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Exceptional sets and Hilbert–Schmidt composition operators

Eva A. Gallardo-Gutiérrez*[,1] and María J. González[2]

Departamento de Matemáticas, Universidad de Cádiz, Apartado 40, 11510 Puerto Real (Cádiz), Spain

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Abstract

It is shown that an analytic map φ of the unit disk into itself inducing a Hilbert–Schmidt composition operator on the Dirichlet space has the property that the set $E_\varphi = \{e^{i\theta} \in \partial\mathbb{D} : |\varphi(e^{i\theta})| = 1\}$ has zero logarithmic capacity. We also show that this is no longer true for compact composition operators on the Dirichlet space. Moreover, such a condition is not even satisfied by Hilbert–Schmidt composition operators on the Hardy space.
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1. Introduction

Let $\mathbb{D}$ denote the open unit disk of the complex plane. The Dirichlet space $\mathscr{D}$ is the space of analytic functions f on $\mathbb{D}$ such that the norm

$$||f||_{\mathscr{D}}^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \, dA(z)$$

*Corresponding author.

E-mail addresses: eva.gallardo@uca.es (E.A. Gallardo-Gutiérrez), majose.gonzalez@uca.es (M.J. González).

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is finite. Here A stands for the normalized Lebesgue area measure of the unit disk. Observe that for a univalent function f, the integral above is just the area of $f(\mathbb{D})$.

If φ is an analytic function on $\mathbb{D}$ with $\varphi(\mathbb{D}) \subset \mathbb{D}$, then the equation

$$C_\varphi f = f \circ \varphi$$

defines a composition operator C_φ on the space of all holomorphic functions on the unit disk $\mathscr{H}(\mathbb{D})$. Furthermore, C_φ acts boundedly on $\mathscr{H}(\mathbb{D})$ endowed with the topology of uniform convergence on compact subsets. On the Dirichlet space $\mathscr{D}$, a necessary condition for C_φ to be bounded is that φ belongs to $\mathscr{D}$. This follows easily from the fact that $C_\varphi z = \varphi$. On the other hand, not all the Dirichlet functions induce bounded composition operators on $\mathscr{D}$. Such functions were characterized by Voas [11] in his thesis. This and other related problems on composition operators on the Dirichlet space have been extensively studied. For a comprehensive treatment of such problems on spaces of analytic functions, like the Dirichlet space, see Cowen and MacCluer's book [3].

In this paper we are interested in the relationship between Hilbert–Schmidt composition operators and the boundary behavior of their inducing symbols. In particular, if $\partial\mathbb{D}$ denotes the unit circle, we focus on the size of the set

$$E_\varphi = \{e^{i\theta} \in \partial\mathbb{D} : |\varphi(e^{i\theta})| = 1\},$$

whenever φ induces a Hilbert–Schmidt composition operator on the Dirichlet space. To this end, recall that if E is a Borel set contained in the unit circle $\partial\mathbb{D}$ and Λ_E denotes the class of distributions of mass 1 on E, i.e., non-negative set functions μ with total mass 1 and support S_μ contained in E, the logarithmic capacity[3] of E is defined by

$$e^{-\inf_{\Lambda_E}\{I(\mu)\}},$$

where $I(\mu)$ denotes the logarithmic energy integral of μ, that is

$$I(\mu) = \int\int \log\frac{1}{|\xi - \eta|}\, d\,\mu(\xi)\, d\,\mu(\eta).$$

Beurling [1] proved that if φ is a Dirichlet function, then the radial limits

$$\varphi(e^{i\theta}) = \lim_{r\to 1^-} \varphi(re^{i\theta})$$

exist except on a set of logarithmic capacity zero (see also [2, p. 55]). So, it makes sense to ask about the logarithmic capacity of the set E_φ when C_φ is Hilbert–Schmidt on $\mathscr{D}$. It will be shown in Section 2 that such a set has logarithmic capacity zero.

[3] Some authors define the logarithmic capacity of E by $(\inf_{\Lambda_E}\{I(\mu)\})^{-1}$. In our case, both definitions are consistent because we deal with sets of logarithmic capacity zero. For more about capacities see [2] and [5].

