha_n}A^j_{ki_0}E_j &= (\alpha_1,\ldots,\alpha_k-1,\ldots,\alpha_n) + \exp(A^j_{ki_0}E_j)\\ &\leqslant \exp(a\partial_1^{\alpha_1}\cdots\partial_k^{\alpha_k-1}\cdots\partial_n^{\alpha_n}A^j_{ki_0}E_j) \leqslant \delta.\end{aligned}$$

So, the characteristic exponent (δ', i'_0) of $\sum_j H'_jE_j$ is less than (δ, i_0) w.r.t the well ordering $\lhd$, which implies the assertion of the proposition. □

The following theorem gives the criterion for complete integrability.

Theorem 17 (Criterion for complete integrability). *With the notations above, the system S is completely integrable if and only if relations* (3) *of Proposition* 16 *are verified.*

Proof. We must show that if the relations are verified then S is a completely integrable system. To prove that S is a completely integrable system, by Proposition 15, it suffices to show that $\{E_i\}_{i=1}^r$ is a Gröbner basis. Let $H \in I$, then $H = \sum_{i=1}^r Q_i E_i$ where the only variables in each monomial in Q_i are multiplicative variables for E_i in S. In particular, the leading exponent of H is the leading exponent of an element of the class of an unique E_i. That is

$$\exp(H) = \exp(P), \quad P \in C_{E_i,S}.$$

On the other hand $\exp(P) \in \exp(E_i) + \mathbb{N}^n$.

The converse being nothing but a particular case of Proposition 16. □

As a consequence of this theorem, we have a finite procedure that, starting with a given system, constructs a Completely integrable system one (or equivalently, for any given generating system of a left ideal I of $\mathscr{D}$, constructs a Janet basis of I). This algorithm should be compared to Buchberger's algorithm for computing Gröbner bases. The algorithm is as follows:

Algorithm 1 (Finding a Completely Integrable System). *Input*: $S = \{E_1, \ldots, E_r\}$ *system in canonical form and complete.*

Output: *The Completely Integrable System.*

1. *For each* $i = 1, \ldots, r$ *and each* k *such that* ∂_k *is non-multiplicative variable for* E_i *in* S, *write*

$$\partial_k E_i = \sum_{j=1}^{r} A_{ki}^j E_j + R_{ik}$$

where

   (a) *Each monomial in* A_{ki}^j *(in* $\partial_1, \ldots, \partial_n$*) is formed only by multiplicative variables for* E_i *in* S.

   (b) $\exp(A_{ki}^j) \leqslant \exp(\partial_k E_i)$ *for* $j = 1, \ldots, r$.

   (c) *If* $R_{ki} \neq 0$ *then* $\mathrm{support}(R_{ki}) \subseteq \mathbb{N}^n \setminus \Delta(S)$.

2. *If all the* R_{ki} *are zero, then* S *is completely integrable.*
3. *If there exist* $R_{ki} \neq 0$, *then we consider the new system* $S_1 = S \cup \{R_{ki}\}$ *and restart.*

This procedure is finite. Indeed, let S_i, $i = 1, 2, \ldots$, be the sequence of systems obtained by applying the above procedure. Let $F_i = \{\exp(E) \mid E \in S_i\}$ be a subset of $\mathbb{N}^n$. Then we have

$$F_1 \subset F_2 \subset \cdots.$$

This sequence is stationary, because $\mathbb{N}^n$ is noetherian. So, the procedure is finite.

3.1. Non-homogeneous systems

In this subsection we will explain how to extend the above results to system of non-homogeneous linear differential equations

$$S : P_1(u) = f_1, \ldots, P_r(u) = f_r,$$

where $P_i \in \mathscr{D},\ 1 \leqslant i \leqslant s,\ f_i \in \mathscr{N}$ and the unknown u belongs to $\mathscr{N}$. We denote by S^h the homogeneous system

$$P_1(u) = 0, \dots, P_r(u) = 0$$

associated to S.

Definition 18. Let S be a system of non-homogeneous linear differential equations. We will say that

- S is complete if S^h is complete.
- S is in canonical form if S^h is in canonical form.

