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CLASSICAL SYMMETRY REDUCTIONS OF THE SCHWARZ–KORTEWEG–DE VRIES EQUATION IN 2+1 DIMENSIONS

M. L. Gandarias,* M. S. Bruzón,* and J. Ramirez*

Classical reductions of a (2+1)*-dimensional integrable Schwarz–Korteweg–de Vries equation are classified. These reductions to systems of partial differential equations in* 1+1 *dimensions admit symmetries that lead to further reductions, i.e., to systems of ordinary differential equations. All these systems have been reduced to second-order ordinary differential equations. We present some particular solutions involving two arbitrary functions.*

Keywords: partial differential equations, Lie symmetries

1. Introduction

There is much current interest in integrable (2+1)-dimensional equations, i.e., equations with two spatial variables and one temporal variable. In this paper, we consider the (2+1)-dimensional integrable generalization of the Schwarz–Korteweg–de Vries (SKdV) equation

$$W_t + \frac{1}{4}W_{xxz} - \frac{W_x W_{xz}}{2W} - \frac{W_{xx}W_z}{4W} + \frac{W_x^2 W_z}{2W^2} - \frac{W_x}{8}\left(\partial_x^{-1}\left(\frac{W_x^2}{W^2}\right)\right)_z = 0, \tag{1}$$

where $\partial^{-1}f = \int f\,dx$. This equation was recently derived by Toda and Yu [1] using the Calogero manner [2]. Although this equation arises in a nonlocal form, it can be written using the transformations

$$W = \phi_x, \qquad \phi = e^{\psi}, \qquad \psi_x = u, \qquad \psi_t = v \tag{2}$$

as

$$\begin{aligned} &4u^2 v_x - 4uu_x v + u^2 u_{xxz} - uu_{xx}u_z - 3uu_x u_{xz} + 3u_x^2 u_z - u^4 u_z = 0,\\ &u_t - v_x = 0. \end{aligned} \tag{3}$$

The machinery of the Lie group theory provides a systematic method for searching for group-invariant solutions. For systems of partial differential equations (PDEs) with 2+1 independent variables, such as system (3), a single group reduction transforms the original system into another system in 1+1 dimensions. But if the system in 1+1 dimensions admits further symmetries, it can be fully reduced to a system of ordinary differential equations (ODEs) by the classical Lie symmetry reductions. All these systems of ODEs can be reduced to second-order ordinary differential equations. In addition, a certain class of solutions can be derived from these ODEs. Most of the required theory and a description of the method can be found in [3], [4].

*Departamento de Matemáticas, Universidad de Cádiz, Cádiz, Spain, e-mail: marialuz.gandarias@uca.es; santos.bruzon@uca.es; pepe.ramirez@uca.es.

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2. Lie symmetries

The classical method for finding symmetry reductions of systems of PDEs is the Lie group method of infinitesimal transformations. Although this method is entirely algorithmic, it often involves a large amount of tedious algebra. In this paper, we use the MACSYMA program symmgrp.max [5] to generate associated determining equations. To apply the classical method to (3), we seek fields of the form

$$\mathbf{v} = X\frac{\partial}{\partial x} + Z\frac{\partial}{\partial z} + T\frac{\partial}{\partial t} + U\frac{\partial}{\partial u} + V\frac{\partial}{\partial v} \tag{4}$$

that leave the set of solutions of (3) invariant. The machinery of Lie group theory provides a systematic method for seeking these special invariant solutions. Having determined the infinitesimals, the symmetry variables are found by solving the invariant surface conditions

$$\Phi_1 \equiv X\frac{\partial u}{\partial x} + Z\frac{\partial u}{\partial z} + T\frac{\partial u}{\partial t} - U = 0, \qquad \Phi_2 \equiv X\frac{\partial v}{\partial x} + Z\frac{\partial u}{\partial z} + T\frac{\partial v}{\partial t} - V = 0. \tag{5}$$

Applying the classical method to system (3) yields a system of equations that leads to a four-parameter Lie group. Associated with this Lie group, we have a Lie algebra that can be represented by the generators

$$\mathbf{v}_1 = \frac{\partial}{\partial t}, \qquad \mathbf{v}_2 = \frac{\partial}{\partial z},$$

$$\mathbf{v}_3 = x\frac{\partial}{\partial x} - 2z\frac{\partial}{\partial z} - u\frac{\partial}{\partial u}, \qquad \mathbf{v}_4 = t\frac{\partial}{\partial t} + z\frac{\partial}{\partial z} - v\frac{\partial}{\partial v}$$

and the infinite-dimensional generator

$$\mathbf{v}_\alpha = \alpha(t)\frac{\partial}{\partial x} - \alpha'(t)u\frac{\partial}{\partial v}.$$