On the other hand, since Hilbert–Schmidt operators are compact operators, the following question arises:

If C_φ is a compact operator on $\mathcal{D}$, is E_φ a set of logarithmic capacity zero?

We will show that this question has a negative answer in Section 3. This will be accomplished by constructing a simply connected domain Ω contained in $\mathbb{D}$ such that the Riemann map ψ that takes $\mathbb{D}$ onto Ω induces a compact composition operator C_ψ on $\mathcal{D}$ but E_ψ has positive logarithmic capacity.

The first results of this type were obtained by Schwartz [10] in 1969. He proved that if C_φ is compact on the Hardy space $\mathcal{H}^2$, the space of analytic functions on $\mathbb{D}$ whose boundary values are in $L^2(\partial\mathbb{D})$, then

$$|\varphi(e^{i\theta})| < 1$$

almost everywhere on $\partial\mathbb{D}$, or equivalently the set E_φ has Lebesgue measure zero. In this case, Fatou's radial limit theorem ensures that $\varphi(e^{i\theta})$ is defined except on a set of Lebesgue measure zero (see for instance [4]).

Finally, using the construction in Section 3, we provide not only a compact but also a Hilbert–Schmidt composition operator C_φ on the Hardy space such that E_φ is a set of positive logarithmic capacity.

2. Hilbert–Schmidt composition operators on $\mathcal{D}$

Recall that a linear operator T on a Hilbert space $\mathcal{H}$ is said to be Hilbert–Schmidt if the series

$$\sum_{n=1}^{\infty} ||Te_n||^2 \tag{1}$$

converges for an orthonormal basis $\{e_n\}$ of $\mathcal{H}$. If this is the case, condition (1) holds for all orthonormal bases of $\mathcal{H}$. Furthermore, it is not difficult to see that every Hilbert–Schmidt operator is a bounded operator.

In the Dirichlet space, Hilbert–Schmidt composition operators are characterized in terms of their inducing symbols by the following lemma.

Lemma 2.1. *C_φ is Hilbert–Schmidt on $\mathcal{D}$ if and only if the integral*

$$\int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2}\, dA(z)$$

is finite.

A standard argument using the fact that $\{z^n/(n+1)^{1/2}\}_{n\geqslant 0}$ is an orthonormal basis on the Dirichlet space and Stirling's formula yields the statement of the lemma (see [3, Chapter 3]).

Now, suppose that φ is a Dirichlet function. As mentioned in the introduction the radial limits

$$\lim_{r\to 1^-} \varphi(re^{i\theta})$$

exist except on a set of logarithmic capacity zero (see [1], and also [2, p. 55]). Thus, the set

$$E_\varphi = \{e^{i\theta} \in \partial\mathbb{D} : |\varphi(e^{i\theta})| = 1\}$$

is well defined. In addition, if φ induces a Hilbert–Schmidt composition operator on $\mathscr{D}$, we have the following conclusion.

Theorem 2.1. *Let φ be a holomorphic self-map of the unit disk $\mathbb{D}$. Assume that C_φ defines a Hilbert–Schmidt operator on the Dirichlet space $\mathscr{D}$. Then E_φ has zero logarithmic capacity.*

Proof. The key point of the proof is to translate the Hilbert–Schmidt condition on the operator into one involving positive harmonic functions. Let us consider the linear fractional transformation

$$\tau(z) = \frac{1+z}{1-z}$$

that maps the unit disk onto the right half plane. Let u denote the real part of $C_\varphi\tau = \tau\circ\varphi$, that is,

$$u(z) = \mathrm{Re}\,\frac{1+\varphi(z)}{1-\varphi(z)}.$$

Clearly, u is a positive harmonic function and therefore the Poisson integral of a positive measure on $\partial\mathbb{D}$. If we write $z = x+iy$ and $\nabla u = (\partial u/\partial x, \partial u/\partial y)$, a simple calculation shows that

$$\frac{2|\varphi'(z)|}{1-|\varphi(z)|^2} = \frac{|\nabla u(z)|}{u(z)}.$$

Upon applying Lemma 2.1 we conclude that the integral

$$\int_{\mathbb{D}} \frac{|\nabla u(z)|^2}{u(z)^2}\, dA(z)$$

is finite, or equivalently,

$$\int_{\mathbb{D}} |\nabla \log u(z)|^2\, dA(z) < \infty. \tag{2}$$

This inequality will be the key to constructing a Dirichlet function to which we will apply Beurling's Theorem. Since $\log u(z)$ is not harmonic on $\mathbb{D}$, we consider instead the Poisson extension of the boundary function $\log u(e^{i\theta})$.