We will denote by E_i the equation $P_i(u) = f_i$ (or $P_i(u) - f_i = 0$). We identify the equation E_i (i.e. $P_i(u) = f_i$) with the couple $(P_i, f_i) \in \mathscr{D} \oplus \mathscr{N}$ and we consider the $\mathscr{D}$-sub-module M of $\mathscr{D} \oplus \mathscr{N}$ generated by $\{(P_1, f_1), \dots, (P_r, f_r)\}$.

Definition 19. Let $S = \{E_1, \dots, E_r\}$ be a complete system in canonical form. Let M be the $\mathscr{D}$-sub-module of $\mathscr{D} \oplus \mathscr{N}$ generated by S. The system S is said to be completely integrable if the following holds:

(1) If $(0, f) \in M$ then $f = 0$.
(2) If $(P, f) \in M$ and $P \neq 0$ then the support of P is not contained in $\mathbb{N}^n \setminus \Delta(S^h)$.

4. Chain of systems

Let $S = \{E_1, \dots, E_r\}$ be a completely integrable system. Hence we have

$$\partial_k E_i = \sum_{j=1}^{r} A_{ki}^j E_j$$

for all non-multiplicative variable ∂_k of E_i (in S).

Let us denote

$$R_{ki}^j = \begin{cases} \partial_k - A_{ki}^i, & j = i, \\ -A_{ki}^j, & j \neq i, \end{cases}$$

and $\mathbf{R}_{ki} = (R_{ki}^1, \dots, R_{ki}^r)$. Then $\mathbf{R}_{ki}$, is a syzygy of $(E_1, \dots, E_r)$, because $\sum_{j=1}^r R_{ki}^j E_j = 0$.

Proposition 20. *With the notations as above, the set $\{\mathbf{R}_{ki}\}$ generates the module of syzygies of $(E_1, \dots, E_r)$.*

Proof. Let us consider the order "to be lower than" (see [7, p. 100])

$$\beta^1 \prec \beta^2 \quad \Leftrightarrow \quad \begin{cases} |\beta^1| < |\beta^2| \\ \text{or} \\ |\beta^1| = |\beta^2| \text{ and } (\beta_n^1, \dots, \beta_1^1) <_{\text{lex}} (\beta_n^2, \dots, \beta_1^2). \end{cases}$$

Such an ordering is called "ranking" in [15].

Let $\mathbf{H} = (H_1,\dots,H_r)$ be a syzygy of $(E_1,\dots,E_r)$, that is to say, $\sum_{i=1}^{r} H_iE_i = 0$, where $H_i \neq 0$ for some i. Let $H_i = h_i\partial^{\beta(i)} + \hat{H}_i$, where $h_i \neq 0$, $\exp(\hat{H}_i) \prec \exp(H_i)$ and $E_i = \partial^{\alpha(i)} + \hat{E}_i$ with $\exp(\hat{E}_i) \prec \exp(E_i)$.

Let us consider

$$\gamma = \gamma(\mathbf{H}) := \max_{\prec}\{\alpha(j) + \beta(j) \mid j \in \{1,\dots,r\}\},$$

$$J = J(\mathbf{H}) := \{j \in \{1,\dots,r\} \mid \alpha(j) + \beta(j) = \gamma(\mathbf{H})\},$$

and

$$m(\mathbf{H}) := \max(J(\mathbf{H})).$$

If for each $j \in \{1,\dots,r\}$ all the variables of $\partial^{\beta(j)}$ are multiplicative for E_j (in S), then $\exp(H_iE_i) \neq \exp(H_jE_j)$ for $i \neq j$ and $\sum_{i=1}^{r} H_iE_i \neq 0$. Hence, $J(\mathbf{H}) \neq \emptyset$ and therefore $m(\mathbf{H}) \geqslant 1$.

Let us write $i = m(\mathbf{H})$ and $\mathbf{H}' = \mathbf{H} - h_i\partial^{\beta(i)-\varepsilon_k}\mathbf{R}_{ki}$ where $\varepsilon_k = \exp(\partial_k)$.