To find all invariant solutions with respect to s-dimensional subalgebras, it suffices to construct invariant solutions for the optimal system of order s. The set of invariant solutions thus obtained is called an *optimal system of invariant solutions*. We only consider one-parameter subgroups. The problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of subalgebras. Although this latter problem can still be quite complicated in general, this classification problem for one-dimensional subalgebras is essentially the same as the problem of classifying the adjoint representation orbits. The construction of the one-dimensional optimal system appears in [4] using a global matrix for the adjoint transformation. Olver [3] uses a slightly different technique, which we follow. To construct the one-dimensional optimal system, we construct the commutator table (Table 1) and the adjoint table (Table 2), which shows the separate adjoint actions of each element in $\mathbf{v}_i$, $i = 1, \dots, 4$, as it acts on all other elements. It is easily constructed by summing the Lie series.

Table 1

	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$
$\mathbf{v}_1$	0	0	0	$\mathbf{v}_2$
$\mathbf{v}_2$	0	0	$-2\mathbf{v}_2$	$\mathbf{v}_2$
$\mathbf{v}_3$	0	$2\mathbf{v}_2$	0	0
$\mathbf{v}_4$	$-\mathbf{v}_1$	$-\mathbf{v}_2$	0	0

Table 2

	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$
$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4 - \varepsilon\mathbf{v}_1$
$\mathbf{v}_2$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3 + 2\varepsilon\mathbf{v}_2$	$\mathbf{v}_4 - \varepsilon\mathbf{v}_2$
$\mathbf{v}_3$	$\mathbf{v}_1$	$e^{-2\varepsilon}\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$
$\mathbf{v}_4$	$e^{\varepsilon}\mathbf{v}_1$	$e^{\varepsilon}\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$

We then consider a general element in a basis of $\mathcal{L}_r$ and ask whether it can be transformed into a new element of a simpler form by iteratively subjecting it to various adjoint transformations. The corresponding generators of the optimal system of subalgebras are

$$\langle \mathbf{v}_1 \rangle, \qquad \langle \mu\mathbf{v}_3 + \mathbf{v}_4 + \mathbf{v}_\alpha \rangle, \qquad \left\langle \mu\mathbf{v}_2 + \frac{1}{2}\mathbf{v}_3 + \mathbf{v}_4 \right\rangle,$$
$$\langle \mu\mathbf{v}_1 + \mathbf{v}_3 \rangle, \qquad \langle \mu\mathbf{v}_1 + \mathbf{v}_2 \rangle, \qquad \langle \mu\mathbf{v}_3 \rangle, \qquad \langle \mathbf{v}_4 \rangle,$$

where $\mu \in \mathbb{R}^*$ is arbitrary. In what follows, we list the corresponding similarity variables and similarity solutions and the systems of PDEs obtained when system (3) is reduced using $\{\mathbf{u}_i\}$, $i = 1, \dots, 6$. These generators are obtained by adding the infinite-dimensional generator $\mathbf{v}_\alpha$ to the generators of the optimal system.

In Table 3, we list the nontrivial optimal system $\{\mathbf{u}_i\}$ with $i = 1, \dots, 6$, where $\mu \in \mathbb{R}^*$ is arbitrary. We also list the corresponding similarity variables and similarity solutions.

