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REDUCTIONS OF THE DISPERSIONLESS KP HIERARCHY

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We present a method for constructing the S-function based on a system of first-order differential equations and use it to analyze reductions of dispersionless integrable hierarchies.

Keywords: hodograph transformations, reductions of dispersionless hierarchies

1. Introduction

Dispersionless integrable systems have awakened more and more interest during the last decade. Some reasons for this increasing interest are their applications to the classification problem in topological field theory [1], the study of systems of hydrodynamic type [2], and the theory of conformal maps [3]. Reductions of these systems were considered in [2], [4].

In this paper, we provide a general scheme for studying the reductions of the dispersionless integrable systems. Our scheme is based on the S-function determination method introduced in [5]. We can thus characterize both the reductions and the hodograph solutions. This scheme is used to study the dispersionless KP (dKP) hierarchy. We also analyze various illustrative examples and obtain large families of solutions of some reductions of the dKP hierarchy.

2. Reductions of the dispersionless KP hierarchy

2.1. Characterization of the reductions. To introduce the dKP hierarchy, we consider a function $z = z(p, \mathbf{t})$ depending on a complex variable p and an infinite set of complex time parameters $\mathbf{t} := (x := t_1, t_2, \dots)$ that admits an expansion of the form

$$z = p + \sum_{n=1}^{\infty} \frac{a_n(\mathbf{t})}{p^n}, \quad p \to \infty. \tag{1}$$

Then the dKP hierarchy [4]–[6] is given by the set of equations

$$\frac{\partial z}{\partial t_n} = \{\Omega_n, z\}, \qquad \Omega_n := (z^n)_+, \quad n \ge 1, \tag{2}$$

where $\{\,\cdot\,,\,\cdot\,\}$ is the usual Poisson bracket defined with respect to the variables p and x and $\Omega_n = (z^n)_+$ denotes the polynomial part of z^n as a function of p. For example,

$$(z)_+ = p, \qquad (z^2)_+ = p^2 + 2a_1, \qquad (z^3)_+ = p^3 + 3pa_1 + 3a_2.$$

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The compatibility conditions for (2),

$$\frac{\partial \Omega_m}{\partial t_n} - \frac{\partial \Omega_n}{\partial t_m} + \{\Omega_m, \Omega_n\} = 0, \quad m \neq n,$$

constitute a hierarchy of nonlinear partial differential equations. For instance, for $m = 3$ and $n = 2$, we obtain the dKP equation

$$(u_{t_3} + 3uu_x)_x = \frac{3}{4} u_{t_2 t_2}, \qquad u := -a_1, \quad y := t_2, \quad t := t_3. \tag{3}$$

It follows [5] that there exists the *S-function* $S = S(z, \mathbf{t})$ such that

$$\frac{\partial S(z)}{\partial t_n} = \Omega_n(p, \mathbf{t}), \quad n \geq 1, \tag{4}$$

$$S(z, \mathbf{t}) = \sum_{n \geq 1} z^n t_n + \sum_{n \geq 1} \frac{S_n(\mathbf{t})}{z^n}, \quad z \to \infty. \tag{5}$$

Setting $n = 1$ in (4) yields

$$p = z + \sum_{n \geq 1} \frac{b_n(\mathbf{t})}{z^n}, \qquad b_n := \frac{\partial S_n}{\partial x}, \tag{6}$$

and it can be proved [5] that the inverted series determines a solution $z = z(p, \mathbf{t})$ of the dKP hierarchy. Conditions (4), which characterize the S-function, constitute a system of compatible Hamilton–Jacobi-type equations, which represents the *semiclassical limit* of the linear system for the wave function of the standard KP hierarchy.

