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AN AGING CONCEPT BASED ON MAJORIZATION

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In this article, we introduce a new dispersion order weaker than the classic dispersion order discussed by Lewis and Thompson (1981). We study the equivalence of this order with the majorization order under the assumption of unimodality. Finally, we use this equivalence to characterize the IFR aging notion for unimodal distributions by means of the notion of decreasing in randomness.

1. INTRODUCTION

The notion of stochastic variability orders have been extensively studied in the literature. Lewis and Thompson [6] introduced a concept of variability through the definition of the dispersion order (L-T sense). Let F and G be two distribution functions; we say that F is less dispersive than G, denoted $F \prec_{\text{Disp}} G$, if any pair of quantiles of G are at least more widely separated as the corresponding quantiles of F. Many useful characterizations of the dispersion order can be found in the literature. For example, Shaked [10] provided a characterization of the dispersion ordering through the existence of an expansion function and he also gave a new interpretation by the number of crossings of the distribution functions. Muñoz-Pérez [7] characterized the dispersion order by the concept of Q-addition of random variables and by the spread function under restriction on the respective quantile function. Rojo and Guo [9] showed

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that F and G are ordered in the L-T sense if and only if the sample spacings are also ordered in the L-T sense. They also showed the preservation of the ordering by monotone convex (concave) transformations and by truncation at the same quantile. Finally, Shaked and Shanthikumar [11] and, later, Pellerey and Shaked [8] showed the relationship between the variability orders and reliability theory; in particular, they provided a characterization of the increasing failure rate (IFR) aging notion by means of the dispersion order.

Although the ordering in the L-T sense is intuitively reasonable, sometimes it involves two main disadvantages when we want to compare some specific distributions. First, suppose that two distribution functions have the same finite interval as their supports; then Shaked [10] and Hickey [5] showed that, in this case, they are not related in terms of the L-T sense unless they are identical. Second, we have that the variables X and $-X$ are, in general, not equally dispersed or even ordered in the L-T sense. However, in the literature, there are other variability orders which do not have these disadvantages; for instance, the convex, peakedness, and dilation orders; see Shaked and Shanthikumar [11].

Parallel to this, there exists the notion of majorization for two probability vectors with n components. Let $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (q_1, \ldots, q_n)$ be two probability vectors. We say that $\mathbf{p}$ is smaller in the majorization sense than $\mathbf{q}$, denoted $\mathbf{p} \prec_M \mathbf{q}$ if $\sum_{k=i}^{n} p_{(i)} \leq \sum_{k=i}^{n} q_{(i)}$ for $i = 1, \ldots, n$, where $p_{(i)}$ denotes the increasing rearrangement of $\mathbf{p}$ and the same for $q_{(i)}$. Hickey [4] generalized the notion of the majorization order for the continuous distribution function; this we will see later. Roughly speaking, the concept of majorization is not properly a concept of dispersion; however, this order measures another interesting statistical property known as randomness. Nevertheless, Hickey [5] related the order in the L-T sense with the concept of majorization and he showed that $\prec_{\text{Disp}}$ implies $\prec_M$ under unimodal distributions, but they are not equivalent.

The purpose of this article is to introduce a new concept of variability order weaker than the dispersion order in the L-T sense which we will call weakly dispersive, denoted $\overset{\text{wd}}{\preccurlyeq}$. In Section 2, we study the $\overset{\text{wd}}{\preccurlyeq}$ order and we obtain some characterizations. We also study the relationship among the dispersion ordering in the L-T sense, the majorization order, and the weakly dispersive order. In particular, we will note that this new concept does not have the disadvantages that the order in the L-T sense has and we also show that the weakly dispersive order is equivalent to the majorization order under the assumption of unimodality. Hence, the weakly dispersive order seems to be a bridge between the dispersion order in the L-T sense and the majorization order. In Section 3, we propose a new concept of aging based on the concept of randomness. Using results of equivalence in Section 2, we characterize the IFR aging notion for unimodal distributions by means of the concept of randomness.

