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The Schwarzian Korteweg–de Vries equation in (2 + 1) dimensions

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Abstract

In this paper, the complete Lie group classification of a (2 + 1)-dimensional integrable Schwarzian Korteweg–de Vries equation is obtained. The reduction to systems of partial differential equations in (1 + 1) dimension is derived from the optimal system of subalgebras. The invariance study of these systems leads to second-order ODEs. These ODEs provide several classes of solutions; all of them are expressible in terms of known functions, some of them are expressible in terms of the second and third Painlevé transcendents. The corresponding solutions of the (2+1)-dimensional equation involve up to three arbitrary smooth functions. They even appear in the form $\rho(z) f(x + \varphi(t))$. Consequently, the solutions exhibit a rich variety of qualitative behaviour. Indeed, by making appropriate choices for the arbitrary functions, we are able to exhibit large families of solitary waves, coherent structures and different types of bound states.

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1. Introduction

In this paper, we discuss the (2 + 1)-dimensional integrable generalization of the Schwarzian Korteweg–de Vries (SKdV) equation

$$W_t + \frac{1}{4}W_{xxz} - \frac{W_x W_{xz}}{2W} - \frac{W_{xx} W_z}{4W} + \frac{W_x^2 W_z}{2W^2} - \frac{W_x}{8}\partial_x^{-1}\left(\frac{W_x^2}{W^2}\right)_z = 0 \qquad (1)$$

where $\partial_x^{-1} f = \int f \, dx$. This equation was derived by Toda and Yu in [28] and was called by them the Schwarz–Korteweg–de Vries (2 + 1)-dimensional equation.

Within the family of Korteweg–de Vries (KdV) related equations, probably the most fundamental equation is the SKdV equation

$$\frac{\phi_t}{\phi_x} + \{\hat{\phi}; x\} = 0 \tag{2}$$

where

$$\{\hat{\phi}; x\} = \left(\frac{\phi_{xx}}{\phi_x}\right)_x - \frac{1}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2 \tag{3}$$

is the Schwarzian derivative [14, 27]. This equation was introduced by Krichever and Novikov in [17] and by Weiss in [30] and is a specialization of the KdV equation which is invariant under the Möbius transformations, i.e. under the group PSL(2). In recent years the SKdV equation has gained a lot of attention, see for example [13, 33]. Dorfman [8] established the bi-symplectic structure of the equation, her work on the bi-Hamiltonian phenomenon in integrable systems appears in her monograph [9].

It is well known that the similarity solutions of integrable nonlinear partial differential equations (PDEs) give rise to Painlevé transcendents [1, 12, 24, 25]. This connection between Painlevé equations and soliton-type equations has led to the Ablowitz, Ramani and Segur (ARS) conjecture [3]. Namely, it was demonstrated that ODEs obtained as the reductions of the well-known soliton equations yield ODEs with the Painlevé property (PP). A modern survey about PP can be found in [7, 20]. Moreover, similarity reductions of the best known soliton equations lead to second-order Painlevé equations [1, 2].

Although similarity reductions of PDEs to ODEs of Painlevé type are very important theoretically, as far as we know, there are only isolated examples for which such connections have been established. In this sense, it is of great interest to consider the similarity reductions of the Schwarzian equations to ODEs. In [21], the similarity reduction of the SKdV equation has been obtained by using the scaling group. The third-order reduced ODE is no longer Möbius invariant; however, it is directly related to the Painlevé II (PII). As was remarked in [21], this ODE, called by Nihjoff Schwarzian PII, does not seem to occur in the class of Chazy equations [6].

In [32], the PP was defined for PDEs and the connection between the PP and the occurrence of the Lax pair and Bäcklund transformation was demonstrated for the KdV equation [30], as well as for some other well-known equations. This connection was formulated in terms of the Schwarzian derivative of the function defining the singular manifold. This formulation is invariant under the Möbius group acting on dependent variables. The PP associated with the Schwarzian derivative leads to the identification of a wide class of nonlinear PDEs that possess, in part, the PP. The Schwarzian Boussinesq equation was studied in [31]. Lou [19] proposed a (1 + 1)-dimensional integrable model with spacetime exchange symmetry with the aim that the model can be changed to a form with the PP. A list of (1 + 1)-dimensional integrable equations and their symmetries has been given in [29].

Several integrable models in the context of (2 + 1)-dimensional equations have been developed. The Schwarzian Kadomtsev–Petviashvili (SKP) equation has been considered in [16, 30]. Dorfman and Nijhoff considered a (2 + 1) analogue of the SKdV equation [10]. Lou [19] proposed two new types of (2 + 1) SKdV extensions with space exchange invariance. Both of them, for solutions of the form $W(x, z, t) = \bar{W}(x + z, t)$, will be reduced back to the SKdV equation (2).

Toda and Yu [28] constructed some new integrable models by using the Calogero method. In this method, the new equation is derived by considering a Lax pair (L, T) of the basic equation and modifying the T operator to include another spatial dimension z. In this manner from the SKdV equation they obtain equation (1). This equation, by integrating with respect to x after the change of $W = \phi_x$, can be expressed as

$$\frac{\phi_t}{\phi_x} + \{\hat{\phi}; x\}_{2+1} = 0 \tag{4}$$

where

$$\{\hat{\phi}; x\}_{2+1} = \left(\frac{\phi_{xx}}{\phi_x}\right)_z - \frac{1}{2}\partial_x^{-1}\left(\frac{\phi_{xx}}{\phi_x}\right)_z^2 \tag{5}$$

is the (2 + 1)-dimensional Schwarzian derivative. Equation (1) is invariant under the Möbius transformation and, for solutions of the form $W(x, z, t) = \bar{W}(x + z, t)$, reduces to the SKdV equation. In [28] the corresponding Lax pair was exhibited and it was proved that it passes the Painlevé test in the sense of the Weiss–Tabor–Carnevale (WTC) method [32].

Although this (2 + 1)-dimensional SKdV equation appears in a non-local form, by using the following transformations,

$$W = \phi_x \qquad \phi = \exp(\psi) \qquad \psi_x = u \qquad \psi_t = v \tag{6}$$

equation (1) can be written as the system of differential equations:

$$4u^2v_x - 4uu_xv + u^2u_{xxz} - uu_{xx}u_z - 3uu_xu_{xz} + 3u_x^2u_z - u^4u_z = 0 \qquad u_t - v_x = 0. \tag{7}$$

In this paper, we apply the Lie group method of infinitesimal transformations to system (7). By using this method, we bring out the unexplored invariance properties and similarity reduced systems of (1 + 1) PDEs of the above system (7). First, we obtain a point transformation group which leaves the system (7) invariant. In order to find all invariant solutions with respect to s-dimensional subalgebras, it is sufficient to construct invariant solutions for an optimal system of order s. The set of invariant solutions obtained in this way is called an *optimal system of invariant solutions*.