Let us calculate $\gamma(\mathbf{H}')$:

$$H'_j = \begin{cases} H_i - h_i\partial^{\beta(i)-\varepsilon_k}(\partial_k - A^i_{ki}), & j = i,\\ H_j - h_i\partial^{\beta(i)-\varepsilon_k}(-A^j_{ki}), & j \neq i. \end{cases}$$

- If $j = i$ then

  $$\exp(H_iE_i - h_i\partial^{\beta(i)-\varepsilon_k}(\partial_kE_i - A^i_{ki}E_i)) \prec \gamma,$$

  because $\exp(A^i_{ki}E_i) \prec \alpha(i) + \varepsilon_k$. Therefore $\gamma(\mathbf{H}') \prec (\mathbf{H})$.
- If $j \neq i$ we must consider two cases:

(1) If $j \in J$, then we have $\exp(A^j_{ki}E_j) \preccurlyeq \gamma$ where $j < i$ and $\exp(H_jE_j) \prec \gamma$. Therefore $\exp(h_i\partial^{\beta(i)-\varepsilon_k}(-A^j_{ki})E_j) \preccurlyeq \gamma$ and so $m(\mathbf{H}') < m(\mathbf{H})$.

(2) If $j \notin J$, let us suppose that the leading monomial of $H_j - h_i\partial^{\beta(i)-\varepsilon_k}A^j_{ki}$ is $\partial^{r(j)}$. Then we can consider two cases:

   (a) If in $\tau(j)$ there exists some non-multiplicative variable for $\alpha(j)$ in $\mathscr{F}$, then $\tau(j) + \alpha(j) \prec \gamma$.

   (b) If each variable in $\tau(j)$ is multiplicative for $\alpha(j)$ in $\mathscr{F}$, then

   $$\sum_{j \notin J} (H_j - h_i\partial^{\beta(i)-\varepsilon_k}A^j_{ki})E_j = 0.$$

Therefore, in all cases we have obtained $(\gamma(\mathbf{H}'), m(\mathbf{H}')) <_{\text{lex}} (\gamma(\mathbf{H}), m(\mathbf{H}))$. Since the set $(\mathbb{N}^n \times \{1,\dots,r\}, <_{\text{lex}})$ is a well ordered set, we can apply recurrence and we obtain the assertions of the proposition. □

Let $S_0 = \{E_1,\dots,E_{r_0}\}$ be a completely integrable system. Then, by Theorem 17, for each equation E_i, and each $\partial_k \notin mult(E_i, S_0)$, we get

$$H_{ki} \equiv \partial_kE_i - \sum_{j=1}^{r_0} A^j_{ki}E_j = 0, \tag{4}$$

where the only variables in each monomial (in the variables $\partial_1,\ldots,\partial_n$) of A^j_{ki} are multiplicative variables for E_j, for $j=1,2,\ldots,r_0$.

Let S_1 be the system of linear partial differential equations in the unknowns $E_1,\ldots,E_{r_0}$ formed with relations (4). The following theorem is verified.

Theorem 21 (Janet [7, p. 113]). *With the notations as above*, S_1 *is a completely integrable system of linear partial differential equations.*

If we apply the same reasoning to the system S_1 we obtain a new system S_2 that also is completely integrable and so on. This process is finite, more concretely.

Theorem 22. *With the notations as above*, *there exist*, *at most*, $n+1$ *systems of type* S_i, *where* n *is the number of independent variables.*

Remark 23. The system S_1 must be considered as the module of the relations between the elements of S_0. Hence, the previous theorem is a precedent of the theorem of Schreyer (see [17]) that assures that the relations (elementary syzygies) between the elements of a Gröbner basis of an ideal of polynomials is a Gröbner basis of the module of syzygies of such ideal.

5. Convergent coefficients. Rank of a connection

The previous results can be extended to the case of systems of linear differential equations with convergent coefficients working at the "generic point". More precisely, if S is such a system (i.e. the coefficients are convergent series at the neighborhood of a point in $\mathbb{C}^n$) we will be able to transform it into a complete system in canonical form, if we locate ourselves at a point in the corresponding convergence dominance, where none of the coefficients in the first members of S_1 are null.