Table 3

	$\mathbf{u}_i$	z_1	z_2	u	v
1	$\mu\mathbf{v}_3 + \mathbf{v}_4 + \mathbf{v}_\alpha$	$xt^{-\mu} - \beta$	$zt^{2\mu-1}$	$t^{-\mu}f$	$t^{-1}(g - \gamma f)$
2	$\mu\mathbf{v}_2 + \frac{1}{2}\mathbf{v}_3 + \mathbf{v}_4 + \mathbf{v}_\alpha$	$xt^{-1/2} - \delta$	$t^{1/2}e^{-z}$	$t^{1/2}f$	$t^{-1}(g - \zeta f)$
3	$\mu\mathbf{v}_1 + \mathbf{v}_3 + \mathbf{v}_\alpha$	$xe^{-t/\mu} - \dfrac{\eta}{\mu}$	$ze^{2t/\mu}$	$e^{-t/\mu}f$	$g - \dfrac{\beta f}{\mu}$
4	$\mu\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_\alpha$	$x - \dfrac{\kappa}{\mu}$	$\mu z - t$	f	$g - \dfrac{\alpha f}{\mu}$
5	$\mathbf{v}_3 + \mathbf{v}_\alpha$	t	$(x+\alpha)^2 z$	$z^{1/2}f$	$z^{1/2}\alpha' f + g$
6	$\mathbf{v}_4 + \mathbf{v}_\alpha$	$x - \rho$	$\dfrac{z}{t}$	f	$\dfrac{g - f\alpha}{t}$

In Table 3,

$$\beta = \int t^{-(\mu+1)}\alpha(t)\,dt, \qquad \gamma(t) = \int t^{-\mu}\alpha'(t)\,dt, \qquad \delta(t) = \int t^{-3/2}\alpha(t)\,dt,$$
$$\zeta(t) = \int t^{-1/2}\alpha'(t)\,dt, \qquad \eta(t) = \int e^{-t/\mu}\alpha(t)\,dt, \qquad \kappa(t) = \int \alpha(t)\,dt,$$
$$\rho(t) = \int t^{-1}\alpha(t)\,dt, \qquad f = f(z_1, z_2), \qquad g = g(z_1, z_2).$$

Table 4

	$E_i^1(f,g,f',g',f'')=0$	$E_i^2(f,g,f',g',g'')=0$
S_1	$4f^2g_{z_1}-4ff_{z_1}g-ff_{z_1z_1}f_{z_2}+3f_{z_1}^2f_{z_2}-$ $-f^4f_{z_2}+f^2f_{z_1z_1z_2}-3ff_{z_1}f_{z_1z_2}=0$	$(2\mu-1)z_2f_{z_2}-g_{z_1}-$ $-\mu z_1f_{z_1}-\mu f=0$
S_2	$z_2\big(ff_{z_1z_1}f_{z_2}-3(f_{z_1})^2f_{z_2}+f^4f_{z_2}-f^2f_{z_1z_1z_2}+$ $+3ff_{z_1}f_{z_1z_2}\big)+4f^2g_{z_1}-4ff_{z_1}g=0$	$z_2f_{z_2}-z_1f_{z_1}-$ $-2g_{z_1}-f=0$
S_3	$4f^2g_{z_1}-4ff_{z_1}g-ff_{z_1z_1}f_{z_2}+3(f_{z_1})^2f_{z_2}-$ $-f^4f_{z_2}+f^2f_{z_1z_1z_2}-3ff_{z_1}f_{z_1z_2}=0$	$2z_2f_{z_2}-z_1f_{z_1}-$ $-\mu g_{z_1}-f=0$
S_4	$\mu(-ff_{z_1z_1}f_{z_2}+3f_{z_1}^2f_{z_2}-f^4f_{z_2}+f^2f_{z_1z_1z_2}-$ $-3ff_{z_1}f_{z_1z_2})+4f^2g_{z_1}-4ff_{z_1}g=0$	$g_{z_1}+f_{z_2}=0$
S_5	$z_2^3(8f^2f_{z_2z_2z_2}-32ff_{z_2}f_{z_2z_2}+24f_{z_2}^3)+$ $+z_2(f^2f_{z_2}-f^5)g+$ $+z_2^2(20f^2f_{z_2z_2}-28ff_{z_2}^2-2f^4f_{z_2})+$ $+16z_2^{3/2}(f^2g_{z_2}-ff_{z_2}g)=0$	$f_{z_1}-2z_2^{1/2}g_{z_2}=0$
S_6	$4f^2g_{z_1}-4ff_{z_1}g+$ $+(-ff_{z_1z_1}+3f_{z_1}^2-ff_{z_1}-f^4)f_{z_2}+$ $+f^2f_{z_1z_1z_2}+(f^2-3ff_{z_1})f_{z_1z_2}=0$	$z_2f_{z_2}+g_{z_1}=0$

In Table 4, we list the systems of PDEs obtained when system (3) is reduced using $\{\mathbf{u}_i\}$, $i=1,\ldots,6$. In what follows, the prime denotes the derivative, $'\equiv d/dz$.