By using the classical Lie method we obtain reductions of the (1+1)-dimensional systems of PDEs, to obtain systems of ODEs and by further reductions to second-order integrable ODEs. The solutions of all of these ODEs are expressible in terms of known functions, some of them can be expressed in terms of the second and third Painlevé transcendents. We also derive exact solutions for the (2 + 1)-dimensional integrable generalization of the SKdV equation. Some of these solutions are soliton solutions, localized on a curve, and they decay exponentially apart from that curve.

2. Lie symmetries

In order to apply the classical method to the (2 + 1)-dimensional system (7), we consider the one-parameter Lie group of infinitesimal transformations in (x, t, z, u, v) given by

$$\begin{aligned} x^* &= x + \varepsilon X(x, z, t, u, v) + \mathcal{O}(\varepsilon^2) \\ z^* &= z + \varepsilon Z(x, z, t, u, v) + \mathcal{O}(\varepsilon^2) \\ t^* &= t + \varepsilon T(x, z, t, u, v) + \mathcal{O}(\varepsilon^2) \\ u^* &= u + \varepsilon U(x, z, t, u, v) + \mathcal{O}(\varepsilon^2) \\ v^* &= v + \varepsilon V(x, z, t, u, v) + \mathcal{O}(\varepsilon^2) \end{aligned} \tag{8}$$

where ε is the group parameter. Then, one requires that this transformation leaves invariant the set of solutions of the system (7). This yields to an overdetermined linear system of equations for the infinitesimals $X(x, z, t, u, v)$, $Z(x, z, t, u, v)$, $T(x, z, t, u, v)$, $U(x, z, t, u, v)$ and

$V(x, z, t, u, v)$. The associated Lie algebra of infinitesimal symmetries is the corresponding set of vector fields of the form

$$\mathbf{v} = X\frac{\partial}{\partial x} + Z\frac{\partial}{\partial z} + T\frac{\partial}{\partial t} + U\frac{\partial}{\partial u} + V\frac{\partial}{\partial v}. \tag{9}$$

Having determined the infinitesimals, the symmetry variables are found by solving the invariant surface conditions

$$\Phi_1 \equiv X\frac{\partial u}{\partial x} + Z\frac{\partial u}{\partial z} + T\frac{\partial u}{\partial t} - U = 0 \qquad \Phi_2 \equiv X\frac{\partial v}{\partial x} + Z\frac{\partial u}{\partial z} + T\frac{\partial v}{\partial t} - V = 0. \tag{10}$$

Applying the classical method to the system (7) yields a system of equations which leads to a four-parameter Lie group. Associated with this Lie group we have a Lie algebra which can be represented by the following generators,

$$\mathbf{v}_1 = \frac{\partial}{\partial t} \qquad \mathbf{v}_2 = \frac{\partial}{\partial z} \qquad \mathbf{v}_3 = x\frac{\partial}{\partial x} - 2z\frac{\partial}{\partial z} - u\frac{\partial}{\partial u} \qquad \mathbf{v}_4 = t\frac{\partial}{\partial t} + z\frac{\partial}{\partial z} - v\frac{\partial}{\partial v}$$

and the infinite-dimensional vector fields of the form

$$\mathbf{v}_\alpha = \alpha(t)\frac{\partial}{\partial x} - \alpha'(t)u\frac{\partial}{\partial v}.$$

In order to obtain all the invariant solutions with respect to one-dimensional subalgebras, we construct the one-dimensional optimal system of subalgebras. The corresponding generators of the optimal system of subalgebras are

$$\langle \mu\mathbf{v}_3 + \mathbf{v}_4\rangle \quad \left\langle \mu\mathbf{v}_2 + \tfrac{1}{2}\mathbf{v}_3 + \mathbf{v}_4\right\rangle \quad \langle \mu\mathbf{v}_1 + \mathbf{v}_3\rangle \quad \langle \mu\mathbf{v}_1 + \mathbf{v}_2\rangle \quad \langle \mu\mathbf{v}_3\rangle \quad \langle \mathbf{v}_4\rangle \quad \langle \mathbf{v}_1\rangle$$

where $\mu \in \mathbb{R}^*$ is arbitrary. In the following, we list the similarity variables and similarity solutions as well as the systems of PDEs obtained when system (7) is reduced by means of $\{\mathbf{u}_i\}, i = 1, \ldots, 7$. These generators $\{\mathbf{u}_i\}, i = 1, \ldots, 7$, are obtained by adding to the generators of an optimal system the infinite-dimensional generator $\mathbf{v}_\alpha$. In the following $' \equiv \mathrm{d}/\mathrm{d}t$.

Reduction 1. By using the generator $\mu\mathbf{v}_3 + \mathbf{v}_4 + \mathbf{v}_\alpha$, we obtain the similarity variables and similarity solutions

$$\begin{aligned} z_1 &= xt^{-\mu} - \int t^{-(\mu+1)}\alpha(t)\,\mathrm{d}t & z_2 &= zt^{2\mu-1} \\ u &= t^{-\mu} f(z_1, z_2) & v &= t^{-1}\left(g(z_1, z_2) - f(z_1, z_2)\int t^{-\mu}\alpha'(t)\,\mathrm{d}t\right) \end{aligned} \tag{11}$$

and the systems of PDEs $\mathbf{S}_1$,

$$4f^2 g_{z_1} - 4ff_{z_1}g - ff_{z_1z_1}f_{z_2} + 3f_{z_1}^2 f_{z_2} - f^4 f_{z_2} + f^2 f_{z_1z_1z_2} - 3ff_{z_1}f_{z_1z_2} = 0$$
$$(2\mu - 1)z_2 f_{z_2} - \mu z_1 f_{z_1} - g_{z_1} - \mu f = 0.$$

Reduction 2. By using the generator $\mu\mathbf{v}_2 + \frac{1}{2}\mathbf{v}_3 + \mathbf{v}_4 + \mathbf{v}_\alpha$, we obtain the similarity variables and similarity solutions

$$\begin{aligned} z_1 &= xt^{-1/2} - \int t^{-3/2}\alpha(t)\,\mathrm{d}t & z_2 &= t^{1/2}\,\mathrm{e}^{-z} \\ u &= t^{-1/2} f(z_1, z_2) & v &= t^{-1}\left(g(z_1, z_2) - \int t^{-1/2}\alpha'(t)\,\mathrm{d}t\, f(z_1, z_2)\right) \end{aligned} \tag{12}$$

and the systems of PDEs $\mathbf{S}_2$,

$$z_2\big(ff_{z_1z_1}f_{z_2} - 3\big(f_{z_1}\big)^2 f_{z_2} + f^4 f_{z_2} - f^2 f_{z_1z_1z_2} + 3ff_{z_1}f_{z_1z_2}\big) + 4f^2 g_{z_1} - 4ff_{z_1}g = 0$$
$$z_2 f_{z_2} - z_1 f_{z_1} - 2g_{z_1} - f = 0.$$