We denote by $\mathcal{O}=\mathbb{C}\{x_1,\ldots,x_n\}=\mathbb{C}\{x\}$ the ring of convergent power series in the variables $x_1,\ldots,x_n$ and coefficients in $\mathbb{C}$. We denote by $\mathcal{D}=\mathcal{O}[\partial_1,\ldots,\partial_n]$ the ring of linear differential operators with coefficients in $\mathcal{O}$.

Recall the monomial order, $\prec$ in $\mathbb{N}^n$, "to be lower than" (see Proposition 20):

$$\beta^1 \prec \beta^2 \iff \begin{cases} |\beta^1| < |\beta^2| \\ \text{or} \\ |\beta^1| = |\beta^2| \text{ and } (\beta^1_n,\ldots,\beta^1_1) <_{\text{lex}} (\beta^2_n,\ldots,\beta^2_1). \end{cases}$$

Let $\prec_J$ be the total ordering on $\mathbb{N}^n\times\mathbb{N}^n$, compatible with sums, defined by

$$(\alpha^1,\beta^1) \prec_J (\alpha^2,\beta^2) \iff \begin{cases} \beta^1 \prec \beta^2 \\ \text{or} \\ \beta^1=\beta^2 \text{ and } \alpha^2 <_{\text{lex}} \alpha^1. \end{cases}$$

Let $P = \sum_{(\alpha,\beta)} a_{\alpha\beta} x^\alpha \partial^\beta$ be a non-zero element of $\mathscr{D}$ where $a_{\alpha\beta} \in \mathbb{C}$. We denote by $\mathscr{N}(P)$ the Newton diagram of P, namely

$$\mathscr{N}(P) = \{(\alpha, \beta) \in \mathbb{N}^{2n} \mid a_{\alpha\beta} \neq 0\}.$$

Following the notation of F. Castro in [1], given a non-zero element $P \in \mathscr{D}$, we define the leading exponent of P (with respect to $\prec_J$) by $\exp_{\prec_J}(P) = \max_{\prec_J} \mathscr{N}(P)$. Let I be a left ideal of $\mathscr{D}$, we denote by $\mathrm{Exp}_{\prec_J}(I)$ the set of leading exponents $\exp_{\prec_J}(P)$ for P in I. A finite subset $\{P_1, \dots, P_r\} \subseteq I$ is said to be a Gröbner basis of I if $\bigcup_{i=1}^r (\exp_{\prec_J}(P_i) + \mathbb{N}^{2n}) = \mathrm{Exp}_{\prec_J}(I)$.

We will consider the projection map

$$\pi : \mathbb{N}^n \times \mathbb{N}^n \to \mathbb{N}^n$$

$$(\alpha, \beta) \to \beta.$$

Remark 24. Note that $\pi(\exp_{\prec_J}(P)) = \exp_{\prec}(P)$ for $P \in \mathscr{D}$.

We recall (see Definition 14) that $\{P_1, \dots, P_r\}$ is a Janet basis of I if the system $S = \{E_1, \dots, E_r\}$, where $E_i \equiv P_i(u) = 0$, is a completely integrable system.

Theorem 25. *Let I be a non-zero left ideal of $\mathscr{D}$ and let $\{P_1, \dots, P_r\}$ be a Janet basis of I, then $\{P_1, \dots, P_r\}$ is a Gröbner basis of I (with respect to $\prec_J$).*

Proof. By definition, it suffices to prove that

$$\mathrm{Exp}_{\prec_J}(I) = \bigcup_{i=1}^{r} (\exp_{\prec_J}(P_i) + \mathbb{N}^{2n}).$$

Clearly, $\bigcup_{i=1}^r (\exp_{\prec_J}(P_i) + \mathbb{N}^{2n}) \subseteq \mathrm{Exp}_{\prec_J}(I)$ since I is left ideal.