Let $P_r(\theta)$ denote the Poisson kernel, that is

$$P_r(\theta) = \mathrm{Re}\left[\frac{1+re^{i\theta}}{1-re^{i\theta}}\right]$$

for $r \in [0,1)$ and $\theta \in [0,2\pi]$, and v the harmonic function

$$v(z) = v(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log u(t) P_r(\theta - t)\, dt.$$

In other words, v is the convolution $\log u * P_r$, and therefore the harmonic extension of $\log u(e^{i\theta})$ to the unit disk $\mathbb{D}$. On the other hand, since harmonic functions minimize the energy integral, it follows that

$$\int_{\mathbb{D}} |\nabla v(z)|^2\, dA(z) \leqslant \int_{\mathbb{D}} |\nabla \log u(z)|^2\, dA(z), \tag{3}$$

and from (2) we get that the integral on the left-hand side above is finite.

Let f be the analytic function on the unit disk such that $f(0) = 0$ and $\mathrm{Re} f = v$. From what we have just shown, it follows that f belongs to the Dirichlet space $\mathscr{D}$. Thus, the radial limits $f(e^{i\theta}) = \lim_{r\to 1^-} f(re^{i\theta})$ exist except on a set of logarithmic capacity zero, and hence the function $\mathrm{Re} f(e^{i\theta}) = \log u(e^{i\theta})$ is finite except on a set of logarithmic capacity zero. Since the sets

$$\{e^{i\theta} \in \partial\mathbb{D} : \ \log u(e^{i\theta}) < \infty\},$$

$$\{e^{i\theta} \in \partial\mathbb{D} : u(e^{i\theta}) > 0\}$$

and

$$\{e^{i\theta} \in \partial\mathbb{D} : |\varphi(e^{i\theta})| < 1\}$$

coincide, the logarithmic capacity of E_φ is zero and the theorem is proved. □

Remark 2.1. The function $\varphi(z) = (1+z)/2$ induces a non-Hilbert–Schmidt composition operator on the Dirichlet space and the set E_φ only has one element. Therefore Theorem 2.1 is just only a necessary condition. This example was first provided by Schwartz [10] to show that his result, which states that if C_φ is compact on the Hardy space, then E_φ is a set of Lebesgue measure zero, is not a sufficient condition either.

3. Compact composition operators on $\mathscr{D}$

The characterization of compact composition operators on the Dirichlet space can be found in [3] in terms of Carleson measures (see also [7] and [12]). In fact, if $n_\varphi(w)$ denotes the multiplicity of φ at w and $S(\xi,\delta)=\{z\in\mathbb{D} : |z-\xi|<\delta\}$ is the Carleson disk centered in $\xi\in\partial\mathbb{D}$ of radius δ, with $0<\delta<1$, then C_φ is compact on $\mathscr{D}$ if and only if

$$\lim_{\delta\to 0}\frac{1}{\delta^2}\int_{S(\xi,\delta)} n_\varphi(w)\,dA(w)=0. \tag{4}$$

In this section we are interested in a different aspect of the subject: the size of E_φ. In particular, we ask if the conclusion of Theorem 2.1 holds when C_φ satisfies the weaker condition of being compact on the Dirichlet space. The answer is negative.

Theorem 3.1. *There exists a compact composition operator C_φ on the Dirichlet space such that the set*

$$E_\varphi=\{e^{i\theta}\in\partial\mathbb{D} : |\varphi(e^{i\theta})|=1\}$$

has positive logarithmic capacity.

Proof. We will construct a simply connected domain Ω contained in the unit disk $\mathbb{D}$. The Riemann map φ that maps $\mathbb{D}$ onto Ω will furnish an example of the required behavior.