Reduction 3. By using the generator $\mu\mathbf{v}_1 + \mathbf{v}_3 + \mathbf{v}_\alpha$, we obtain the similarity variables and similarity solutions

$$\begin{aligned} z_1 &= x\,\mathrm{e}^{-t/\mu} - \frac{1}{\mu}\int \mathrm{e}^{\frac{-t}{\mu}}\alpha(t)\,\mathrm{d}t & z_2 &= z\,\mathrm{e}^{2t/\mu} \\ u &= \mathrm{e}^{-t/\mu} f(z_1, z_2) & v &= g(z_1, z_2) - \frac{1}{\mu}\int t^{-(\mu+1)}\alpha(t)\,\mathrm{d}t\, f(z_1, z_2) \end{aligned} \tag{13}$$

and the systems of PDEs $\mathbf{S}_3$,

$$\begin{aligned} &4f^2 g_{z_1} - 4ff_{z_1}g - ff_{z_1z_1}f_{z_2} + 3(f_{z_1})^2 f_{z_2} - f^4 f_{z_2} + f^2 f_{z_1z_1z_2} - 3ff_{z_1}f_{z_1z_2} = 0 \\ &2z_2 f_{z_2} - z_1 f_{z_1} - \mu g_{z_1} - f = 0. \end{aligned}$$

Reduction 4. By using the generator $\mu\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_\alpha$ we obtain the similarity variables and similarity solutions

$$\begin{aligned} z_1 &= x - \frac{1}{\mu}\int \alpha(t)\,\mathrm{d}t & z_2 &= \mu z - t \\ u &= f(z_1, z_2) & v &= g(z_1, z_2) - \frac{1}{\mu}\alpha f(z_1, z_2) \end{aligned} \tag{14}$$

and the systems of PDEs $\mathbf{S}_4$,

$$\begin{aligned} &\mu\left(-ff_{z_1z_1}f_{z_2} + 3f_{z_1}^2 f_{z_2} - f^4 f_{z_2} + f^2 f_{z_1z_1z_2} - 3ff_{z_1}f_{z_1z_2}\right) + 4f^2 g_{z_1} - 4ff_{z_1}g = 0 \\ &g_{z_1} + f_{z_2} = 0. \end{aligned}$$

Reduction 5. By using the generator $\mathbf{v}_3 + \mathbf{v}_\alpha$ we obtain the similarity variables and similarity solutions

$$\begin{aligned} z_1 &= t & z_2 &= (x + \alpha(t))^2 z \\ u &= z^{1/2} f(z_1, z_2) & v &= z^{1/2}\alpha'(t) f(z_1, z_2) + g(z_1, z_2) \end{aligned} \tag{15}$$

and the systems of PDEs $\mathbf{S}_5$,

$$\begin{aligned} &z_2^3\left(8f^2 f_{z_2z_2z_2} - 32ff_{z_2}f_{z_2z_2} + 24f_{z_2}^3\right) + z_2^2\left(20f^2 f_{z_2z_2} - 28ff_{z_2}^2 - 2f^4 f_{z_2}\right) \\ &\qquad + 16z_2^{3/2}\left(f^2 g_{z_2} - ff_{z_2}g\right) + z_2\left(4f^2 f_{z_2} - f^5\right) g = 0 \\ &f_{z_1} - 2z_2^{1/2} g_{z_2} = 0. \end{aligned}$$

Reduction 6. By using the generator $\mathbf{v}_4 + \mathbf{v}_\alpha$ we obtain the similarity variables and similarity solutions

$$\begin{aligned} &z_1 = x - \int t^{-1}\alpha(t)\,\mathrm{d}t \qquad z_2 = \frac{z}{t} \qquad u = f(z_1, z_2) \\ &v = \frac{1}{t}(g(z_1, z_2) - f(z_1, z_2)\alpha(t)) \end{aligned} \tag{16}$$

and the systems of PDEs $\mathbf{S}_6$,

$$\begin{aligned} &4f^2 g_{z_1} - 4ff_{z_1}g + \left(-ff_{z_1z_1} + 3f_{z_1}^2 - ff_{z_1} - f^4\right) f_{z_2} + f^2 f_{z_1z_1z_2} + \left(f^2 - 3ff_{z_1}\right) f_{z_1z_2} = 0 \\ &z_2 f_{z_2} + g_{z_1} = 0. \end{aligned}$$

Reduction 7. By using the generator $\mathbf{v}_1 + \mathbf{v}_\alpha$ we obtain the similarity variables and similarity solutions

$$\begin{aligned} z_1 &= x - \int \alpha(t)\,\mathrm{d}t \qquad & z_2 &= z \\ u &= f(z_1, z_2) \qquad & v &= g(z_1, z_2) - \alpha f(z_1, z_2) \end{aligned} \tag{17}$$

and the systems of PDEs $\mathbf{S}_7$,

$$-ff_{z_1z_1}f_{z_2} + 3f_{z_1}^2 f_{z_2} - f^4 f_{z_2} + f^2 f_{z_1z_1z_2} - 3ff_{z_1}f_{z_1z_2} - 4ff_{z_1}g = 0 \qquad g_{z_2} = 0.$$

3. Symmetry reductions to ODEs and exact solutions

In several cases, the reduced systems of (1 + 1)-dimensional PDEs admit symmetries which lead to further reductions to systems of ODEs. We use again the techniques of Lie group theory. Moreover, all these systems of ODEs can be reduced to second-order ODEs. The next step will consist of using solutions of the second-order ODEs together with the corresponding symmetry reductions to construct exact solutions for the (2 + 1)-dimensional equation (1).

Among these ODEs, equation (30) provides soliton solutions for our (2 + 1)-dimensional integrable equation. These soliton solutions include some arbitrary functions.

(1) System $\mathbf{S_1}$ admits the following symmetries:

$$\mathbf{v}_{11} = \frac{\partial}{\partial z_1} - \mu f \frac{\partial}{\partial g} \qquad \mathbf{v}_{12} = -z_1 \frac{\partial}{\partial z_1} + 2z_2 \frac{\partial}{\partial z_2} + f \frac{\partial}{\partial f}.$$

By using $c\mathbf{v}_{11} + \mathbf{v}_{12}$ we obtain the similarity variables and similarity solutions

$$w = -\sqrt{z_2}(z_1 - c) \qquad f = \sqrt{z_2}h(w) \qquad g = -c\mu\sqrt{z_2}h(w) + k(w) \tag{18}$$

and the system of ODEs

$$2k' - wh' - h = 0$$
$$w(h^2h''' - 4hh'h'' + 3h'^3 - h^4h') - 8h^2k' + 8hh'k + 2h^2h'' - 3hh'^2 - h^5 = 0.$$

This system is equivalent to the following second-order ODE,

$$h'' = \frac{3}{2}\frac{(h')^2}{h} + \frac{h^3}{2} + h\left(2 - \frac{k_1}{w^2}\right) \tag{19}$$

and

$$k = \frac{wh}{2} + k_1.$$

The change of variables $h^2 = v^{-1}$, $v = \lambda(w)V(\xi(w))$ leads to

$$V'' = \frac{3}{4}\frac{(V')^2}{V} - 1$$

see [15, p 337], where λ must satisfy

$$\lambda'' - \frac{3}{4}\frac{(\lambda')^2}{\lambda} + 2\lambda\left(2 - \frac{k_1}{w^2}\right) = 0.$$

By making $\lambda = \zeta^4$ we obtain that ζ is the solution of the linear equation

$$\zeta'' + \frac{1}{2}\left(2 - \frac{k_1}{w^2}\right)\zeta = 0$$

whose solutions can be expressed in terms of the Bessel functions.