From (5) and (6), it is clear that a function S admitting an expansion of form (5) satisfies (4) iff

$$\left(\frac{\partial S(z)}{\partial t_n} \right)_- = 0, \quad n \geq 1. \tag{7}$$

Hereafter, we assume that S is a function of either z or p and let $S(z)$ and $S(p)$ denote the corresponding functions ($S(z, \mathbf{t}) = S(p(z, \mathbf{t}), \mathbf{t})$). Furthermore, we let $S(p) = S_+(p) + S_-(p)$ denote the decomposition of $S(p)$ in terms of positive and negative powers of p. From (5) and (6), we have

$$S_+(p) = \sum_{n \geq 1} \Omega_n(p, \mathbf{t}) t_n. \tag{8}$$

Hence, Eq. (7) gives

$$\left(\frac{\partial S(p)}{\partial p} \frac{\partial p}{\partial t_n} + \frac{\partial S_-(p)}{\partial t_n} \right)_- = 0, \quad n \geq 1. \tag{9}$$

We now consider N-reductions of the dKP hierarchy for which $z = z(p, \mathbf{t})$ depends on $\mathbf{t}$ only through N functions $\mathbf{u} = (u_1(\mathbf{t}), \ldots, u_N(\mathbf{t}))$. We characterize these reductions in terms of systems of equations for $p = p(z, \mathbf{u})$ of the form

$$\frac{\partial p}{\partial u_i} = R_i(p, \mathbf{u}), \quad i = 1, \ldots, N, \tag{10}$$

or, equivalently, in terms of $z = z(p, \mathbf{u})$,

$$\frac{\partial z}{\partial u_i} + R_i(p, \mathbf{u}) \frac{\partial z}{\partial p} = 0, \quad i = 1, \ldots, N. \tag{11}$$

We assume the following conditions for the functions R_i.

1. The functions R_i are rational functions of p, which have singularities only at N simple poles $p_i = p_i(\mathbf{u})$, $i = 1, \ldots, N$, and vanish at $p = \infty$. Therefore, they admit the expansions

$$R_i(p, \mathbf{u}) = \sum_{j=1}^{N} \frac{r_{ij}(\mathbf{u})}{p - p_j(\mathbf{u})}. \tag{12}$$

2. The functions R_i satisfy the compatibility conditions for (11),

$$\frac{\partial R_i}{\partial u_j} - \frac{\partial R_j}{\partial u_i} + R_j \frac{\partial R_i}{\partial p} - R_i \frac{\partial R_j}{\partial p} = 0, \quad i \neq j. \tag{13}$$

We also assume that $S_-(p)$ depends on $\mathbf{t}$ only through the functions $\mathbf{u} = \mathbf{u}(\mathbf{t})$. Therefore, condition (9) can be written as

$$\left(\frac{\partial S(p)}{\partial p} R_i + \frac{\partial S_-(p)}{\partial u_i} \right)_- = 0. \tag{14}$$

To construct solutions of the reductions, we assume that S satisfies the conditions

$$\frac{\partial S}{\partial p}(p_i) = 0, \quad i = 1, \ldots, N. \tag{15}$$

We now let $E = E(p, \mathbf{u})$ denote any entire function in p satisfying

$$E(p_i, \mathbf{u}) = F_i(\mathbf{u}), \quad i = 1, \ldots, N, \tag{16}$$

where

$$F_i(\mathbf{u}) := \frac{\partial S_-}{\partial p}(p_i).$$

In terms of this function, we find that (14) is equivalent to

$$\frac{\partial S_-(p)}{\partial u_i} + R_i \frac{\partial S_-(p)}{\partial p} = (E R_i)_-. \tag{17}$$

Moreover, using (13) we find that the compatibility conditions for (17) are

$$\frac{\partial (E R_i)_-}{\partial u_j} - \frac{\partial (E R_j)_-}{\partial u_i} + R_j \frac{\partial (E R_i)_-}{\partial p} - R_i \frac{\partial (E R_j)_-}{\partial p} = 0, \quad i \neq j. \tag{18}$$

Taking into account that

$$(E R_j)_- = \sum_{k=1}^{N} \frac{r_{jk} F_k}{p - p_k}, \tag{19}$$

we find that Eqs. (18) constitute a set of consistency conditions for the functions F_j.