Fix p in $(0,1)$ and consider the sequence $\{a_n\}_{n\geqslant 0}=\{\frac{1}{4}p^{n^2}\}_{n\geqslant 0}$. First, we consider the Cantor set $E=\bigcap_{n\geqslant 0}E_n\subset[0,\pi]$, where its n-th approximation E_n consist of 2^n open intervals I_n of length a_n. Let $J_n^{k)}$ denote the intervals in the complement of E_n, that is,

$$E_{n-1}\backslash E_n=\bigcup_{k=1}^{2^{n-1}} J_n^{k)}.$$

Observe that each of intervals $J_n^{k)}$ are closed.

Let $R_n^{k)}$ be the closed rectangle (Fig. 1) supported on the interval $J_n^{k)}$ of height a_{n-2}.

Set $\mathscr{R}=\bigcup_n\bigcup_k R_n^{k)}$ and let Δ be the set

$$\Delta=\{z\in\mathbb{D} : |z|<1/2\}\cup\{z=re^{i\theta}\in\mathbb{D} : \theta\in(0,\pi)\}.$$

Then the simply connected domain Ω we are looking for is $\Delta\backslash\mathscr{R}$ (see Fig. 2).

Let φ be a Riemann map that takes $\mathbb{D}$ onto Ω and $\varphi(0)=0$. Since $\varphi(\mathbb{D})$ has finite area, it follows that φ belongs to the Dirichlet space $\mathscr{D}$. In addition, C_φ is also bounded on $\mathscr{D}$ since φ is a univalent self-map of the disk.

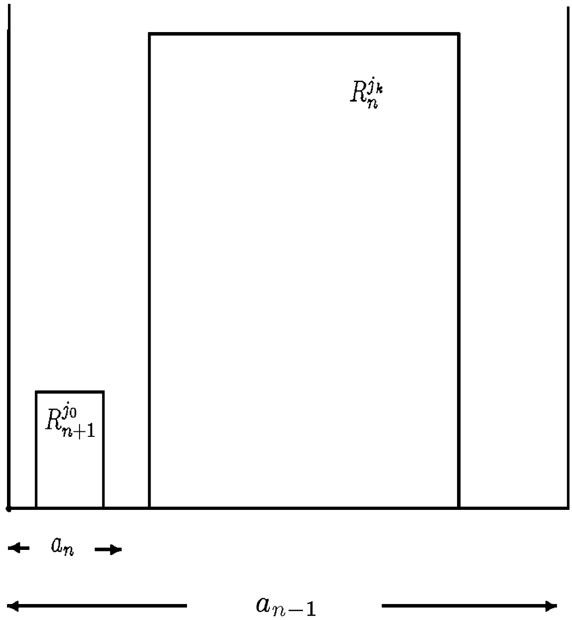

Fig. 1. Rectangles R_n^j.

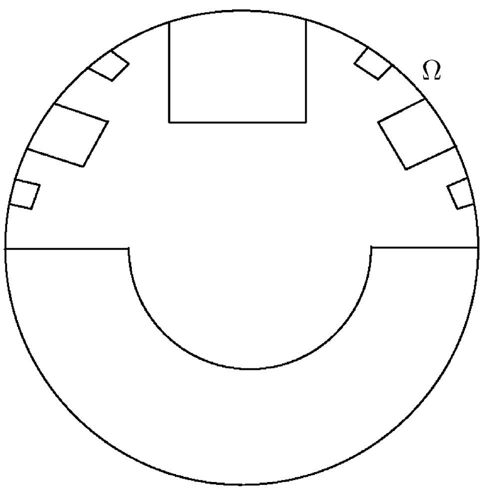

Fig. 2. Domain Ω.

First, we prove that φ induces a compact composition operator C_φ on $\mathscr{D}$. For that purpose, we show that φ satisfies the condition in (4).

Let $\xi = e^{i\theta}$ be in $\partial\mathbb{D}$. We may suppose that $\theta \in (0, \pi)$, otherwise condition (4) is trivially verified. Let $S(\xi, \delta)$ be the Carleson disk of center ξ and radius $0 < \delta < 1/4$.