2. THE WEAKLY DISPERSIVE ORDER: MOTIVATION AND PROPERTIES

From this point forward, let X and Y be random variables which are defined on the same probability space $(\Omega, \mathcal{A}, P)$ and let μ be the Lebesgue measure on Borel's σ-algebra. We will always consider continuous distribution functions.

DEFINITION 1: *Let X and Y be two random variables with distribution functions F and G, respectively. Then, X is said to be less weakly dispersive than Y, denoted* $X \overset{\text{wd}}{\preccurlyeq} Y$ *or, equivalently,* $F \overset{\text{wd}}{\preccurlyeq} G$, *if for all* $\varepsilon > 0$, *it holds that*

$$\sup_{x_0}[F(x_0+\varepsilon)-F(x_0)] \geq \sup_{y_0}[G(y_0+\varepsilon)-G(y_0)].$$

We will say that X and Y are equally dispersed in the weakly dispersive sense, denoted as $X \overset{\text{wd}}{=} Y$, if $X \overset{\text{wd}}{\preccurlyeq} Y$ and $Y \overset{\text{wd}}{\preccurlyeq} X$ holds. It is easily seen that the relation $\overset{\text{wd}}{\preccurlyeq}$ is a partial order in the set of distribution functions for real random variables.

The expression $\sup_{x_0}[F(x_0+\varepsilon)-F(x_0)]$ is well known in the literature and it is called the Lévy concentration function. For example, Averous, Fougéres, and Meste [1] proposed a tailweight ordering for unimodal distributions based on this function. The interpretation of Definition 1 is meaningful through the notion of the concentration of probability. Roughly speaking, we can say that if we have an interval of length ε in the support of the variable Y, then we can always find another interval, with the same length, in the support of the variable X which concentrates more probability than the first one. Of course, the supremum is always well defined, but it is not always true that it is achieved at a point of the support of the variable. However, this is true if the distribution function is continuous on its support and the support is a closed interval.

We have to emphasize that the $\overset{\text{wd}}{\preccurlyeq}$ order defines dispersion as a concentration of probability in intervals with the same length. Intuitively, the $\overset{\text{wd}}{\preccurlyeq}$ order is different than the classic dispersion order in the L-T sense. For two distribution functions F and G denote by F^{-1} and G^{-1}, respectively, the corresponding left-continuous quantile function (i.e., $F^{-1}(u) = \inf\{x : F(x) \geq u\}$ and the same for G^{-1}). Formally, $F <_{\text{Disp}} G$, or, equivalently, $X <_{\text{Disp}} Y$, if and only if

$$F^{-1}(\beta) - F^{-1}(\alpha) \leq G^{-1}(\beta) - G^{-1}(\alpha) \quad \text{whenever } 0 < \alpha < \beta < 1.$$

Here, for the distribution F, we fix an interval of the form $(F^{-1}(\alpha), F^{-1}(\beta))$ which accumulate a probability $\beta - \alpha$ and we compare its length with the analogous interval for the variable Y. This comparison is stronger than the weakly dispersive order, as we will see later.

A dispersion order can be expected to satisfy properties of preservation under location and some homogeneous scale transformations. The following results show that.

First, we denote by $<_{\text{st}}$ the classical stochastic order and we denote by $\overset{\text{st}}{=}$ equality in distribution. We also say that a function h is an expansion or an expansive function if it holds that

$$|h(x) - h(y)| \geq |x - y| \quad \forall x, y \in \mathbb{R}.$$

THEOREM 1: *Let X and Y be two continuous random variables with distribution functions F and G, respectively, such that* $Y \overset{\text{st}}{=} h(X)$, *where* $h(\cdot)$ *is a real monotone expansive function. Then,* $X \overset{\text{wd}}{\preccurlyeq} Y$.

PROOF: Let x_0 be a point in $\mathbb{R}$ and let $\varepsilon > 0$ be arbitrary. By assumption, h is a monotone function. First, suppose that h is increasing.