On the other hand let $(\alpha, \beta) \in \mathrm{Exp}_{\prec_J}(I)$ then there exists $P \in I$ such that $\exp_{\prec_J}(P) = (\alpha, \beta)$. By Remark 24, we have $(\alpha, \beta) = (\alpha, \exp_{\prec}(P))$. Since $\{P_1, \dots, P_r\}$ is a Janet basis of I (with respect to $\prec$), we have $\beta = \alpha_i + \gamma$ where $\alpha_i = \exp_{\prec}(P_i)$ and $\gamma \in \mathbb{N}^n$. So

$$(\alpha, \beta) = (\alpha, \alpha_i + \gamma) = (\alpha, \gamma) + (0, \alpha_i).$$

As $\{P_1, \dots, P_r\}$ is in canonical form (see Definition 1), we have $\exp_{\prec_J}(P_i) = (0, \alpha_i)$ and so $(\alpha, \beta) \in \bigcup_{i=1}^r (\exp_{\prec_J}(P_i) + \mathbb{N}^{2n})$. □

Let $\{P_1, \dots, P_r\}$ be a system of operators of $\mathscr{D}$ and let $(\Delta_1, \dots, \Delta_r, \bar{\Delta})$ (resp. $(\nabla_1, \dots, \nabla_r, \bar{\nabla})$) be the partition of $\mathbb{N}^n$ (resp. $\mathbb{N}^{2n}$) associated with $(\exp_{\prec}(P_1), \dots, \exp_{\prec}(P_r))$ (resp. $(\exp_{\prec_J}(P_1), \dots, \exp_{\prec_J}(P_r))$) where

- $\Delta_1 = \exp_{\prec}(P_1) + \mathbb{N}^n$ (resp. $\nabla_1 = \exp_{\prec_J}(P_1) + \mathbb{N}^{2n}$).
- $\Delta_i = (\exp_{\prec}(P_i) + \mathbb{N}^n) \setminus \bigcup_{j=1}^{i-1} \Delta_j$ (resp. $\nabla_i = (\exp_{\prec_J}(P_i) + \mathbb{N}^{2n}) \setminus \bigcup_{j=1}^{i-1} \nabla_j$) for $i = 2, \dots, r$.
- $\bar{\Delta} = \mathbb{N}^n \setminus \bigcup_{i=1}^r \Delta_i$ (resp. $\bar{\nabla} = \mathbb{N}^{2n} \setminus \bigcup_{i=1}^r \nabla_i$).

Lemma 26. *With the notations as above,*

$$\nabla_i = \mathbb{N}^n \times \Delta_i \quad \text{and} \quad \bar{\nabla} = \mathbb{N}^n \times \bar{\Delta}.$$

Proof. Since $\exp_{\prec_J}(P_i) = (0, \exp_{\prec}(P_i))$, we have $\nabla_i = \mathbb{N}^n \times \Delta_i$ for all $i = 1, \ldots, r$. □

Theorem 27. *Let the notations be as above. Let $\{P_1, \ldots, P_r\}$ be a Janet basis of I and we consider $M = \mathscr{D}/I$. Then*

$$dim_{\mathbb{C}((x))}(M \otimes_{\mathscr{O}} \mathbb{C}((x))) = \sharp\left(\mathbb{N}^n \setminus \left(\bigcup_{i=1}^{r} \Delta_i\right)\right).$$

Proof. We will prove that $\{\partial^\alpha + I\}_{\alpha \in \bar{\Delta}}$ is a basis of M as an $\mathscr{O}$-module.

Let $P \in \mathscr{D}$. We have $P = \sum_\beta a_\beta(x)\partial^\beta$ where $a_\beta(x) \in \mathbb{C}\{x\}$, that is $a_\beta(x) = \sum_\alpha a_{\alpha\beta}x^\alpha$ where $a_{\alpha\beta} \in \mathbb{C}$.

By the division theorem in $\mathscr{D}$ (see [1]), there exists an unique element $(Q_1, \ldots, Q_r, R) \in \mathscr{D}^{r+1}$ such that

$$P = \sum_{i=1}^{r} Q_i P_i + R,$$

where either $R = 0$ or $\mathscr{N}(R) \subseteq \bar{\nabla}$. If we denote $\mathscr{N}_\partial(R) = \pi(\mathscr{N}(R))$ then, either $R = 0$ or

$$\mathscr{N}_\partial(R) \subseteq \pi(\bar{\nabla}) = \bar{\Delta},$$

that is, $R = \sum_{\beta \in \bar{\Delta}} a_\beta(x)\partial^\beta$ with $a_\beta(x) \in \mathbb{C}\{x\}$. On other hand, as $P - R = \sum_{i=1}^{r} Q_i P_i \in I$, then $\forall (P + I) \in M$ we have $P + I \equiv \sum_{\beta \in \bar{\Delta}} a_\beta(x)\partial^\beta + I$.