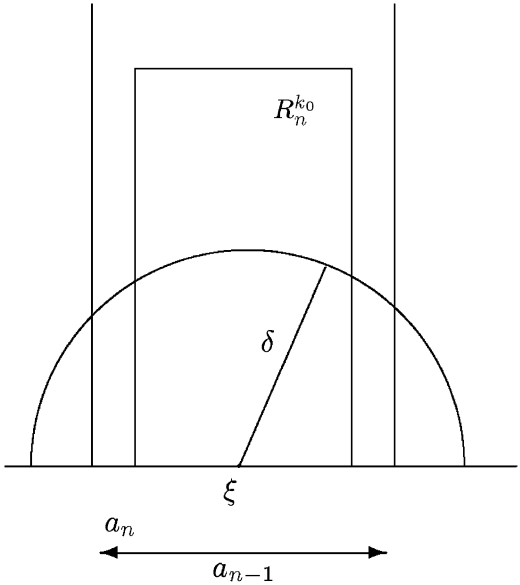

Fig. 3.

There exists n such that $a_{n-1} \leqslant \delta < a_{n-2}$. By construction, there exists k_0 such that $S(\xi, \delta)$ meets the rectangle $R_n^{k_0)}$. Therefore, the area of $S(\xi, \delta) \cap \varphi(\mathbb{D})$ is less than $2a_n\delta$ (see Fig. 3).

Since φ is univalent, it follows that

$$\frac{1}{\delta^2} \int_{S(\xi,\delta)} n_\varphi(w)\, dA(w) < \frac{2a_n\delta}{\delta^2} \leqslant \frac{2a_n}{a_{n-1}}.$$

Now, the limit $\lim_{n\to\infty} a_{n+1}/a_n = 0$ because $a_n = \frac{1}{4}p^{n^2}$ and $0 < p < 1$.

Finally, it only remains to estimate the logarithmic capacity of the set $E_\varphi = \{e^{i\theta} \in \partial\mathbb{D} : |\varphi(e^{i\theta})| = 1\}$. Observe that $E_\varphi = \varphi^{-1}(E)$, where E is the Cantor set constructed at the beginning of the proof. Moreover, E has positive logarithmic capacity because the series

$$\sum_{n=1}^{\infty} 2^{-n} \log \frac{1}{a_n}$$

is convergent (see [2, p. 29]). The problem here is that logarithmic capacity is not invariant under conformal mappings, that is, sets of logarithmic capacity zero can be carried onto positive logarithmic capacity sets conformally. So, we need to relate the logarithmic capacity of a set to a conformally invariant quantity: extremal length.

Recall that the module of a family of curves Γ contained in a domain $\mathscr{R}$ is defined by

$$M(\Gamma, \mathscr{R}) = \inf_{\rho} \int \int_{\mathscr{R}} \rho^2(z)\, dA(z),$$

where the infimum is taking over the positive functions ρ such that

$$\int_{\gamma} \rho(z)|dz| \geqslant 1$$

for every curve γ in the family Γ. Such functions ρ are called admissible for the family Γ. The extremal length of the family Γ in $\mathscr{R}$ is defined by

$$\lambda(\Gamma) = \frac{1}{M(\Gamma, \mathscr{R})}.$$

On the other hand, if F is a set contained in $\partial\mathbb{D}$, $0 < r \leqslant 1/3$ is a given number and $\Gamma(r)$ is the family of curves in the annulus $\{z \in \mathbb{D} : r < |z| < 1\}$ which connect the circle $|z| = r$ to F, then the logarithmic capacity of F and the extremal length of the family $\Gamma(r)$ are related by Pfluger's Theorem in the following way:

$$\frac{\sqrt{r}}{1+r}\operatorname{cap}(F) \leqslant e^{-\pi\lambda(\Gamma(r))} \leqslant \frac{\sqrt{r}}{1-r}\operatorname{cap}(F),$$

where $\operatorname{cap}(F)$ denotes the logarithmic capacity of F. For this and many other results concerning extremal length, we refer to [5] and [8].

Therefore, we just need to get estimates on extremal length in the domain Ω to apply Pfluger's Theorem to the set $E_\varphi = \varphi^{-1}(E)$.