If we consider the interval $(x_0, x_0 + \varepsilon)$, it is seen from the definition of expansion that $h(x_0 + \varepsilon) - h(x_0) \geq \varepsilon$. Denote $y_0 = h(x_0)$; then it is clear that $y_0 + \varepsilon \leq h(x_0 + \varepsilon)$. In addition, from the supposition that h is increasing and by the assumption that h is an expansion, it holds that h has to be strictly increasing. Then,

$$F(x_0 + \varepsilon) - F(x_0) = \text{Prob}\{h(X) \leq h(x_0 + \varepsilon)\} - \text{Prob}\{h(X) \leq h(x_0)\},$$

and by the assumption that $h(X) =_{\text{st}} Y$, it holds that the last expression is equal to

$$G(h(x_0 + \varepsilon)) - G(h(x_0)),$$

and using the definition of y_0, we have that

$$G(h(x_0 + \varepsilon)) - G(h(x_0)) \geq G(y_0 + \varepsilon) - G(y_0).$$

Since x_0 and ε are arbitrary, then by the definition of supremum we have the implication.

Now suppose that h is decreasing. It is only necessary to prove that $X \overset{\text{wd}}{=} -X$, because if we have that h is a monotone decreasing expansion, then $-h$ is a monotone increasing expansion; thus, $-h(X) =_{\text{st}} -Y$ and $X \overset{\text{wd}}{\preccurlyeq} -Y \overset{\text{wd}}{=} Y$.

Note that $X \overset{\text{wd}}{=} -X$ is easily obtained from

$$F_{-X}(x + \varepsilon) - F_{-X}(x) = F_X(-x) - F_X(-x - \varepsilon). \qquad \blacksquare$$

Theorem 1 provides many possible comparisons. In particular if we take a linear expansion function (i.e., of the form $h(x) = ax + b$ where $|a| \geq 1$), then it holds that $X \overset{\text{wd}}{\preccurlyeq} aX + b$. For example, we can say that X is equal weakly dispersive to $-X$ and is less weakly dispersive than $-3X$. These last two examples are not always possible in the L-T sense. Moreover, X is equal weakly dispersive to $X + c$ for all $c \in \mathbb{R}$; thus the relation $\overset{\text{wd}}{\preccurlyeq}$ is location independent.

Another interesting property is related to the supports. If a random variable is ordered in dispersion with respect to other variables, then it can be expected that the supports will be ordered too.

THEOREM 2: *Let X and Y be random variables with supports $S_X = (a_X, b_X)$ and $S_Y = (a_Y, b_Y)$ being finite intervals. If $X \overset{\text{wd}}{\preccurlyeq} Y$, then the ranges of the supports are ordered; that is, $b_X - a_X \leq b_Y - a_Y$.*

PROOF: If we take $b_Y - a_Y = \varepsilon$, the proof follows directly from Definition 1. $\blacksquare$

The weakly dispersive order satisfies another desirable closure property; that is, for two independent continuous random variables, we have that X is less weakly dispersive than $X + Y$. The classical dispersive order in the L-T sense does not have this property. It has this property if and only if X has a logconcave density; see Shaked and Shanthikumar [11].

THEOREM 3: *Let X and Y be two independent continuous random variables with distribution functions F_X and F_Y, respectively. Then,* $X \overset{\text{wd}}{\preccurlyeq} X + Y$.

PROOF: From Götze and Zaitsev [3, p. 772], for example, we know that

$$\sup_{z_0}[F_{X+Y}(z_0+\varepsilon) - F_{X+Y}(z_0)]$$
$$\leq \min\left\{\sup_{x_0}[F_X(x_0+\varepsilon) - F_X(x_0)], \quad \sup_{y_0}[F_Y(y_0+\varepsilon) - F_Y(y_0)]\right\};$$

therefore the stated result follows. ■

As we mentioned in Section 1, one of the main disadvantages of the dispersion order in the L-T sense is that we cannot compare two distribution functions having the same finite interval as their supports unless they are identical. This can be seen by employing the argument used in Hickey [5]. Without loss of generality, let the common support be the unit interval. If $F <_{\text{Disp}} G$, it then follows that for all x in (0,1), $F(x) \geq G(x)$ and $1 - F(x) \geq 1 - G(x)$, and so $F(x) = G(x)$. The next results show that we can compare many distributions with the same finite interval as their support in the weakly dispersive sense. Let F be a distribution function; we will denote the left and right end points of their support by l_F and r_F, respectively; that is,

$$l_F = \inf\{x : x \in \text{supp(F)}\} \quad \text{and} \quad r_F = \sup\{x : x \in \text{supp(F)}\}.$$

THEOREM 4: *Let F and G be two continuous distribution functions such that* $-\infty < a = l_F = l_G$. *If G is a concave function on their support and* $F <_{\text{st}} G$, *then* $F \overset{\text{wd}}{\preccurlyeq} G$.