(2) System $\mathbf{S_2}$ admits the following symmetries:

$$\mathbf{v}_{21} = z_2 \frac{\partial}{\partial z_2} \qquad \mathbf{v}_{22} = 2\frac{\partial}{\partial z_1} - f\frac{\partial}{\partial g} + \frac{\partial}{\partial f}.$$

By using $c\mathbf{v}_{21} + \mathbf{v}_{22}$ we obtain the similarity variables and similarity solutions

$$w = \frac{e^{cz_1}}{z_2^2} \qquad f = h(w) \qquad g = k(w) - \frac{z_1 f}{2} \tag{20}$$

and the system of ODEs

$$ck' + h' = 0$$

$$c^2w^3(h^2h''' - 4hh'h'' + 3h'^3) + c^2w^2(3h^2h'' - 4hh'^2) + cw(2h^2k' - 2hh'k) - wh^4h' + c^2wh^2h' - h^3 = 0.$$

This system is equivalent to the following second-order ODE,

$$h^4 + 2h^2\log(w) + 3c^2w^2(h')^2 - 2ch(2c_1 + cw(h' + wh'')) - 2k_1h^2 = 0 \tag{21}$$

and

$$k = -\frac{1}{c}h + k_1.$$

Setting $h = \frac{1}{Y}$ and $w = e^z$ we obtain

$$Y'' = \frac{(Y')^2}{2Y} + \frac{2c_1Y^2}{c} - \frac{zY}{c^2} + \frac{k_2Y}{c^2} - \frac{1}{2c^2Y}.$$

By making the change of variables $Y(z) = \alpha V(Z)$, with $Z = \beta z$, we obtain

$$V'' = \frac{1}{2}\frac{(V')^2}{V} + 4aV^2 - ZV - \frac{1}{2}\frac{1}{V}$$

with $\alpha^2 = 1$, $\beta^2 = -c^{-2}$ and $\frac{2c_1\alpha}{c\beta^2} = 4a$. The solutions can be written in terms of the second Painlevé equation (PII) [15].

(3) System $\mathbf{S_3}$ admits the following symmetries:

$$\mathbf{v}_{31} = a\frac{\partial}{\partial z_1} - f\frac{\partial}{\partial g} \qquad \mathbf{v}_{32} = z_1\frac{\partial}{\partial z_1} - 2z_2\frac{\partial}{\partial z_2} - f\frac{\partial}{\partial f}.$$

By using $c\mathbf{v}_{31} + \mathbf{v}_{32}$ we obtain the similarity variables and similarity solutions

$$w = z_2^{1/2}(z_1 + c\mu) \qquad f = hz_2^{1/2} \qquad g = ch(w)z_2^{1/2} + k(w) \tag{22}$$

and the system of ODEs

$$k' = 0 \qquad w(h^2h''' - 4hh'h'' + 3h'^3 - h^4h') + 8h^2k' - 8hh'k + 2h^2h'' - 3hh'^2 - h^5 = 0. \tag{23}$$

On dividing by h^4 and integrating once with respect to w we arrive at PIII

$$h'' - \frac{h'^2}{h} + \frac{h'}{w} - h^3 + \frac{4k_1}{w} - \frac{k_2h^2}{w} = 0.$$

Some particular solutions are

$$h = \pm\frac{1}{w(c_1 - \log(w))} \qquad h = \pm\frac{2c_1c_2w^{c_2-1}}{c_1{}^2 - w^{2c_2}} \qquad h = \pm\frac{c_2}{w\cos(c_1 + c_2\log(w))}. \tag{24}$$

From (24), by considering the transformations (6) as well as the corresponding symmetry reductions (22) and (13), we obtain that a family of solutions for the (SKdV) equation in (2 + 1) dimensions (1) (after some simplifications) can be written as

$$W = \frac{2\mu\rho(z)}{(x - \varphi(t))(\log(z^{\mu}(x - \varphi(t))^{2\mu}) - c_1)^2} \tag{25}$$

$$W = \frac{\rho(z)}{(x-\varphi(t))(-1+\sin(c_1+\log(z^{\mu}(x-\varphi(t))^{2\mu})))} \tag{26}$$

$$W = \frac{2c_1c_2\rho(z)(x-\varphi(t))^{-1+c_2}}{(c_1 - z^{c_2/2}(x-\varphi(t))^{c_2})^2} \tag{27}$$

with $\varphi(t) = \frac{\mathrm{e}^{\frac{t}{\mu}}}{\mu}\int \mathrm{e}^{\frac{-t}{\mu}}\alpha(t)\,\mathrm{d}t$ and $\rho(z)$ arbitrary functions.

(4) System $\mathbf{S_4}$ admits the symmetries

$$\mathbf{v}_{41} = \frac{\partial}{\partial z_1} \qquad \mathbf{v}_{\beta} = \beta(z_2)\frac{\partial}{\partial z_2} - \beta'(z_2)g\frac{\partial}{\partial g}$$

by using $c\mathbf{v}_{41} + \mathbf{v}_{\beta}$ we obtain the similarity variables and similarity solutions

$$w = z_1 - c\int\frac{\mathrm{d}z_2}{\beta(z_2)} \qquad f = h \qquad g = \frac{1}{\beta(z_2)}k(w) \tag{28}$$

and the system of ODEs

$$k' - ch' = 0 \qquad \mu c(4hh'h'' - h^2h''' - 3h'^3 + h^4h') + 4h^2k' - 4hh'k = 0. \tag{29}$$

This system is equivalent to the following second-order ODE,

$$2\mathrm{d}h + c\mu(h^4 + (h')^2 - hh'') - k_1h^3 = 0 \tag{30}$$

and

$$k = ch + k_1.$$

The general solution can be written in terms of the Jacobian elliptic functions, so there exist solutions such as

$$h = \frac{a_1}{a_2 + sn^2(a_4 + a_3x|a_5)}$$

with a_1, a_2, a_3, a_4, arbitrary constants and $a_5 = -\frac{1}{a_2} - \frac{a_1^2}{4a_2^2a_3^2(1+a_2)}$.