So, fix $r = 1/4$ and consider the family of curves Γ in the annulus $\{z \in \mathbb{D} : 1/4 < |z| < 1\}$ which join the circle $\{z \in \mathbb{D} : |z| = 1/4\}$ to E_φ. Let $\tilde{\Gamma}$ be the family of curves

$$\tilde{\Gamma} = \{\varphi(\gamma) : \gamma \in \Gamma\}.$$

Thus $\tilde{\Gamma}$ consist of all the curves contained in Ω that connect $\varphi(\{z : |z| = 1/4\})$ with the Cantor set E. Now, the extremal length of $\tilde{\Gamma}$ is the same as the module of the family $\mathscr{F}$ of disconnected curves in Ω that separate the sets $\varphi(\{z : |z| = 1/4\})$ and E (see [8, p. 197]). Hence,

$$\lambda(\Gamma) = \lambda(\tilde{\Gamma}) = M(\mathscr{F}, \Omega).$$

By Pfluger's Theorem, we just need to show that $M(\mathscr{F}, \Omega)$ is finite. So, for $z = re^{i\theta} \in \Omega$, set

$$\rho(z) = \begin{cases} \dfrac{1}{2^{n+2}a_{n+2}} & \text{if } a_n < 1 - r < a_{n-1}, \\ 1 & \text{if } 1/4 < 1 - r < 1/2. \end{cases}$$

Note that ρ is admissible for the family $\mathscr{F}$. In particular, we show that ρ is admissible for the curves $\gamma_r = \{z \in \Omega : |z| = r\}$ with $1/2 < r < 1$ which are, roughly speaking, the curves in $\mathscr{F}$ of minimum length. It is easy to check that the length of any other curve $\gamma \in \mathscr{F}$ increases enough so the integral $\int_\gamma \rho |dz|$ is also bigger than 1.

Let $\gamma_r = \{z \in \Omega : |z| = r\}$ and let n be such that $a_n < 1 - r < a_{n-1}$. The length $\ell(\gamma_r)$ of the curve γ_r is $\ell(\gamma_r) \simeq 2^{n+1} a_{n+1}$, that is, there exist C_1, C_2 universal constants such that

$$C_1 2^{n+1} a_{n+1} \leqslant \ell(\gamma_r) \leqslant C_2\ 2^{n+1} a_{n+1}.$$

Thus, we have

$$\int_{\gamma_r} \rho |dz| \simeq \frac{2^{n+1} a_{n+1}}{2^{n+2} a_{n+2}} > 1,$$

because $a_n = \frac{1}{4} p^{n^2}$ and $0 < p < 1$.

On the other hand,

$$\begin{aligned} \int \int_\Omega \rho^2(z)\, dA(z) &= \int_0^1 \int_{\gamma_r} \rho^2(z) |dz|\, dr \\ &= \sum_{n > 1} \int_{a_n}^{a_{n-1}} \frac{1}{(2^{n+2} a_{n+2})^2} \ell(\gamma_r)\, dr \\ &\leqslant \sum_{n > 1} \frac{2^{n+1} a_{n+1} a_{n-1}}{(2^{n+2} a_{n+2})^2} \\ &= \frac{1}{(2p^2)^3} \sum_{n > 1} \frac{1}{(2p^8)^n}. \end{aligned}$$

Now, if we choose $(1/2)^{1/8} < p < 1$, the series above is convergent. This shows that the module of $\mathscr{F}$ is finite. So, the logarithmic capacity of E_φ is positive, which completes the proof. $\square$

Remark 3.1. Compact composition operators on the Dirichlet space are also compact on the Hardy space (see [7] and [12]). Therefore, Theorem 3.1 provides a compact operator C_φ on the Hardy space such that E_φ has positive logarithmic capacity.