PROOF: By assumption, G is a nondecreasing concave function; therefore,

$$G(x+\Delta) - G(x) \geq G(y+\Delta) - G(y)$$

whenever $x \leq y$ and $\Delta > 0$. If we take y_0 in $\mathbb{R}$ and $\varepsilon > 0$, it holds that

$$G(y_0+\varepsilon) - G(y_0) \leq G(a+\varepsilon) - G(a) = G(a+\varepsilon).$$

In addition, from $F <_{\text{st}} G$, we have that $F(x) \geq G(x)$ for all x in $\mathbb{R}$. In particular, it holds that

$$G(a+\varepsilon) \leq F(a+\varepsilon).$$

Since y_0 and ε are arbitrary, then, by the definition of supremum, we have the implication. ■

Similarly, we can prove the following theorem.

THEOREM 5: *Let F and G be two continuous distribution functions such that* $+\infty > b = r_F = r_G$. *If G is a convex function on their support and* $G <_{\text{st}} F$, *then* $F \overset{\text{wd}}{\preccurlyeq} G$.

Note that the supposition in Theorems 4 and 5 about the concavity and convexity of the distribution G, respectively, means that the distribution G is unimodal.

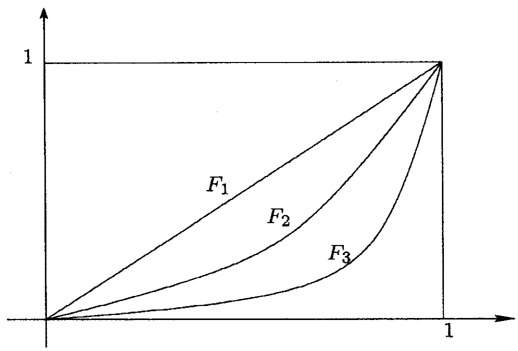

FIGURE 1. Distribution having the same interval support.

From Theorems 4 and 5, we can compare a large number of distributions having the same finite interval as their supports. For example, if we look at Figure 1, the functions F_i, for $i = 1, 2$, and 3, correspond to three distribution functions in the same finite interval (0,1), and it is easy to see from Theorem 5 that $F_3 \overset{\text{wd}}{\preccurlyeq} F_2 \overset{\text{wd}}{\preccurlyeq} F_1$.

3. THE RELATIONSHIP WITH OTHER ORDERS IN DISPERSION

As an obvious consequence of the previous results, the dispersive order in the L-T sense and the weakly dispersive order are not equivalent. The $<_{\text{Disp}}$ order is strictly stronger; this follows from the next result.

THEOREM 6: *Let X and Y be two continuous random variables with distribution functions F and G, respectively. If $F <_{\text{Disp}} G$, then $F \overset{\text{wd}}{\preccurlyeq} G$.*

PROOF: Note that for continuous distribution functions, $F <_{\text{Disp}} G$ means that there exists a function ϕ such that $Y =_{\text{st}} \phi(X)$ and ϕ satisfies that $\phi(x') - \phi(x) \geq x' - x$ whenever $x \leq x'$; see Shaked and Shanthikumar [11]. Hence, ϕ is an increasing expansion function. The stated result now follows from Theorem 1. ■

We can say that this implication can be expected because the dispersion order is, in general, a strong order. To obtain new implications with other known orders, we first need some definitions.

There are different ways to study the notion of unimodal density on $\mathbb{R}$. We consider one of the most general ways based on the following definition (see Sudhakar and Kumar [12, p. 2]).