Suppose $\sum_{\beta \in \bar{\Delta}} a_\beta(x)\partial^\beta \equiv 0 \pmod{I}$ with $a_\beta(x) \in \mathbb{C}\{x\}$ and $a_\beta(x) \neq 0$ for some β, then $0 \neq \sum_{\beta \in \bar{\Delta}} a_\beta(x)\partial^\beta \in I$ so, (see Theorem 25)

$$\exp_{\prec_J}\left(\sum_{\beta \in \bar{\Delta}} a_\beta(x)\partial^\beta\right) \in \operatorname{Exp}_{\prec_J}(I) = \bigcup_{i=1}^{r}(\nabla_i + \mathbb{N}^{2n}).$$

By Remark 24 and Lemma 26, we have

$$\exp_{\prec}\left(\sum_{\beta \in \bar{\Delta}} a_\beta(x)\partial^\beta\right) = \pi\left(\exp_{\prec_J}\left(\sum_{\beta \in \bar{\Delta}} a_\beta(x)\partial^\beta\right)\right) \in \pi(\operatorname{Exp}_{\prec_J}(I))$$

$$= \bigcup_{i=1}^{r}\pi(\nabla_i + \mathbb{N}^{2n}) = \bigcup_{i=1}^{r}(\Delta_i + \mathbb{N}^n).$$

So

$$\exp_{\prec}\left(\sum_{\beta \in \bar{\Delta}} a_\beta(x)\partial^\beta\right) \in \bigcup_{i=1}^{r}(\Delta_i + \mathbb{N}^n)$$

which is a contradiction. □

6. The groups $Ext^m_{\mathscr{D}}(M_0, \mathscr{O})$

In this section we calculate the *Ext* groups of a completely integrable system and we prove that they are null except for the first one. In what follows $\mathscr{D}$ will denote the rings of linear differential operators with coefficients in $\mathbb{C}\{x_1,\dots,x_n\}$.

6.1. $Ext^0_{\mathscr{D}}(M_0, \mathscr{O})$

Let S_0 be the following completely integrable system in the unknown u

$$S_0 \equiv \begin{cases} P_1(u) = 0 \\ P_2(u) = 0 \\ \vdots \\ P_{r_0}(u) = 0, \end{cases}$$

where $P_i = \partial^{\alpha(i)} - \hat{P}_i$ for $i = 1,\dots,r_0$ and $\exp(\hat{P}_i) \prec \exp(P_i)$, where $\prec$ is the order "to be lower than".

We consider the following exact sequence with $M_0 = \mathscr{D}/\mathscr{D}I$, where $\mathscr{D}I = \mathscr{D}(P_1,\dots,P_{r_0})$

$$\mathscr{D}^{r_0} \xrightarrow{\phi_0} \mathscr{D} \to M_0 \to 0 \quad \text{with } \phi_0(Q_1,\dots,Q_{r_0}) = \sum_{i=1}^{r_0} Q_i P_i.$$

We denote by r_1 the number of relations[2] that the system S_0 must verify to be completely integrable (see Theorem 17). These relations $\{\mathbf{R}_1,\dots,\mathbf{R}_{r_1}\}$ constitute a generating system for $\ker(\phi_0)$ (see Proposition 20). We consider the exact sequence

$$\mathscr{D}^{r_1} \xrightarrow{\phi_1} \mathscr{D}^{r_0} \xrightarrow{\phi_0} \mathscr{D} \to M_0 \to 0, \tag{5}$$

where $\phi_1(Q_1,\dots,Q_{r_1}) = \sum_{i=1}^{r_1} Q_i \mathbf{R}_i$.