4. A final remark

In this section we discuss whether the conclusion of Theorem 2.1 also holds for Hilbert–Schmidt composition operators on the Hardy space $\mathscr{H}^2$. A characterization of such operators in term of their inducing symbols can be found in [9] (see also [3, p. 146]). In fact, φ induces a Hilbert–Schmidt composition operator on $\mathscr{H}^2$ if and only if

$$\int_{\partial\mathbb{D}} \frac{1}{1-|\varphi(\xi)|^2}\, d\xi < \infty. \tag{5}$$

Alternatively, because the usual norm on the Hardy space is equivalent to the norm

$$\|f\|^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)\, dA(z) \quad (f \in \mathscr{H}^2),$$

where A stands for the normalized Lebesgue measure of the unit disk, it follows that C_φ is Hilbert–Schmidt on $\mathscr{H}^2$ if and only if the integral

$$\int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^3} (1-|z|^2)\, dA(z)$$

is finite (see [3, Chapter 3] for more references and results). This condition along with the fact that a self-map φ of the unit disk satisfies that

$$\frac{1-|z|}{1-|\varphi(z)|} \leqslant \frac{1+|\varphi(0)|}{1-|\varphi(0)|} \quad (z \in \mathbb{D})$$

yields easily the following conclusion.

Proposition 4.1. *Let φ be a holomorphic self-map of the unit disk and assume that C_φ is Hilbert–Schmidt on the Dirichlet space. Then C_φ is Hilbert–Schmidt on the Hardy space.*

Although there are Hilbert–Schmidt composition operators C_φ on the Hardy space such that E_φ is a set of logarithmic capacity zero, we will prove that this is no longer true for all Hilbert–Schmidt composition operators on $\mathscr{H}^2$. Actually, we have the following

Theorem 4.1. *There exists a Hilbert–Schmidt composition operator C_φ on the Hardy space such that E_φ has positive logarithmic capacity.*

Proof. Let us consider the domain Ω constructed in Theorem 3.1. We will prove that the Riemann map φ that sends the unit disk $\mathbb{D}$ onto Ω induces a Hilbert–Schmidt composition operator on $\mathscr{H}^2$.

Let $\{a_n\}$ be the sequence used to construct the domain Ω and consider A_n the set on $\partial\mathbb{D}$ defined by

$$A_n = \{e^{i\theta} \in \partial\mathbb{D} : a_{n+1} < 1 - |\varphi(e^{i\theta})| \leqslant a_n\}.$$

First, observe that the integral in (5) is finite if the series

$$\sum_{n=1}^{\infty} \frac{1}{a_{n+1}} |A_n|$$

is convergent. Here $|A_n|$ denotes the Lebesgue measure of A_n. Now, the fact that the harmonic measure $\omega(0, A_n, \mathbb{D})$ of the set A_n in the unit disk $\mathbb{D}$ at the origin and the Lebesgue measure of A_n are related by $|A_n| = 2\pi\omega(0, A_n, \mathbb{D})$, plus the invariance of harmonic measure under conformal mappings, implies that C_φ is Hilbert–Schmidt on $\mathscr{H}^2$ if the series

$$\sum_{n=1}^{\infty} \frac{1}{a_{n+1}} \omega(\varphi(0), \varphi(A_n), \Omega) \tag{6}$$

is convergent. To estimate the harmonic measure of $\varphi(A_n)$ in Ω at $\varphi(0) = 0$, consider the simply connected domain (Fig. 4)

$$\Omega'_n = \{w \in \Omega : 1 - |w| > a_n\}$$

and the set on the boundary of Ω'_n

$$I_n = \{w \in \partial\Omega'_n : 1 - |w| = a_n\}.$$

By the maximum principle, we have

$$\omega(0, \varphi(A_n), \Omega) \leqslant \omega(0, I_n, \Omega'_n). \tag{7}$$

Suppose temporarily that the following claim is already proved.

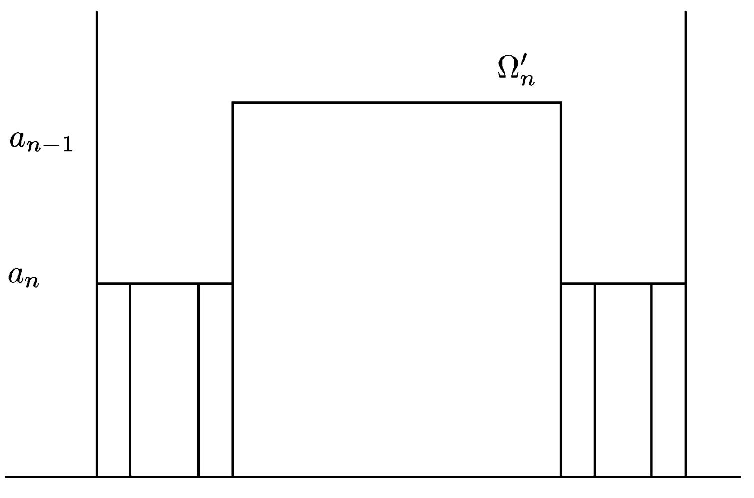

Fig. 4. Domain Ω'_n.