DEFINITION 2: *A real random variable X or its distribution function F is called unimodal about a mode (or vertex) m_F if F is convex on $(-\infty, m_F)$ and concave on (m_F, ∞).*

A simple consequence of Definition 2 is that if F is unimodal about m_F, then apart from a possible mass at m_F, F is absolutely continuous, and, if this is the case, then the unimodality of F about m_F is equivalent to the existence of a density f, which is nondecreasing on $(-\infty, m_F)$ and nonincreasing on (m_F, ∞). Note that this density function f may be constant in a set of positive measure or may not be continuous in a countable number of points on its support. It is easily seen from Definition 2 that the set $\{x : f(x) \geq c\}$ is always an interval for all $c > 0$.

As a direct consequence of the concept of unimodal distribution, if we have that F is a symmetric and unimodal distribution about the mode m_F, then it is well known that $\sup_{x_0}[F_X(x_0 + \varepsilon) - F_X(x_0)]$ is equal to $F_X(m_F + \varepsilon/2) - F_X(m_F - \varepsilon/2)$ for all $\varepsilon > 0$ (see Sudhakar and Kumar [12, Corr. 1]). If we have X and Y as two random variables with F and G as two symmetric and unimodal distribution functions about m_F and m_G, respectively, such that $F \overset{\text{wd}}{\leqslant} G$, then it is easily seen that F is more peaked about its mode than G; that is, it holds that $|X - m_F| <_{\text{st}} |Y - m_G|$. Hence, from the classical characterization of the stochastic order, it follows that

$$\mathbb{E}(g(|X - m_F|)) \leq \mathbb{E}(g(|Y - m_G|))$$

for all increasing function g for which the expectation exists.

We are now interested in studying the relationship between the weakly dispersive order and the notion of majorization for continuous distributions. This concept was introduced by Hickey [4] and it was used to compare continuous distribution functions in terms of randomness. Using the notation in Hickey [5], we say that G is at least as dispersed as F in the majorization sense, denoted $G <_M F$, if

$$\int_0^t g^*(y)\,dy \leq \int_0^t f^*(x)\,dx \quad \forall t, \tag{1}$$

where f^* and g^* are the decreasing rearrangements of f and g, respectively; that is,

$$f^*(x) = \sup\{c : m(c) > x\}, \quad x > 0, \tag{2}$$

with $m(c) = \mu\{x : f(x) > c\}$, μ denoting Lebesgue measure, and f the corresponding density function of F. Note that the wider side of the majorization symbol is placed against the less dispersed distribution.

Hickey [5] studied the relationship between the majorization order and the dispersive order in the L-T sense. If F and G are two distribution functions with G unimodal distribution, then $F <_{\text{Disp}} G \Rightarrow G <_M F$. The following theorem shows the relation between the weak dispersion order and the majorization order. We show that they are equivalent under the assumption of unimodal distributions.

THEOREM 7: *Let F and G be continuous distribution functions, with f and g being their unimodal density functions, respectively. Then, $F \overset{\text{wd}}{\leqslant} G$ if and only if $G <_M F$.*

PROOF: To prove the equivalence, it is only necessary to show that

$$\int_0^t f^*(s)\,ds = \sup_x [F_X(x + \varepsilon) - F_X(x)] \tag{3}$$

for all F unimodal distribution functions about the mode m_0.

By the unimodality assumption on F, we know that the supremum of $\phi_t(x) = F(x+t) - F(x)$ is achieved in a real value $x_1 \in \mathbb{R}$ such that the mode, m_0, is in the interval $[x_1, x_1 + t]$ (see Sudhakar and Kumar [12, Thm. 1.7]). From the definition of unimodality, it also holds that there exists a density function f which is nondecreasing on $(-\infty, m_0)$ and nonincreasing on (m_0, ∞). Without loss of generality, we can consider f to be a left-continuous function.

Therefore, from (2), it is easily shown that

$$f^*(t) = \min\{f(x_1), f(x_1 + t)\}, \tag{4}$$

where x_1 is such that

$$F(x_1 + t) - F(x_1) = \sup_x [F(x+t) - F(x)].$$

Note that we use the minimum in (4) because f may not be continuous in x_1 or $x_1 + t$. If f is continuous, obviously $f(x_1) = f(x_1 + t)$.