We apply $Hom_{\mathscr{D}}(-, \mathscr{O})$ to (5),

$$0 \to Hom_{\mathscr{D}}(\mathscr{D}, \mathscr{O}) \xrightarrow{\phi_0^*} Hom_{\mathscr{D}}(\mathscr{D}^{r_0}, \mathscr{O}) \xrightarrow{\phi_1^*} Hom_{\mathscr{D}}(\mathscr{D}^{r_1}, \mathscr{O})$$

and we get

$$0 \to \mathscr{O} \xrightarrow{\phi_0^*} \mathscr{O}^{r_0} \xrightarrow{\phi_1^*} \mathscr{O}^{r_1},$$

where

$$\phi_0^*(u) = \begin{pmatrix} P_1 \\ \vdots \\ P_{r_0} \end{pmatrix}(u), \quad u \in \mathscr{O}$$

[2] We recall that this number depends only on the number of the non-multiplicative variables of each equation E_i in the system $\{P_1,\dots,P_{r_0}\}$.

and

$$\phi_1^*(f_1,\dots,f_{r_0}) = \begin{pmatrix} \mathbf{R}_1 \\ \vdots \\ \mathbf{R}_{r_1} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_{r_0} \end{pmatrix} \quad (f_1,\dots,f_{r_0}) \in \mathcal{O}^{r_0}.$$

We have $Ext^0_{\mathscr{D}}(M_0,\mathcal{O}) = \ker(\phi_0^*)$. Therefore $Ext^0_{\mathscr{D}}(M_0,\mathcal{O})$ is the set of convergent power serie solutions of the system S_0.

Remark 28. Completely integrable systems have an unique analytic solution, depending on certain initial conditions. These initial conditions are obtained from the first members of the system. In [10], we give an algorithm that, under certain initial conditions, allows us to calculate the serie solution of a given completely integrable system. So for a completely integrable system $S = \{E_1,\dots,E_r\}$ where $E_i \equiv P_i(u) = 0$ and $u \in \mathcal{N}$, there exists a function $\psi \in \mathbb{C}\{x_1,\dots,x_n\}$ such that $E_i(\psi) = 0$ for all $1 \leqslant i \leqslant r$.

6.2. $Ext^m_{\mathscr{D}}(M_0,\mathcal{O})$, $m \geqslant 1$

In this subsection we will prove that $Ext^m_{\mathscr{D}}(M_0,\mathcal{O}) = 0$ where $m \geqslant 1$. We recall that

$$Ext^1_{\mathscr{D}}(M_0,\mathcal{O}) = \frac{\ker(\phi_1^*)}{\mathrm{Im}(\phi_0^*)}.$$

Let $(f_1,\dots,f_{r_0}) \in \ker(\phi_1^*)$. Then $\phi_1^*(f_1,\dots,f_{r_0}) = 0 \in \mathcal{O}^{r_1}$ and

$$\begin{pmatrix} \mathbf{R}_1 \\ \vdots \\ \mathbf{R}_{r_1} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_{r_0} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix};$$

so, $(f_1,\dots,f_{r_0})$ verifies the same relations as $(P_1,\dots,P_{r_0})$ and consequently (see Section 3.1), the system

$$\begin{aligned} &P_1(u) = f_1 \\ &\vdots \\ &P_{r_0}(u) = f_{r_0}, \end{aligned}$$

is completely integrable. Hence there exists $\psi \in \mathcal{O}$ such that $P_i(\psi) = f_i$ for $i = 1,\dots,r_0$, that is $\phi_0^*(\psi) = (f_1,\dots,f_{r_0})$. Therefore $(f_1,\dots,f_{r_0}) \in \mathrm{Im}(\phi_0^*)$ and $\ker(\phi_1^*)/\mathrm{Im}(\phi_0^*) = 0$.

The identity $Ext^1_{\mathscr{D}}(M_0,\mathcal{O}) = 0$ is also true for modules M_0 associated with completely integrable systems with several unknowns.