3. Invariance analysis of (1+1)-dimensional systems

In several cases, the reduced systems of 1+1 PDEs admit symmetries that lead to further reductions to systems of ODEs. We again use the Lie group theory techniques. The system S_i, $i=1,\ldots,6$, admits the symmetries

$$\mathrm{S}_1:\quad \mathbf{v}_{11}=\frac{\partial}{\partial z_1}-\mu f\frac{\partial}{\partial g},\qquad \mathbf{v}_{12}=-z_1\frac{\partial}{\partial z_1}+2z_2\frac{\partial}{\partial z_2}+f\frac{\partial}{\partial f},$$

$$\mathrm{S}_2:\quad \mathbf{v}_{21}=z_2\frac{\partial}{\partial z_2},\qquad \mathbf{v}_{22}=2\frac{\partial}{\partial z_1}-f\frac{\partial}{\partial g}\frac{\partial}{\partial f},$$

$$\mathrm{S}_3:\quad \mathbf{v}_{31}=a\frac{\partial}{\partial z_1}-f\frac{\partial}{\partial g},\qquad \mathbf{v}_{32}=z_1\frac{\partial}{\partial z_1}-2z_2\frac{\partial}{\partial z_2}-f\frac{\partial}{\partial f},$$

$$\mathrm{S}_4:\quad \mathbf{v}_{41}=\frac{\partial}{\partial z_1},\qquad \mathbf{v}_{\beta}=\beta(z_2)\frac{\partial}{\partial z_2}-\beta'(z_2)g\frac{\partial}{\partial g},$$

$$\mathrm{S}_5:\quad \mathbf{v}_{51}=\frac{\partial}{\partial z_1},\qquad \mathbf{v}_{52}=2z_1\frac{\partial}{\partial z_1}+2z_2\frac{\partial}{\partial z_2}-f\frac{\partial}{\partial f}-2g\frac{\partial}{\partial g}.$$

1. For system S_1, using $c\mathbf{v}_{11} + \mathbf{v}_{12}$, we obtain the similarity variable and similarity solutions

$$w = -\sqrt{z_2}\,(z_1 - c), \qquad f = \sqrt{z_2}\,h(w), \qquad g = -c\mu\sqrt{z_2}\,h(w) + k(w)$$

and the system of ODEs

$$2k_w - wh_w - h = 0,$$
$$w(h^2 h_{www} - 4hh_w h_{ww} + 3h_w^3 - h^4 h_w) - 8h^2 k_w + 8hh_w k + 2h^2 h_{ww} - 3hh_w^2 - h^5 = 0.$$

This system can be reduced to the second-order ODE

$$w^2\big(4h^2 + h^4 + 3(h')^2 - 2hh''\big) - 2k_1 h^2 = 0.$$

Setting $k_1 = 0$, we obtain

$$h = \frac{2}{c_2 \pm \sqrt{c_2{}^2 + 1}\,\sin 2(w + c_1)}. \tag{6}$$

2. For system S_2, using $c\mathbf{v}_{21} + \mathbf{v}_{22}$, we obtain the similarity variable and similarity solutions

$$w = \frac{e^{cz_1}}{z_2^2}, \qquad f = h(w), \qquad g = k(w) - \frac{z_1 f}{2}$$

and the system of ODEs

$$ck_w + h_w = 0,$$
$$c^2 w^3 (h^2 h_{www} - 4hh_w h_{ww} + 3h_w^3) + c^2 w^2 (3h^2 h_{ww} - 4hh_w^2) +$$
$$+ cw(2h^2 k_w - 2hh_w k) - wh^4 h_w + c^2 wh^2 h_w - h^3 = 0.$$

This system can be reduced to the second-order ODE

$$h^4 + 2h^2 \log(w) + 3c^2 w^2 (h')^2 - 2ch\big(2c_1 + cw(h' + wh'')\big) - 2k_1 h^2 = 0.$$

Setting $h = 1/Y$ and $w = e^z$, we obtain

$$Y'' - \frac{(Y')^2}{2Y} - \frac{2c_1 Y^2}{c} + \frac{zY}{c^2} - \frac{k_2 Y}{c^2} + \frac{1}{2c^2 Y} = 0.$$

