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Symmetry Reductions for a Dissipation-Modified KdV Equation

M. S. BRUZÓN AND M. L. GANDARIAS
Departamento de Matemáticas, Universidad de Cádiz
P.O. BOX 40, 11510 Puerto Real, Cádiz, Spain
marialuz.gandarias@uca.es

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Abstract—In this paper we give a group classification for a dissipation-modified Korteweg-de Vries equation by means of the Lie method of the infinitesimals. We prove that, by using the nonclassical method, we get several new solutions which are unobtainable by Lie classical symmetries. We obtain nonclassical symmetries that reduce the dissipation-modified Korteweg-de Vries equation to ordinary equations with the Painlevé property. These solutions have not been derived elsewhere by the singular manifold method.

1. INTRODUCTION

When heating an open horizontal liquid layer from the air side, as a first instability of the motionless state one expects oscillatory Marangoni-Bénard convection rather than a stationary convective pattern. The disturbances generated by such an instability are actually capillary-gravity waves sustained by the Marangoni thermocapillary effect. In the case of a shallow liquid layer when the wavelength of motions is much larger than the depth of the layer, the propagation of some of these waves can be described by nonlinear equations [1]. These equations include the Korteweg-de Vries (KdV) equations supplemented by additional terms of the Kuramoto-Sivashinsky type,

$$u_t + \alpha_1 \left(\frac{u^2}{2}\right)_x + \alpha_2 u_{xxx} + \beta u_x + \alpha_3 u_{xx} + u_{xxxx} + \alpha_5 \left(\frac{u^2}{2}\right)_{xx} = 0, \tag{1}$$

describing nonlinear convection and the input of energy produced by Marangoni forces on the long scales together with energy dissipation on short scales, where $u(x,t)$ denotes surface deformation, α_i and β are coefficients, and ϵ is a smallness parameter.

The application of Lie transformations group theory for the construction of solutions of nonlinear partial differential equations (PDEs) is one of the most active fields of research in the theory of nonlinear PDEs and applications. The fundamental basis of the technique is that, when a differential equation is invariant under a Lie group of transformations, a reduction transformation

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exists. In order to determine solutions of PDE (1) that are not equivalent by the action of the group, we must calculate the one-dimensional optimal system [2]. Most of the required theory and description of the method can be found in [2].

Motivated by the fact that symmetry reductions for many PDEs are known that are not obtained by using the classical Lie group method, there have been several generalizations of the classical Lie group method for symmetry reductions. The notion of nonclassical symmetries was first introduced by Bluman and Cole [3] to study the symmetry reductions of the heat equation. Since then, a great number of papers have been devoted to the study of nonclassical symmetries of nonlinear PDEs in both one and several dimensions (see, e.g., [4–7]).

Recently there have been some doubts about the efficiency of the nonclassical or conditional symmetries in order to obtain new exact solutions of some classes of nonlinear partial differential equations. In [8], Zhdanov and Lahno stated that, for parabolic PDEs, the nonclassical method seems inefficient in the sense that, once new nonclassical symmetries have been obtained, performing the symmetry reductions gives rise to invariant solutions that do in fact correspond to Lie symmetries. The aim of this paper is to prove that the nonclassical method applied to (1) gives rise to *new* characteristic solutions of (1) which are not group-invariant and, consequently, cannot be obtained by Lie classical symmetries. This result is a counterexample of the statement done in [8].

An ODE is said to possess the Painlevé property (PP) when all the movable singularities are simple poles [9–11]. In [12], Estévez and Gordoa developed a method for identifying the nonclassical symmetries of PDEs using the singular manifold method (SMM) based on the PP as a tool. They propose the following conjecture: "The singular manifold method allows us to identify the nonclassical symmetries that reduce the original equation to an ODE with the Painlevé property."