Let us consider the set of relations $\{\mathbf{R}_1,\dots,\mathbf{R}_{r_1}\}$ as a system of homogeneous linear partial differential equations in the unknowns E_i. We associate with this system, the $\mathscr{D}$-module M_1 defined by the exact sequence

$$\mathscr{D}^{r_1} \xrightarrow{\phi_1} \mathscr{D}^{r_0} \to M_1 \to 0 \quad \text{with } \phi_1(Q_1,\dots,Q_{r_1}) = \sum_{i=1}^{r_1} Q_i \mathbf{R}_i.$$

This system is completely integrable (see Theorem 21). We denote by r_2 the number of relations that we need to verify that the system is completely integrable. These relations $\{\mathbf{S}_1,\dots,\mathbf{S}_{r_2}\}$ (elementary syzygies) constitute a generating system for $\ker(\phi_1)$ (see Proposition 20).

So we can consider the exact sequence

$$\mathscr{D}^{r_2} \xrightarrow{\phi_2} \mathscr{D}^{r_1} \xrightarrow{\phi_1} \mathscr{D}^{s_0} \rightarrow M_1 \rightarrow 0,$$

where $\phi_2(Q_1,\dots,Q_{r_2}) = \sum_{i=1}^{r_2} Q_i\mathbf{S}_i$.

In this way we can build the following exact sequence, which is finite (see Theorem 22):

$$\cdots \rightarrow \mathscr{D}^{r_2} \xrightarrow{\phi_2} \mathscr{D}^{r_1} \xrightarrow{\phi_1} \mathscr{D}^{r_0} \xrightarrow{\phi_0} \mathscr{D} \rightarrow M_0 \rightarrow 0. \tag{6}$$

In order to calculate $Ext^2_{\mathscr{D}}(M_0,\mathscr{O})$, we apply $Hom_{\mathscr{D}}(-,\mathscr{O})$ to the reduced resolution of the $\mathscr{D}$-module M_0, which has been calculated in (6) and we obtain

$$0 \rightarrow \mathscr{O} \xrightarrow{\phi_0^*} \mathscr{D}^{r_0} \xrightarrow{\phi_1^*} \mathscr{O}^{r_1} \xrightarrow{\phi_2^*} \mathscr{O}^{r_2} \rightarrow \cdots,$$

where

$$\phi_0^*(u) = \begin{pmatrix} P_1 \\ \vdots \\ P_{s_0} \end{pmatrix}(u) \quad \text{for } u \in \mathscr{O}$$

and

$$\phi_i^*(f_1,\dots,f_{r_{i-1}}) = \begin{pmatrix} \mathbf{R}_1 \\ \vdots \\ \mathbf{R}_{r_i} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_{r_{i-1}} \end{pmatrix}, \quad (f_1,\dots,f_{r_{i-1}}) \in \mathscr{O}^{r_{i-1}} \text{ and } i \geqslant 1.$$

We have

$$Ext^1_{\mathscr{D}}(M_1,\mathscr{O}) = Ext^2_{\mathscr{D}}(M_0,\mathscr{O})$$

and so $Ext^2_{\mathscr{D}}(M_0,\mathscr{O}) = 0$.

Proposition 29. *With the previous notations*

$$Ext^m_{\mathscr{D}}(M_0,\mathscr{O}) = Ext^1_{\mathscr{D}}(M_{m-1},\mathscr{O}), \quad m \geqslant 2.$$

Proof. By applying the above process we have

$$Ext^{m-i}_{\mathscr{D}}(M_i,\mathscr{O}) = Ext^{m-(i+1)}_{\mathscr{D}}(M_{i+1},\mathscr{O}), \quad 0 \leqslant i \leqslant m-2,$$

which completes the proof. □

As an straightforward consequence we obtain

Corollary 30. *With the previous notation*

$$Ext^m_{\mathscr{D}}(M_0, \mathcal{O}) = 0, \quad \forall m \geqslant 1.$$

We have proved that if S is a completely integrable system, then $Ext^i_{\mathscr{D}}(M, \mathcal{O}) = 0$ for $i \geqslant 1$, where M is the finitely generated $\mathscr{D}$-module associated with the system S. The converse is not true as we can see in the example:

$$S \equiv x\partial_x u = -u.$$

This system is not completely integrable since S is not in canonical form, but $Ext^i_{\mathscr{D}}(M, \mathcal{O}) = 0$ with $i \geqslant 1$.

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