Following the analysis in [6], we can write the solutions for $c = 1$, $k_2 = 0$, and $k_1 \neq 0$ as

$$k_1 Y = V' + V^2 + \frac{1}{2}z$$

in terms of the second Painlevé equation (PII)

$$V'' = 2V^3 + zV - k_1 - \frac{1}{2}.$$

3. For system S_3, using $c\mathbf{v}_{31} + \mathbf{v}_{32}$, we obtain the similarity variable and similarity solutions

$$w = z_2^{1/2}(z_1 + c\mu), \qquad f = hz_2^{1/2}, \qquad g = ch(w)z_2^{1/2} + k(w)$$

and the system of ODEs

$$
\begin{aligned}
&k_w = 0,\\
&w(h^2 h_{www} - 4hh_w h_{ww} + 3h_w^3 - h^4 h_w) + 8h^2 k_w - 8hh_w k + 2h^2 h_{ww} - 3hh_w^2 - h^5 = 0.
\end{aligned} \tag{7}
$$

This system can be reduced to the second-order ODE

$$wh^4 + w(h')^2 - h(4c + h' + wh'') - k_1 h^3 = 0.$$

Integrating once with respect to w for $c = 0$ and $c_3 = 0$ leads to the Painlevé III equations

$$h'' - \frac{h'^2}{h} + \frac{h'}{w} - h^3 = 0,$$

and a solution is

$$h = \pm \frac{1}{w(c_1 - \log w)}. \tag{8}$$

4. For system S_4, using $c\mathbf{v}_{41} + \mathbf{v}_\beta$, we obtain the similarity variable and similarity solutions

$$w = z_1 - c\int \frac{dz_2}{\beta(z_2)}, \qquad f = h, \qquad g = \frac{1}{\beta(z_2)}k(w)$$

and the system of ODEs

$$
\begin{aligned}
&k_w - ch_w = 0,\\
&\mu c(4hh_w h_{ww} - h^2 h_{www} - 3h_w^3 + h^4 h_w) + 4h^2 k_w - 4hh_w k = 0.
\end{aligned} \tag{9}
$$

This system can be reduced to the second-order ODE

$$2dh + c\mu\big(h^4 + (h')^2 - hh''\big) - k_1 h^3 = 0.$$

For $d = 0$, a solution is

$$h = \pm \frac{d_2}{1 + e^{d_2(w - d_1)}}, \tag{10}$$

where $k_1 - c\mu d_2 = 0$.

5. For system S_5, using $c\mathbf{v}_{51} + \mathbf{v}_{52}$, we obtain the similarity variable and similarity solutions

$$w = \frac{z_1 + c}{z_2}, \qquad f = z_2^{-1/2}h(w), \qquad g = z_2^{-1}k(w)$$

and the system of ODEs

$$
\begin{aligned}
&2wk_w + 2k + h_w = 0,\\
&w^3(4h^2 h_{www} - 16hh_w h_{ww} + 12h_w^3) + w^2(12h^2 h_{ww} - 16hh_w^2) +\\
&\qquad + w(8h^2 k_w - 8hh_w k - h^4 h_w + 4h^2 h_w) + 4h^2 k = 0.
\end{aligned}
$$

This system can be reduced to the second-order ODE

$$\frac{2}{w} + \frac{h^2}{2} + \frac{6w^2(h')^2}{h^2} - \frac{4w(h' + wh'')}{h} + k_1 = 0.$$

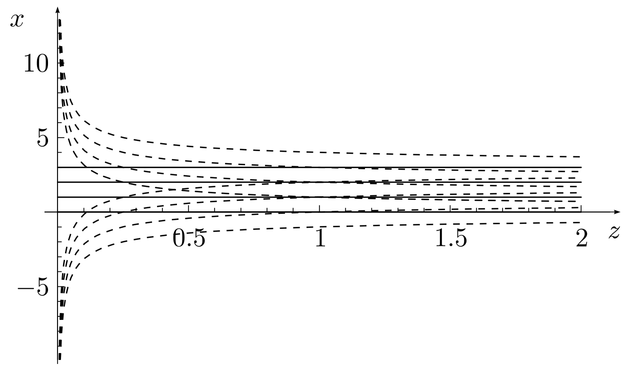


Fig. 1. Graphs of curves for which the solution is singular.