We have found several particular solutions with a suitable choice of the parameters and the more interesting ones are

$$\begin{aligned} &h = d_1 && h = \pm\frac{d_2}{1+\mathrm{e}^{d_2(w-d_1)}} \\ &h = \pm\frac{2d_2}{(w-d_1)^2 - {d_2}^2} && h = \frac{d_2}{\cos(d_2(w-d_1))} \end{aligned} \tag{31}$$

where d_1 and d_2 are arbitrary constants.

From (31), by considering the transformations (6) as well as the corresponding symmetry reductions (28) and (14), we obtain that the corresponding family of solutions for the (SKdV) equation in (2 + 1) dimensions (1) (after some simplifications) can be written as

$$W = d_1\exp(d_1x)\rho(z,t) \tag{32}$$

$$W = \frac{\rho(z)}{\cosh^2(d_1 + d_2x + \varphi(t) + \delta(-t+\mu z))} \tag{33}$$

$$W = \frac{\rho(z)}{(d_1 - x + \varphi(t) + \delta(-t+\mu z))^2} \tag{34}$$

$$W = \frac{\rho(z)}{1 + \sin(d_2(d_1 - x + \varphi(t) + \delta(-t+\mu z))} \tag{35}$$

with $\varphi(t) = \frac{1}{\mu}\int \mathrm{e}^{\frac{-t}{\mu}}\alpha(t)\,\mathrm{d}t$ and $\delta(-t+\mu z) = c\int \frac{1}{\beta(z_2)}\,\mathrm{d}z_2$ and $\rho = \rho(z,t)$, arbitrary functions.

(5) System $\mathbf{S_5}$ admits the following symmetries:

$$\mathbf{v}_{51} = \frac{\partial}{\partial z_1} \qquad \mathbf{v}_{52} = 2z_1\frac{\partial}{\partial z_1} + 2z_2\frac{\partial}{\partial z_2} - f\frac{\partial}{\partial f} - 2g\frac{\partial}{\partial g}.$$

By using $c\mathbf{v}_{51} + \mathbf{v}_{52}$ we obtain the similarity variables and similarity solutions

$$w = \frac{z_1 + c}{z_2} \qquad f = z_2^{-1/2}h(w) \qquad g = z_2^{-1}k(w) \tag{36}$$

and the system of ODEs

$$2wk' + 2k + h' = 0$$

$$w^3(4h^2h''' - 16hh'h'' + 12h'^3) + w^2(12h^2h'' - 16hh'^2) + w(8h^2k' - 8hh'k - h^4h' + 4h^2h') + 4h^2k = 0.$$

This system is equivalent to the following second-order ODE,

$$-h'' + \frac{3}{2}\frac{(h')^2}{h} - \frac{h'}{w} + \frac{h^3}{8w^2} + \frac{h}{2w^3} + \frac{k_1 h}{4w^2} = 0 \tag{37}$$

and

$$k = -\frac{h}{2w} + \frac{k_1}{w}.$$

By making the change of variables $h^2(w) = v^{-1}(w)$, $v(w) = \lambda(w)Y(Z)$ with $Z = \xi(w)$ we obtain

$$Y'' = \frac{3}{4}\frac{(Y')^2}{Y} - 1$$

(see [15]) where $\lambda(w) = \frac{1}{(2w\xi'2)^2}$ and $\xi(w)$ must satisfy

$$\{\xi; w\} = \frac{2+k_1}{w^2} + \frac{1}{2w^3}$$

and $\{\xi; w\}$ is the Schwarzian derivative. On dividing by $\sqrt{\xi'}$ and making the change of variables $\sqrt{\xi'} = \psi^{-1}$ we obtain that ψ must satisfy the following linear equation,

$$\psi'' + \frac{1}{2}\left(\frac{k_1+2}{4w^2} + \frac{1}{2w^3}\right)\psi = 0$$

whose solutions can be expressed in terms of Bessel functions.

(6) System $\mathbf{S_6}$ does not admit Lie symmetries.

(7) System $\mathbf{S_7}$ admits the following symmetries:

$$\mathbf{v}_{71} = \frac{\partial}{\partial z_1} \qquad \mathbf{v}_{72} = z_1\frac{\partial}{\partial z_1} + -f\frac{\partial}{\partial f} - 2g\frac{\partial}{\partial g}$$

$$\mathbf{v}_\beta = \beta(z_2)\frac{\partial}{\partial z_2} - \beta'(z_2)g\frac{\partial}{\partial g}.$$

By using $c\mathbf{v}_{71} + \mathbf{v}_{72} + \mathbf{v}_\beta$ we obtain the similarity variables and similarity solutions

$$w = \log(z_1 + c) - \int\frac{\mathrm{d}z_2}{\beta(z_2)} \qquad f = \frac{h(w)}{z_1 + c} \qquad g = \frac{1}{\beta(z_2)}\exp\left(-2\int\frac{\mathrm{d}z_2}{\beta(z_2)}\right)k(w) \tag{38}$$

and the system of ODEs

$$k' = 0 \qquad (4hh' - 4h^2)k\,\mathrm{e}^{2w} - 4hh'h'' + h^2h''' + 3h'^3 - h^4h' = 0. \tag{39}$$

By making the change of variables $e^{2w} = z, h(w) = Y(z)$, dividing by $u^4 z$ and integrating once with respect to z we obtain

$$Y'' = \frac{(Y')^2}{Y} + \frac{1}{4z^2}Y^3 + \frac{k_1}{8z^2}Y^2 - \frac{Y}{z} + \frac{k}{2z} = 0.$$

This is the Painlevé III equation (1906)

$$Y'' = \frac{(Y')^2}{Y} + \frac{\alpha Y^2 + \gamma Y^3}{4z^2} + \frac{\beta}{4z} + \frac{\delta}{4Y}$$

with $\alpha = \frac{k_1}{2}, \beta = 2k, \gamma = 1, \delta = 0$.

By using $\mathbf{v}_{71} + \mathbf{v}_\beta$ we obtain the similarity variables and similarity solutions

$$w = z_1 - \int \frac{dz_2}{\beta(z_2)} \qquad f = h \qquad g = \frac{1}{\beta(z_2)}k(w) \tag{40}$$

and the system of ODEs

$$k' = 0 \qquad -4hh'h'' + h^2h''' + 3h'^3 - h^4h' + 4h^2k' - 4hh'k = 0. \tag{41}$$

This system can be transformed into the following second-order autonomous ODE,

$$h'' = \frac{3}{2}\frac{(h')^2}{h} + \frac{h^3}{2} - \frac{k_2}{2}h + 4c \tag{42}$$

and

$$k = c.$$

The solution can be written in terms of the elliptic functions.

By using $\mathbf{v}_\beta$ we obtain the similarity variables and similarity solutions

$$w = z_1 \qquad f = h \qquad g = \beta(z_2)c. \tag{43}$$

If we set $c = 0$ then h becomes arbitrary. By considering the transformations (6) as well as the corresponding symmetry reductions (43) and (17), we obtain the corresponding family of solutions for the (SKdV) equation in (2 + 1) dimensions (1),

$$W = \rho(z)f(x - \varphi(t)) \tag{44}$$

where ρ, φ and f are arbitrary functions.