First, let us assume that $f(\cdot)$ does not possess flat zones; that is, there is not a set of positive measures where f is constant. Then, the supremum of $\phi_s(x)$ holds in only one real value for each $s \in [0, t]$. Consequently, using (3.2) in Hickey [4], it holds that

$$\int_0^t f^*(s)\, ds = \int_{f(x) \geq f^*(t)} f(x)\, dx,$$

and by the assumption of unimodality, the set $\{x : f(x) \geq f^*(t)\}$ is an interval. In addition, from (4), the set $\{x : f(x) \geq f^*(t)\}$ corresponds to the interval $(x_1, x_1 + t)$; hence, (3) holds.

Finally, assume that f possesses flat zones. It means that the supremum of $\phi_s(x)$ may not be unique. Without loss of generality, we will only consider a flat zone; that is, there exist intervals $I_1 = (x_1, x_2)$ and $I_2 = (x_3, x_1 + t)$, where $x_2 < m_0$ and $x_3 > m_0$ and $f(x) = c$ for all x in $I_1 \cup I_2$.

Then, the rearrangement $f^*(s)$ has flat zones too, and it is easy to see that f^* is decreasing on $(0, x_3 - x_2)$ and $f^*(s) = c$ in $(x_3 - x_2, t)$. Hence, if we set $x_3 - x_2 = t_1$, it holds that

$$\begin{aligned}\int_0^t f^*(s)\, ds &= \int_0^{t_1} f^*(s)\, ds + c\mu((t_1, t)) \\ &= \int_{x_2}^{x_3} f(x)\, d\mu + c\mu(I_1 \cup I_2) \\ &= \int_{x_1}^{x_1+t} f(x)\, d\mu\end{aligned}$$

and, hence, (3) holds. ■

In general, if we do not assume unimodality, the majorization and the weak dispersion order are not equivalent, as we can see in the following example.

Example 1: Let X and Y be random variables with density functions f and g given by

$$f(x) = \begin{cases} x+1 & \text{if } -1 \leq x < 0 \\ 1-x & \text{if } 0 \leq x \leq 1 \end{cases}$$

and

$$g(x) = \begin{cases} -x & \text{if } -1 \leq x < 0 \\ x & \text{if } 0 \leq x \leq 1, \end{cases}$$

respectively. It is easy to compute that

$$m_f(c) = m_g(c) = \mu\{x : g(x) > c\} = (2-2c)I_{[0,1]}(c).$$

Then, $f^*(x) = g^*(x) = (1-x/2)I_{[0,2)}(x)$. Therefore, we have that $F =_M G$. However, it holds that $F \overset{\text{wd}}{\preccurlyeq} G$, but $G \overset{\text{wd}}{\not\preccurlyeq} F$; thus, $F \overset{\text{wd}}{\neq} G$. It is easy to prove that $F \overset{\text{wd}}{\preccurlyeq} G$ by just looking at the graphs of the densities, but note that if we take $\varepsilon = 1$, we have that

$$\sup_{\forall y_0}\{G(y_0+1) - G(y_0)\} = \frac{1}{2}$$

and the supremum is achieved at $y_0 = 0$. However, the supremum

$$\sup_{\forall x_0}\{F(x_0+1) - F(x_0)\} = \frac{3}{4}$$

and it is achieved at $x_0 = -\frac{1}{2}$, so $G \overset{\text{wd}}{\not\preccurlyeq} F$.

4. APPLICATION TO THE CONCEPT OF AGING

Let X be a lifetime random variable, with distribution function F such that $F(0) = 0$. Given a unit which has survived up to time t, its residual life is given by

$$X_t = \{X - t | X > t\},$$

and let $X_{(t,\infty)}$ be the truncated random variable

$$X_{(t,\infty)} = \{X | X > t\}.$$

In the context of lifetime distributions, the variable $X_{(t,\infty)}$ represents the life for a unit which has survived up to time t, and it is related to the residual life by