Claim. *Let d and h be positive numbers. Consider the rectangles $\mathcal{R}_1 = [-1 - d/2, -d/2] \times [0, h]$ and $\mathcal{R}_2 = [d/2, 1 + d/2] \times [0, h]$. Let $\mathbb{R}^2_+ = \{(x, y) : y > 0\}$ and $\tilde{\Omega} \subset \mathbb{R}^2_+$ be the domain bounded by the rectangles $\mathcal{R}_1$ and $\mathcal{R}_2$. If $I = (-d/2, d/2)$ and $z_0 = 2hi$, then*

$$\omega(z_0, I, \tilde{\Omega}) \leqslant Ce^{-\pi h/d},$$

where C is a universal constant (see Fig. 5).

Thus, it follows that

$$\omega(0, I_n, \Omega'_n) \leqslant C2^{n+1} e^{-\pi(a_{n-1} - a_n)/a_{n+1}}.$$

This along with (7) implies that the series in (6) converges if

$$\sum_{n=1}^{\infty} \frac{2^{n+1}}{a_{n+1}} e^{-\pi(a_{n-1} - a_n)/a_{n+1}} < \infty.$$

This follows easily upon applying any criterion for convergence of positive series since $a_n = \frac{1}{4} p^{n^2}$ with $(1/2)^{1/8} < p < 1$. Therefore, we conclude that C_φ is Hilbert–Schmidt on the Hardy space.

It remains to prove the claim.

Proof of Claim. We will use an extremal length argument similar to the one in [6].

Let $\mathcal{F}$ be the family of curves in $\tilde{\Omega}$ separating z_0 from I and let ρ be admissible for the family $\mathcal{F}$.

For $0 \leqslant t \leqslant h$, consider the line γ_t which joins both rectangles (see Fig. 5). Using Hölder's inequality and the fact that ρ is admissible if follows that

$$1 \leqslant \left(\int_{\gamma_t} \rho \, ds \right)^2 \leqslant \left(\int_{\gamma_t} \rho^2 \, ds \right) \ell(\gamma_t),$$

where $\ell(\gamma_t)$ denotes the length of γ_t. Therefore, we have

$$\int \int_{\tilde{\Omega}} \rho^2 \, ds \, dt \geqslant \frac{h}{d}.$$

Thus, the module $M(\mathcal{F})$ of the family $\mathcal{F}$ is bounded below by h/d. This along with Beurling's Theorem which relates the harmonic measure and the module of the family $\mathcal{F}$ by

$$\omega(z_0, I, \tilde{\Omega}) \leqslant Ce^{-\pi M(\mathcal{F})}$$

(see [5, p. 100]), yields the claim. $\square$

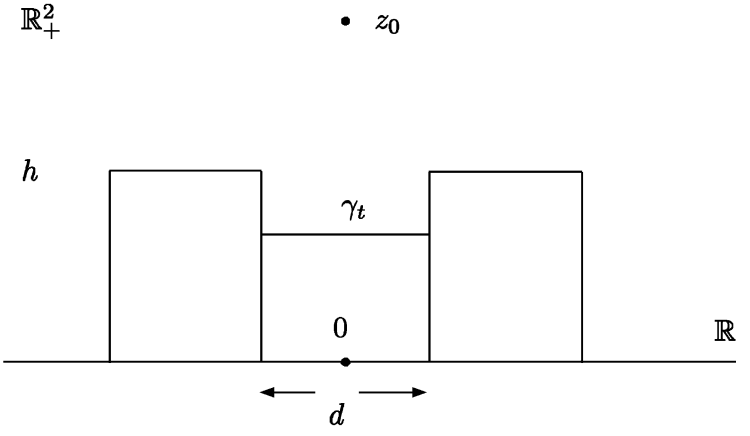

Fig. 5.

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