4. Some explicit solutions

In the following, we perform an analysis for the wide class of explicit solutions that we have obtained by using classical symmetry reductions. The entrance of some arbitrary functions $\rho(z), \varphi(t), \delta(t + \mu z)$ and $f(x + \varphi(t))$ allows a wide variety of qualitative and physical behaviour for these solutions. In particular, we have obtained stationary solutions, solutions which are regular for any value of the variable t, solutions which become singular on a finite or an infinite number of curves moving with t, etc.

The most interesting solutions are the soliton solutions:

- The solutions given by (34) are localized on the curve $d_1 + d_2x + \varphi(t) + \delta(-t + \mu z)) = 0$ and decay exponentially apart from the curve. By choosing $\varphi(t), \delta(t - \mu z)$ as constants in (34) we obtain stationary solutions. If we choose $\varphi(t)$ and $\delta(t + \mu z)$ as bounded, the maximum points will be localized in a bounded domain for x. We can find coherent structures by making $\delta(t + \mu z) = 0, \varphi(t) = t$. Note that, by setting $\rho(z)$ in a convenient form, we can also module the amplitude of the soliton. We choose

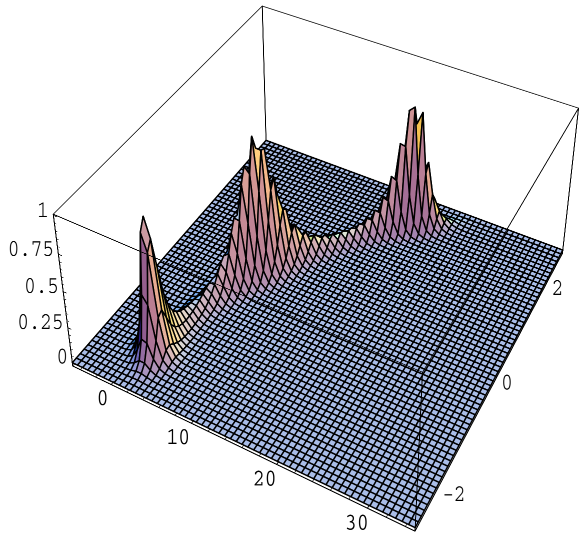

Figure 1. Three-soliton bound state $t = -2$.

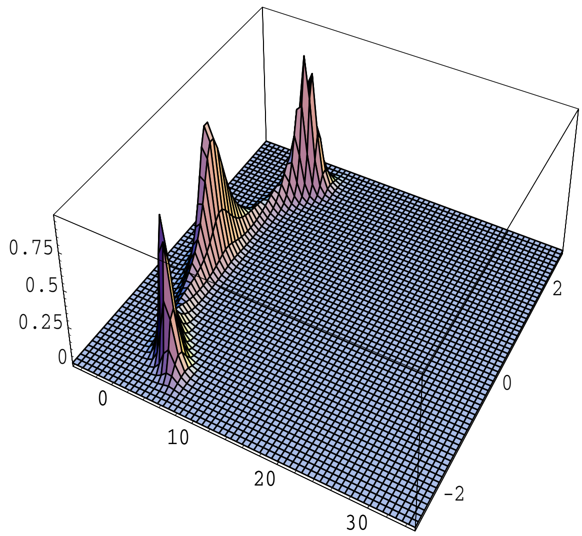

Figure 2. Three-soliton bound state $t = 0$.

$\rho(z) = \cosh^{-1}(z(z^2-4))$, $\delta(t+\mu z) = \cosh^{-2}(t-z)$ and $\varphi(t) = \cosh(t^2-4)$ which leads to a three-soliton bound state. In figures 1–3 we plot these solutions for $t = -2, 0, 4$. Indeed, setting $\rho(z) = \cosh^{-1}(\sin(z))$ we can find an infinite-soliton bound state as well as an n-soliton bound state if we take an adequate n-degree polynomial $p(z)$ in $\rho(z) = \cosh^{-1}(p(z))$. In figures 4–6 we can see a 'snake soliton' given by

$$\sin(2z)\cosh^{-2}(z/2)\cosh^{-2}(x - 3\sin(z-t)).$$

We list some other special interesting cases.

- Solution (25) becomes singular on two different curves. Solution (26) becomes singular on an infinite number of curves in the XZ plane, moving with t. These curves are given

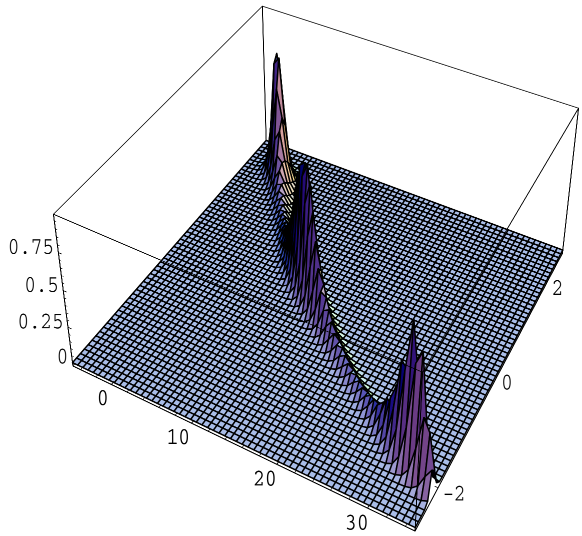

Figure 3. Three-soliton bound state $t = 4$.

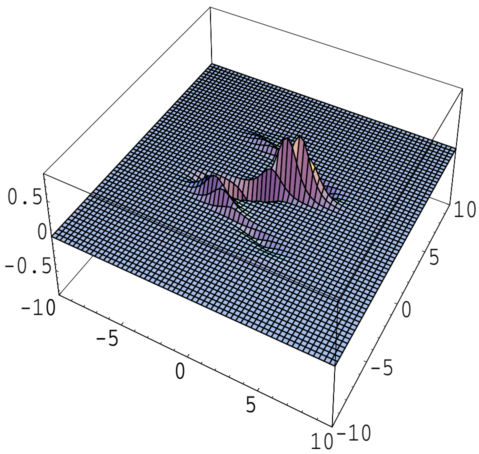

Figure 4. Snake soliton $t = 0$.

by $x = \varphi(t)$ and $x = \varphi(t) + \exp\left(-\frac{c_1}{2c_2^2} + \frac{(4k+1)\pi}{4c_2^2}\right)\frac{1}{\sqrt{z}}$, with k an integer number. In figure 7, by choosing $c_1 = 0, c_2 = 1, \varphi(t) = t$ and $\rho(z) = 1$, we plot some of the singularities corresponding to (26). We can also choose $\varphi(t)$ in such a way that the solution asymptotically approaches the stationary solution.