4. About particular solutions

We give a family of solutions of Eq. (1) in this section. To construct these solutions, we use the above solutions for the ODEs together with the corresponding symmetry reductions. From (8), transformations (2), and the corresponding symmetry reductions

$$w = z_2^{1/2}(z_1 + c\mu), \qquad f = hz_2^{1/2}, \qquad g = ch(w)z_2^{1/2} + k(w),$$

$$z_1 = xe^{-t/\mu} - \Omega(t), \qquad z_2 = ze^{2t/\mu},$$

$$u = e^{-t/\mu} f(z_1, z_2), \qquad v = g(z_1, z_2) - \frac{1}{\mu}\int t^{-(\mu+1)}\alpha(t)\, dt f(z_1, z_2),$$

where

$$\Omega(t) = \frac{1}{\mu}\int e^{-t/\mu}\alpha(t)\, dt,$$

we obtain the corresponding family of solutions for the SKdV equation in 2+1 dimensions

$$W = \frac{\Phi(z)}{\left(x - e^{t/\mu}\Omega(t)\right)\left(\log\left(z^{\mu}\left(x - e^{t/\mu}\Omega(t)\right)^{2\mu}\right) + c_1\right)^2}. \tag{11}$$

These solutions can be written as

$$W = \frac{-2\mu^2\Phi(z)}{\left(x - \Lambda(t)\right)\left(\log\left(z^{\mu}\left(x - \Lambda(t)\right)^{2\mu}\right) - c_1\mu\right)^2}. \tag{12}$$

In Fig. 1, with $\Lambda(t) = t$, $\Phi(z) = -1/2$, $c_1 = 0$, and $\mu = 1$, we plot the curves for which solution (12) becomes singular for $t = 0, 1, 2, 3, 4$. In Fig. 2, for the same choice of the arbitrary functions and constants, we can see a plot of solution (12). We can clearly appreciate two of the curves in which the solution is singular. The third one can be seen better in Fig. 3.

From (10), considering transformations (2) and the corresponding symmetry reductions

$$w = z_1 - \Psi(z_2), \qquad f = h, \qquad g = \frac{1}{\beta(z_2)}k(w),$$

$$z_1 = x - \Upsilon(t), \qquad z_2 = \mu z - t,$$

$$u = f(z_1, z_2), \qquad v = g(z_1, z_2) - \frac{1}{\mu}\alpha f(z_1, z_2),$$

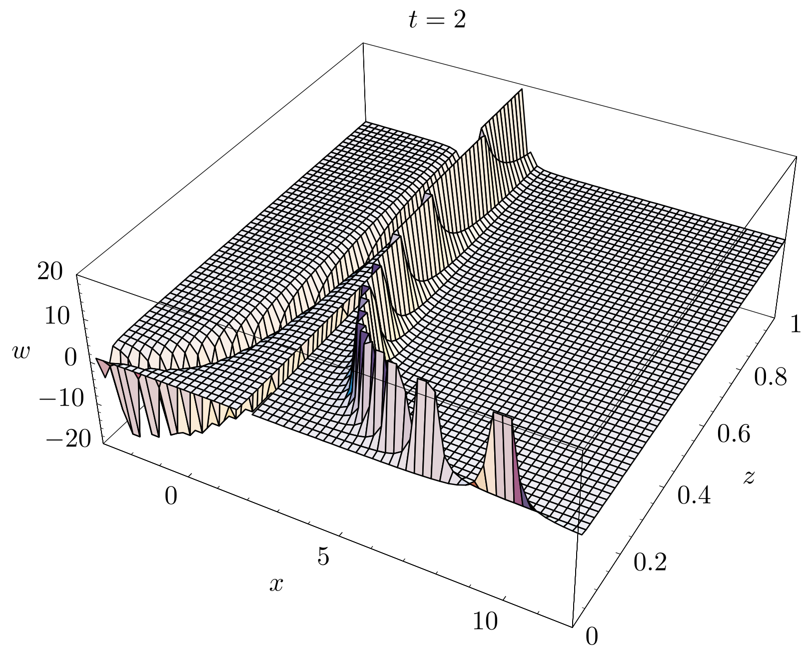


Fig. 2. A solution of SKdV Eq. (1).