In [13], Cerveró and Zurrón have analyzed the nonclassical symmetries with $\tau \neq 0$ as well as the contact symmetries of (1) with $\beta = \alpha_3 = 0$, $\alpha_4 = 1$, $\alpha_2 = \lambda$, $\alpha_5 = 6$, and $\alpha_1 = 6\lambda + 5\gamma$. In this work, the Galilean invariance of the equation was shown and the solutions arising from the related ODE were classified in terms of the physical parameters. Based on the fact that all the equations studied up to 1996 have shown a direct correspondence between the nonclassical method and the SMM, they claim, by using SMM, that the *only* nonclassical symmetries admitted by (1) are translations. Nevertheless, for (1), we have derived *two new* nonclassical symmetries for which the corresponding associated similarity reductions leads to two different ODEs of Painlevé type.

2. LIE SYMMETRIES AND OPTIMAL SYSTEMS

We consider the classical Lie group symmetry analysis of (1). Invariance of (1) under a Lie group of point transformations, with infinitesimal generator

$$V = \xi(x,t,u)\,\partial_x + \tau(x,t,u)\,\partial_t + \eta(x,t,u)\,\partial_u, \tag{2}$$

leads to a set of ten determining equations for the infinitesimals $\xi(x,t,u)$, $\tau(x,t,u)$, and $\eta(x,t,u)$, by using the MACSYMA program `symmgrp.max` [14]. The solutions of this system depend on the parameters of the equation. For general α_1, α_2, α_3, α_5, and β, the only symmetries admitted by (1) are the group of space and time translations, which are defined by the infinitesimal generators

$$V_1 = \partial_x, \qquad V_2 = \partial_t.$$

In Table 1, we list the only cases for which (1) has extra symmetries.

For the sake of completeness, we next provide the generators of the nontrivial one-dimensional optimal system, similarity variables and similarity solutions and the corresponding reduced equations (in Table 2), where $\lambda, \mu, \gamma \in \mathbb{R}^*$ are arbitrary. In Table 2, for $i = 0, 2.2$, we have reduced

Table 1. Each row shows the parameters and the extra symmetries, where $k = \alpha_2/16$.

i	Values	V_3^i	V_4^i	V_5^i
1	$\alpha_1 = \alpha_2 = 0$, $\alpha_5 \neq 0$	$(x + 3\beta t)\,\partial_x + 4t\partial_t - 2\left(u + \frac{\alpha_3}{\alpha_5}\right)\partial_u$		
2	$\alpha_1 \neq 0,\ \alpha_5 = 0$	$\alpha_1 t\partial_x + \partial_u$		
3	$\alpha_1 \neq 0$, $\alpha_2 = \alpha_3 = \alpha_5 = 0$	$\alpha_1 t\partial_x + \partial_u$	$(x + 3\beta t)\,\partial_x + 4t\frac{\partial}{\partial t} - 3u\partial_u$	
4	$\alpha_1 = \alpha_5 = 0$, $8\alpha_3 - 3\alpha_2^2 \neq 0$	$u\partial_u$	$\alpha(x,t)\partial_u$	
5	$\alpha_1 = \alpha_5 = 0$, $8\alpha_3 - 3\alpha_2^2 = 0$	$u\partial_u$	$\frac{1}{4}\left[x + 3\left(\beta - \alpha_2^2 k\right)t\right]\partial_x + t\frac{\partial}{\partial t} + k\left[\beta t - x\right]u\partial_u$	$\alpha(x,t)\partial_u$

Table 2. Each row shows the infinitesimal generators U_i of the optimal systems, as well as their similarity variables z_i and similarity solutions u_i, where $\delta_1 = (3/4)\alpha_2^2 k - \beta$, $\delta_2 = \alpha_2^2 k - \beta$, $\rho = \beta - \lambda$.