- Solution (27), depending on the choice of c_1 and c_2, becomes a singular solution. In particular, if c_2 is a rational number the solution becomes singular on an algebraic curve. It is interesting that this possesses quite rich structure because of the entrance of arbitrary functions. For example, by choosing $c_1 = -1$, $c_2 = 4$, $\varphi(t) = t$ and $\rho(z) = \sin(z)$, the solution becomes

$$W = \frac{(x-t)^3 \sin(z)}{(1 + (t-x)^4 z^2)^2}$$

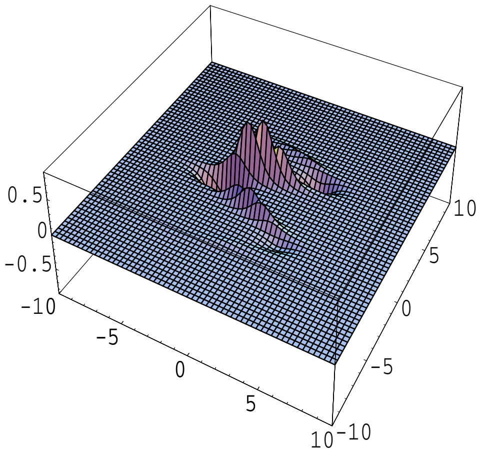

Figure 5. Snake soliton $t = 1$.

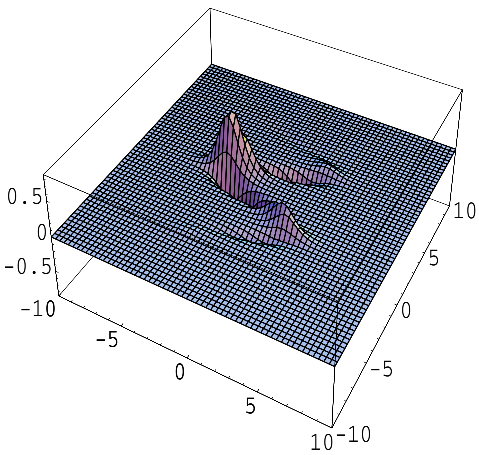

Figure 6. Snake soliton $t = 2$.

which is a bounded regular solution for any values of x, t and z with an infinite number of maximum and minimum points. These points are given as the solutions of the following system: $z = \frac{3}{2}\tan(z)$, $x = t \pm \left(\frac{3}{5}\right)^{1/4} z^{-1/2}$. We can see these facts in figure 8.

- Solution (32), due to the fact that $\rho(t, z)$ is an arbitrary function, exhibits a great variety of qualitative behaviour. For example, if we choose $\rho(t, z) = \frac{1}{t^2+z^2}$ solution (32) with $t > 0$ is singular, and for $t = 0$ becomes regular. Another interesting case is found if we choose

$$\rho(t, z) = \frac{1}{(t^2 + z^2 - 1)(\sin(t) + \sin(z) - 2))}.$$

Thus solution (32) has, depending on the values of the variable t, none, one, two or an infinite number of singularities. We can also obtain a family of regular solutions, all of them modulated by $\exp(x)$.

- Due to the fact that f is an arbitrary function in (44), we can select a lot of different functions to generate particular phenomena. For example, a kink is usually generated by

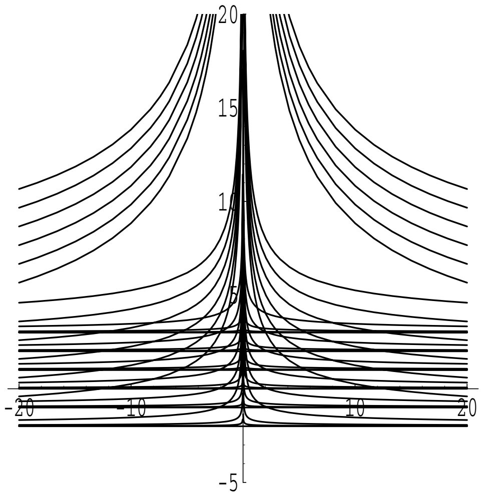

Figure 7. Curves for which the solution is singular, $t = -2, -1, \ldots, 3$; $k = -2, -1, \ldots, 2$.

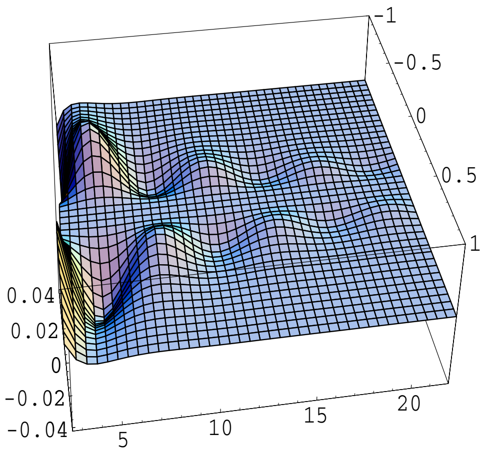

Figure 8. Maxima and minima.

an arctan or tanh, but now we can also obtain kinks using the logistic function. In figure 9 we choose f and ρ such that

$$W = \frac{\exp(z)}{1+\exp(z^2)} \frac{\exp(x-t^2)}{1+\exp(x-t^2)}$$

and in figure 10 we can see a curved kink with

$$W = \frac{\exp(z)}{1+\exp(z)} \frac{\exp(x-t^2)}{1+\exp(x-t^2)}.$$

We also obtain bound states in figure 11 by choosing

$$W = \cosh^{-2}(z)\cosh^{-2}((x+t)^4 - 1)$$

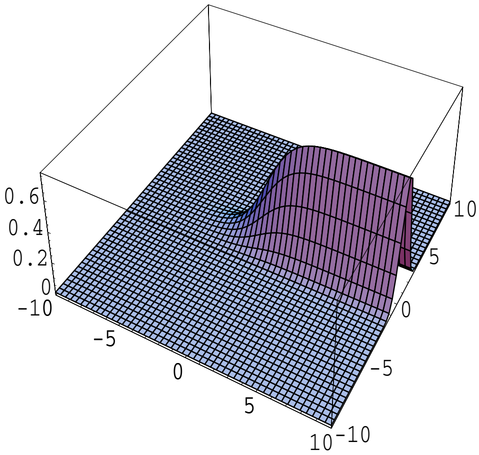

Figure 9. Kink.