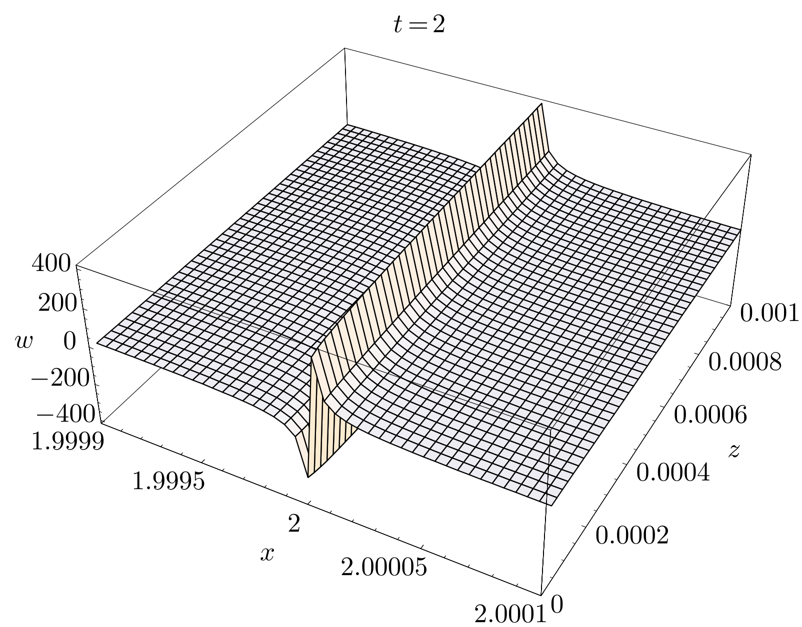


Fig. 3. A solution of SKdV equation (1).

we obtain the corresponding solution for the (SKdV) equation in 2+1 dimensions

$$W = \frac{d_2 \Phi(z)}{2\cosh^2\big(d_2\big(d_1 - x + \Upsilon(t) + \Psi(-t + \mu z)\big)\big)}, \tag{13}$$

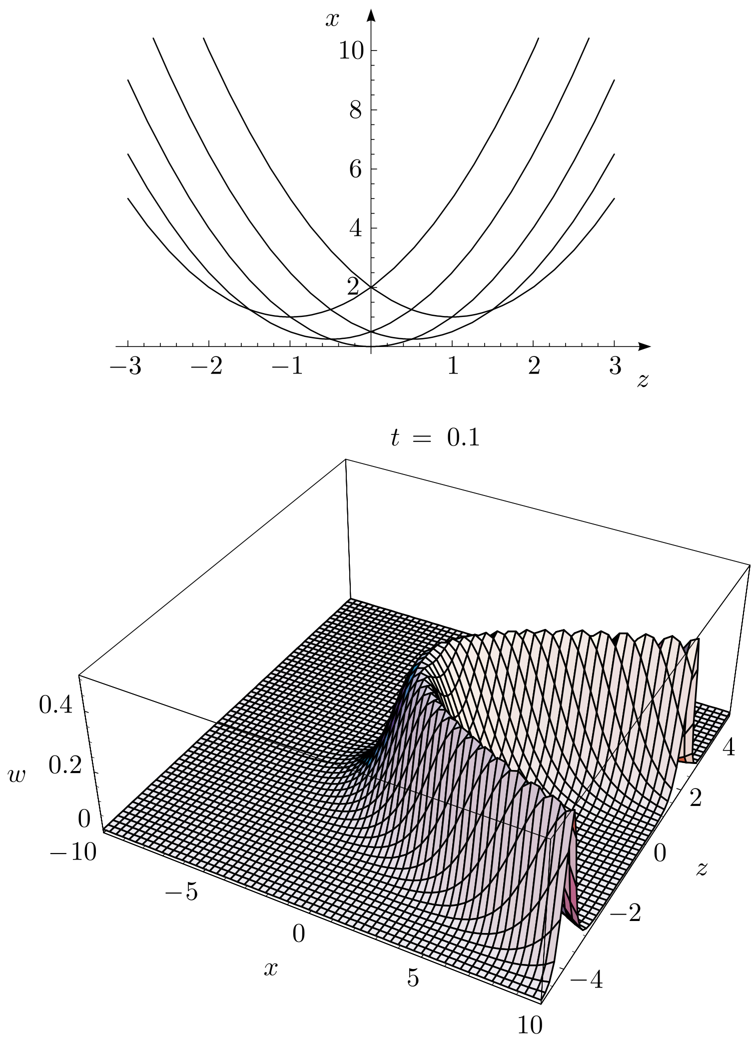


Fig. 4. A solution of SKdV equation (1).