i	U_i	z_i	u_i	ODE_i
0	$\lambda V_1 + V_2$	$x - \lambda t$	$h(z)$	$h''' + \alpha_2 h'' + \alpha_3 h' + \rho h + \frac{\alpha_1}{2}h^2 + \alpha_5 hh' = k$
1	V_3^1	$(x - \beta t)t^{-1/4}$	$t^{-1/2}h(z) - \frac{\alpha_3}{\alpha_5}$	$4y''' - zy + 2\alpha_5 (y')^2 = k_1 z + k_2$
2.1	V_3^2	t	$\frac{x}{\alpha_1 t} + h(z)$	$\alpha_1 th' + \alpha_1 h + \beta = 0$
2.2	$V_2 + V_3^2$	$x - \frac{\alpha_1}{2}t^2$	$h(z) + t$	$h''' + \alpha_2 h'' + \alpha_3 h' + \frac{\alpha_1}{2}h^2 + \beta h + z = k$
3	V_4^3	$(x - \beta t)t^{-1/4}$	$t^{-3/4}h(z)$	$4h'''' + 4\alpha_1 hh' - zh' - 3h = 0$
4	$\lambda V_1 + V_2 + \gamma V_3^4$	$x - \lambda t$	$h(z)\exp(\gamma t)$	$h'''' + \alpha_2 h''' + \alpha_3 h'' + \rho h' + \gamma h = 0$
5	$\mu V_3^5 + V_4^5$	$(x + \delta_2 t)t^{-1/4}$	$h(z)t^{\mu}e^{-4k(x+\delta_1 t)}$	$4h'''' - zh' + 4\mu h = 0$

the order by integrating once with respect to z; for $i = 1$, by means of the change of variables: $h(z) = y'(z)$, we have reduced the order by integrating twice with respect to z.

ODE_{21} can be easily integrated and we find the following exact solution of (1):

$$u = \frac{x}{\alpha_1 t} + \frac{k}{t} - \frac{\beta}{\alpha_1}.$$

The linear equation ODE_4, which can be transformed into the Laguerre-Forsyth canonical form, $h'''' + a_1 h' + a_0 h = 0$ (see, e.g., [7]), has a symmetry algebra of maximal order 8 for $\beta - \lambda = \alpha_2^3/16$ and $\gamma = \alpha_2^4/256$.

3. NONCLASSICAL SYMMETRIES

The basic idea of the method is that the PDE (1) is augmented with the invariance surface condition

$$\Phi \equiv \xi\frac{\partial u}{\partial x} + \tau\frac{\partial u}{\partial t} - \eta = 0, \tag{3}$$

which is associated to the vector field (2). By requiring that both (1) and (3) are invariant under the transformation with infinitesimal generator (2), an overdetermined nonlinear system of equations for the infinitesimals $\xi(x,t,u)$, $\tau(x,t,u)$, and $\eta(x,t,u)$ is obtained. We can distinguish two different cases.

In the case $\tau \neq 0$, the nonclassical method applied to (1) gives rise only to the classical symmetries.

In the case $\tau = 0$, without loss of generality, we may set $\xi = 1$ and the determining equation for the infinitesimal η is

$$\begin{gathered}
\alpha_5\eta_{xx}u + \alpha_1\eta_x u + \alpha_5\eta^2\eta_{uu}u + 2\alpha_5\eta\eta_{ux}u\eta_{xxxx} + \alpha_2\eta_{xxx} + 4\eta\eta_{uu}\eta_{xx} + 4\eta_{ux}\eta_{xx} \\
+\alpha_3\eta_{xx} + 3\eta_{uu}\left(\eta_x\right)^2 + 6\eta^2\eta_{uuu}\eta_x + 12\eta\eta_{uux}\eta_x + 10\eta\eta_u\eta_{uu}\eta_x + 3\alpha_2\eta\eta_{uu}\eta_x + 6\eta_{uxx}\eta_x \\
+4\eta_u\eta_{ux}\eta_x + 3\alpha_2\eta_{ux}\eta_x + 3\alpha_5\eta\eta_x + b\eta_x + \eta^4\eta_{uuuu} + 4\eta^3\eta_{uuux} + 6\eta^3\eta_u\eta_{uuu} \\
+\alpha_2\eta^3\eta_{uuu} + 6\eta^2\eta_{uuxx} + 12\eta^2\eta_u\eta_{uux} + 3\alpha_2\eta^2\eta_{uux} + 4\eta^3\left(\eta_{uu}\right)^2 + 12\eta^2\eta_{ux}\eta_{uu} \\
+7\eta^2\left(\eta_u\right)^2\eta_{uu} + 3\alpha_2\eta^2\eta_u\eta_{uu} + \alpha_3\eta^2\eta_{uu} + 4\eta\eta_{uxxx} + 6\eta\eta_u\eta_{uxx} + 3\alpha_2\eta\eta_{uxx} + 8\eta\left(\eta_{ux}\right)^2 \\
+4\eta\left(\eta_u\right)^2\eta_{ux} + 3\alpha_2\eta\eta_u\eta_{ux} + 2\alpha_3\eta\eta_{ux} + 2\alpha_5\eta^2\eta_u + \eta_t + \alpha_1\eta^2 = 0.
\end{gathered} \tag{4}$$