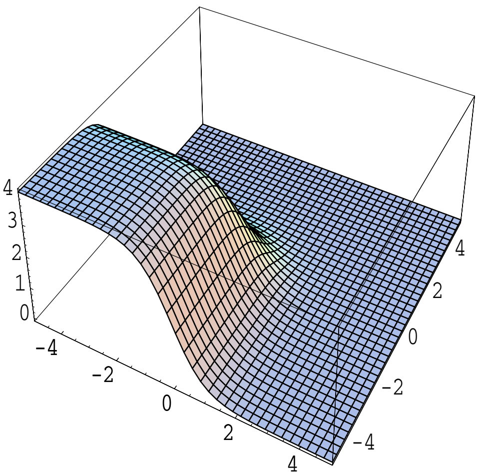

Figure 10. Curved kink.

in (44). We can consider solutions in the form

$$W = \rho(z) \sum_{i=1}^{n} f_i(x - \varphi(t))$$

which show different shapes. Choosing $\varphi(t) = 0$ we can obtain a great variety of stationary solutions of the form $W = \rho(z) f(x)$. In figure 12 we can see a bowl solution with

$$\rho = (\tanh(z + 15) - \tanh(z - 15) + 0.5(-\tanh(z + 10) + \tanh(z - 10)))$$
$$f = (\tanh(x + 15) - \tanh(x - 15) + 0.5(-\tanh(x + 10) + \tanh(x - 10))).$$

In figure 13 we plot the basin solution

$$W = -(\tanh(z + 10) - \tanh(z - 10))(\tanh(x + 10) - \tanh(x - 10)).$$

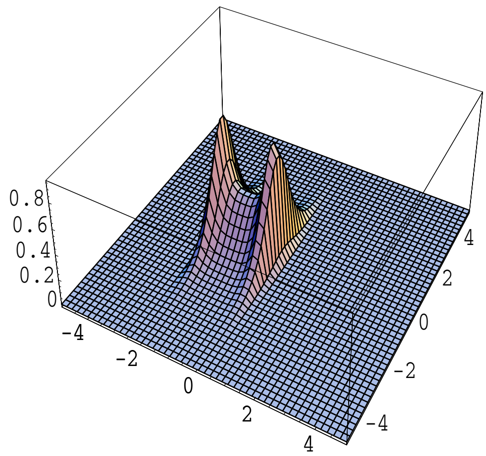

Figure 11. Bound state.

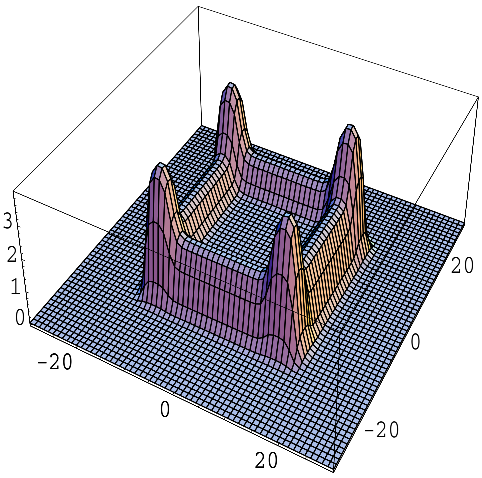

Figure 12. Bowl solution.

Choosing W as

$\rho = (\cosh^{-2}(z+9) - \cosh^{-2}(z+3) + \cosh^{-2}(z-3) - \cosh^{-2}(z-9))$

$f = (\cosh^{-2}(x+9) - \cosh^{-2}(x+3) + \cosh^{-2}(x-3) - \cosh^{-2}(x-9))$

in figure 14 we show a multisoliton.

- Due to arbitrariness in the choice of f we can obtain compactons, if we take f as a compact support $\mathcal{C}^\infty$ function. For example, we can use the well-known function

$$\chi(x) = \begin{cases} \exp(-(x-1)^{-2})\exp(-(x+1)^{-2}) & x \in (-1,1) \\ 0 & \text{elsewhere.} \end{cases}$$

In this way, we can obtain compact support solutions with the same shape as previous solutions. In a similar way instead of the later logistic function, we can take

$$\frac{\int_{-1}^{x} \chi(\zeta)\,\mathrm{d}\zeta}{\int_{-1}^{1} \chi(\zeta)\,\mathrm{d}\zeta}$$

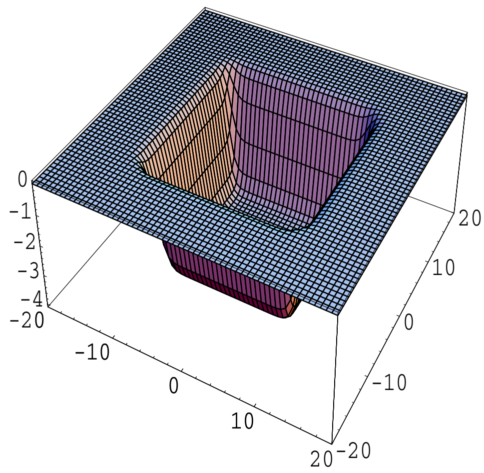

Figure 13. Basin solution.

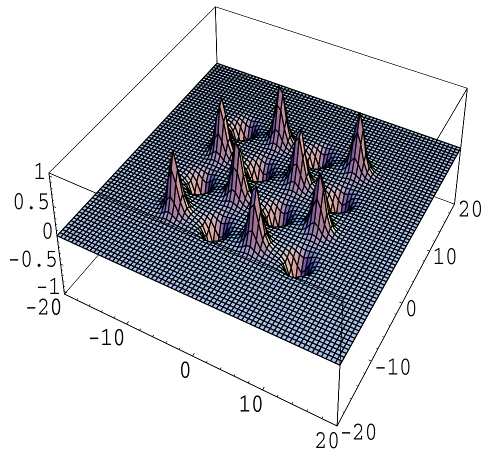

Figure 14. Multisoliton.

which is 0 when $x < -1$ and 1 when $x > 1$ to obtain kinks, plateaux and so on. On the other hand, it is clear that all the solutions evolve without change of form in x.

5. Conclusions

In this work, we have discussed symmetry reductions and exact solutions for the (2 + 1)-dimensional integrable generalization of the Schwarzian Korteweg–de Vries equation. By using the classical Lie method, we have obtained systems of PDEs in (1 + 1) dimensions and systems of ODEs, and by further reductions we obtained second-order integrable ODEs whose solutions are all expressible in terms of known functions, some of them expressible in terms of the second and third Painlevé transcendents. For the SKdV equation in (2+1) dimensions, we obtained families of solutions which have a rich variety of qualitative behaviour. This is due to the freedom in the choice of the arbitrary functions $\rho(z)$, $\varphi(t)$, $\delta(t + \mu z)$ and f. Among them we have obtained singular solutions, regular solutions which are not bounded in infinity and bounded solutions. The most impressive of these solutions are the soliton solutions. Because of these arbitrary functions which are included in the single soliton solution (33), the solution is localized on a curve and the curve may have quite a free form. We also find coherent structures as n-soliton bound states. By selecting the arbitrary functions appropriately in (44) we are able to generate particular phenomena such as kinks, bound states, a great variety of stationary solutions such as basin solutions, bowl solutions and multidromions. By choosing f as a compact support $\mathcal{C}^{\infty}$ function, we also get compactons.

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