$$X_t = X_{(t,+\infty)} - t. \quad \textbf{(5)}$$

The stochastic process $\{X_t, t \geq 0\}$ has been studied in the literature to characterize different aging concepts. In particular, Pellerey and Shaked [8] characterized the notion of increasing failure rate by means of the dispersion order; that is, the distribution X is IFR if and only if $X_t \prec_{\text{Disp}} X_s$ for all $0 \leq s < t$. For distribution functions which admit a density function, say f, the concept of increasing failure rate means that the failure rate function, denoted by $r(t)$ and defined as

$$r(t) = \lim_{h\to 0} \frac{1}{h} \text{Prob}\{X < t + h|_{X>t}\} = \frac{f(t)}{\bar{F}(t)},$$

where $\bar{F}(t) = 1 - F(t)$, is increasing. In other words, the intensity of failure of a device in an infinitesimal amount of time is increasing when the time is increasing. For more details about the IFR notion of aging, see Barlow and Proschan [2].

For our purposes, we define a new concept of aging, which we will call decreasing in randomness.

Definition 3: *The stochastic process $\{X_t, t \geq 0\}$, defined earlier, is said to be decreasing in randomness if*

$$X_{t_1} \prec_M X_{t_2} \quad \forall 0 \leq t_1 < t_2. \tag{6}$$

Hickey [6] noted that $X_{t_1} \prec_M X_{t_2}$ means that X_{t_1} has at least as much randomness as X_{t_2}. Thus, we can say that the stochastic process $\{X_t, t \geq 0\}$ has the property of decreasing in randomness when the randomness of the residual lifetime is decreasing when the time is increasing.

As an application of the characterization of the majorization order in Theorem 7, we characterize the IFR unimodal distributions in term of randomness.

Theorem 8: *Let X be a unimodal random variable and let $\{X_t, t \geq 0\}$ be the residual lifetime stochastic process as described earlier. Then, X is an IFR distribution if and only if $\{X_t, t \geq 0\}$ is decreasing in randomness.*

Proof: First, note that the assumption that X is unimodal implies the unimodality of the residual life distribution X_t for all t.

If X is IFR, then $X_t \prec_{\text{Disp}} X_s$ for $0 \leq s < t$, and from Theorem 6, it follows that $X_t \overset{\text{wd}}{\preccurlyeq} X_s$ for $0 \leq s < t$. From the unimodality of the residual life distributions and from Theorem 7, it follows that $X_s \prec_M X_t$.

To prove the sufficient condition, note that under the assumption of unimodality, the density function is nondecreasing in $0 < t < m_0$; thus, it is easy to check that the failure rate function is always increasing for all $0 < s < t \leq m_0$, where m_0 represents the mode of X. Now, let s and t be such that $m_0 < s < t$. By assumption, it holds that $X_s \prec_M X_t$; thus, from Theorem 7, it follows that $X_t \overset{\text{wd}}{\preccurlyeq} X_s$. Using the definition of weakly dispersive, it follows that

$$\sup_x [F_{X_t}(x + \varepsilon) - F_{X_t}(x)] \geq \sup_y [F_{X_s}(y + \varepsilon) - F_{X_s}(y)]$$

for all $\varepsilon > 0$. The fact that both residual lifetime distributions X_t and X_s, for $m_0 < s < t$, are unimodal and the mode is their left end point of the support implies that the supremum is achieved at the mode. Therefore,

$$F_{X_t}(0 + \varepsilon) - F_{X_t}(0) \geq F_{X_s}(0 + \varepsilon) - F_{X_s}(0)$$

for all $\varepsilon > 0$. Dividing by ε and taking the limit to zero, it holds that

$$f_{X_t}(0) = \frac{f(t)}{\bar{F}(t)} \geq f_{X_s}(0) = \frac{f(s)}{\bar{F}(s)};$$

hence, $r(s) \leq r(t)$. ■

Note that there are many distribution functions that are decreasing in randomness. In particular, if we take F to be a distribution function with log-concave density, then it is well known that F is IFR (see Sudhakar and Kumar [12, Thm. 9.6]). The condition of log-concave density means that F belongs to the class of strong unimodal densities which is strictly included in the set of unimodal densities (see Sudhakar and Kumar [12, Thm 1.10]).

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