The complexity of this equation appears for many $\tau = 0$ symmetries [6]; this is the reason why we cannot solve (4) in general. Thus we proceed, by making an ansatz on the form of $\eta(x,t,u)$ to solve (4). Due to the invariance under temporal and spatial translations, we take $t + t_0 = t$ and $x + x_0 = x$ without loss of generality.

In [13], they have obtained, by using SMM, that the *only* nonclassical symmetries admitted by (1) are *translations*. In the following, we compare these results with our results by using the nonclassical method. In Table 3, we list *new nonclassical* generators $\eta = \eta(x,t)$ corresponding to specific sets of values for α_j, $j = 1,2,3,5$, and β, the corresponding similarity solutions and the corresponding ODEs. By solving these ODEs, we obtain the following exact special solutions of (1):

$$u_1 = \frac{x}{\alpha_1 t} - \frac{\alpha_5 \ln t}{\alpha_1^2 t} + \frac{k}{t} - \frac{\beta}{\alpha_1}, \qquad u_2 = \frac{x^2}{6\alpha_5 t} + \frac{k}{t^{1/3}} - \frac{2\beta x - \beta^2 t + 6\alpha_3}{6\alpha_5}.$$

Table 3. Each row shows the infinitesimals, similarity solutions and ODE$_i$.

i	Values	$\eta(x,t)$	u_i	ODE$_i$
1	$\alpha_1 \neq 0$	$\dfrac{1}{\alpha_1 t}$	$\dfrac{x}{\alpha_1 t} + h(t)$	$\alpha_1^2 t^2 h' + \alpha_1^2 t h + \alpha_1\beta t + \alpha_5 = 0$
2	$\alpha_1 = 0,\ \alpha_5 \neq 0$	$\dfrac{x}{3\alpha_5 t} - \dfrac{\beta}{3\alpha_5}$	$\dfrac{x^2}{6\alpha_5 t} - \dfrac{\beta x}{3\alpha_5} + h(t)$	$9\alpha_5 t h' + 3\alpha_5 h - 2\beta^2 t + 3\alpha_3 = 0$

We make the following remarks.

- For $\alpha_1 \neq 0$ and $\alpha_5 \neq 0$, (1) does not admit any classical symmetry but translations. Consequently u_1, which *is not* a travelling wave reduction, cannot be obtained by Lie classical symmetries. If $\alpha_5 = 0$ and $\beta = 0$, the solution u_1 blows up in finite time.
- For $\alpha_5 \neq 0$ and $\alpha_2 \neq 0$ and $\alpha_1 = 0$, u_2 cannot be obtained by Lie classical symmetries. If $\beta = \alpha_3 = k = 0$, the solution u_2 blows up in finite time.

Therefore, for (1), by using the nonclassical method, we have obtained *two new* symmetries that lead to ODEs which satisfy the PP. These symmetries were not obtained in [13] by using the SMM. Therefore, for (1), the nonclassical method is more general than the SMM and the SMM does not identify all the nonclassical symmetries that reduce the equation to ODEs with the PP.

4. CONCLUDING REMARKS

In this paper, we have seen a classification of symmetry reductions of a dissipation-modified Korteweg-de Vries (1) using the classical Lie method of infinitesimals.

We have proved that for (1), the nonclassical method is susceptible to obtain symmetry reductions which are unobtainable by using the Lie classical method and the exact solutions obtained are not group invariant solutions. Consequently, in contradiction to the statement in [8], we have proved that (1) is an equation for which the nonclassical method with $\tau = 0$ is "efficient".

We have compared the symmetry reductions of this equation by using the nonclassical method with those derived in [13] by using the SMM. For this equation, we have derived two new nonclassical symmetries that reduce the equation to ODEs with the Painlevé property and were not obtained in [13] by the SMM. Therefore, for this equation, the nonclassical method and the SMM lead to different reductions.

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