where

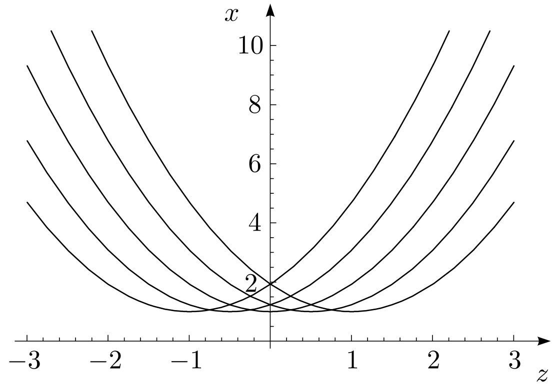
$$\Psi(z_2) = c\int \frac{dz_2}{\beta(z_2)}, \qquad \Upsilon(t) = \frac{1}{\mu}\int \alpha(t)\,dt$$

and $\Phi(z)$ is an arbitrary function. It is interesting that this family of solitonic solutions has a rich structure due to the arbitrary functions $\Upsilon(t)$, $\Psi(z_2)$, and $\Phi(z)$ with $z_2 = \mu z - t$. Solution (13) for $d_2 = -1/2$, $d_1 = 0$, $\Phi(z) = -2$, $\Upsilon(t) = t^2$, and $\Psi(-t + \mu z) = (t + z)^2$ is plotted in Fig. 4. In this figure, we also plot a family of parabolic curves due to the choice of Ψ. These curves represent the time evolution of the maxima of the solitonic solutions.

In Fig. 5, solution (13) for $d_2 = 2$, $d_1 = 1$, $\Phi(z) = \sin^2(z)$, $\Upsilon(t) = 0$, and $\Psi(-t + \mu z) = (t - z)^2$ is plotted. In this figure, we also plot another family of parabolic curves showing the maxima of the solitonic solutions. We can observe different behaviors of the solution due to the choice of $\Phi(z)$.

5. Conclusions

In this paper, we have discussed the (2+1)-dimensional integrable generalization of the SKdV equation. Through this invariance analysis, we obtain a set of six (1+1)-dimensional systems of PDEs. The invariance



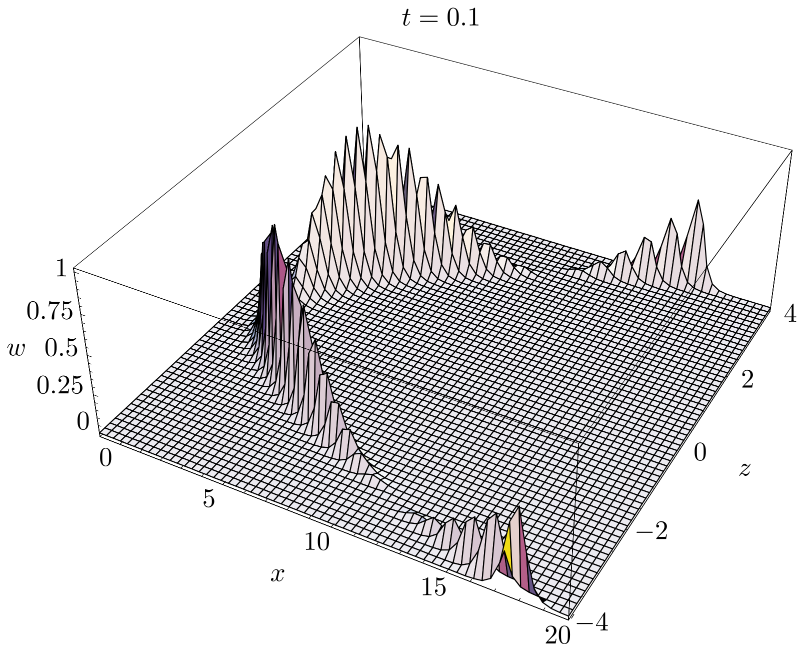


Fig. 5. A solution of SKdV equation (1).

study of these systems leads to a set of systems of ODEs. All these systems have been reduced to second-order ordinary differential equations. We have obtained different families of particular solutions that have a very rich qualitative behavior. We have found that Eq. (1) has unbounded solutions and solitonic solutions.

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