Summarizing, if we start with a set of functions $R_i(p, \mathbf{u})$ and $F_i(\mathbf{u})$ $(i = 1, \ldots, N)$ satisfying (13) and (18), we can obtain a solution of the N-reduction of the dKP hierarchy from (11). Moreover, from (15) and (16), we have

$$\frac{\partial S_+}{\partial p}(p_i) + F_i(\mathbf{u}) = 0$$

or, equivalently,

$$\sum_{n=1}^{\infty} v_{in}(\mathbf{u}) t_n + F_i(\mathbf{u}) = 0, \qquad v_{in} := \frac{\partial \Omega_n}{\partial p}(p_i), \quad i = 1, \ldots, N. \tag{20}$$

This system provides implicit relations for the functions $\mathbf{u} = \mathbf{u}(\mathbf{t})$ characterizing the N-reductions. Finally, from (5) and (17), we see that S_- can be obtained recursively. Because of the form of implicit relations (20), we call these solutions the *hodograph solutions*.

2.2. Bourlet integrability. In [7], the inverse problem technique was used to construct S-functions in order to solve the initial value problem for several dispersionless models. Our analysis provides an alternative standpoint for determining S, which is based on systems of differential equations (10) and (17). The S-function is then characterized by a set of *spectral data* $\{p_i(\mathbf{u}), r_{ij}(\mathbf{u}), F_i(\mathbf{u})\colon 1 \le i,\, j \le N\}$. Moreover, from (12) and (19), we find that compatibility conditions (13) and (18) are equivalent to the following consistency conditions for the spectral data:

$$
\begin{aligned}
&r_{ik}\frac{\partial p_k}{\partial u_j} - r_{jk}\frac{\partial p_k}{\partial u_i} = \sum_{l\neq k}\frac{r_{jl}r_{ik} - r_{il}r_{jk}}{p_k - p_l},\\
&\frac{\partial r_{ik}}{\partial u_j} - \frac{\partial r_{jk}}{\partial u_i} = 2\sum_{l\neq k}\frac{r_{jk}r_{il} - r_{ik}r_{jl}}{(p_k - p_l)^2},\\
&r_{ik}\frac{\partial F_k}{\partial u_j} - r_{jk}\frac{\partial F_k}{\partial u_i} = \sum_{l\neq k}\frac{r_{jl}r_{ik} - r_{il}r_{jk}}{(p_k - p_l)^2}(F_k - F_l),
\end{aligned}
\tag{21}
$$

where $i \neq j$. The first two groups of equations in system (21) characterize the reductions of the dKP hierarchy, while the whole system determines the set of hodograph solutions.

It can be shown that system (21) is consistent in the sense that the relations

$$
\begin{aligned}
\frac{\partial}{\partial u_m}\frac{\partial p_k}{\partial u_l} &= \frac{\partial}{\partial u_l}\frac{\partial p_k}{\partial u_m},\\
\frac{\partial}{\partial u_m}\frac{\partial r_{ik}}{\partial u_l} &= \frac{\partial}{\partial u_l}\frac{\partial r_{ik}}{\partial u_m},\\
\frac{\partial}{\partial u_m}\frac{\partial F_k}{\partial u_l} &= \frac{\partial}{\partial u_l}\frac{\partial F_k}{\partial u_m}
\end{aligned}
$$

hold by virtue of Eqs. (21). We can also see that system (21) is equivalent to the system

$$
\begin{aligned}
&\frac{\partial p_k}{\partial u_i} = \frac{1}{r_{s_k k}}\left(r_{ik}\frac{\partial p_k}{\partial u_{s_k}} - \sum_{l\neq k}\frac{r_{s_k l}r_{ik} - r_{il}r_{s_k k}}{p_k - p_l}\right), \quad i < s_k,\\
&\frac{\partial p_k}{\partial u_i} = -\frac{1}{r_{s_k k}}\sum_{l\neq k}\frac{r_{s_k l}r_{ik} - r_{il}r_{s_k k}}{p_k - p_l}, \quad i > s_k,\\
&\frac{\partial F_k}{\partial u_i} = \frac{1}{r_{s_k k}}\left(r_{ik}\frac{\partial F_k}{\partial u_{s_k}} - \sum_{l\neq k}\frac{r_{s_k l}r_{ik} - r_{il}r_{s_k k}}{(p_k - p_l)^2}(F_k - F_l)\right), \quad i < s_k,\\
&\frac{\partial F_k}{\partial u_i} = -\frac{1}{r_{s_k k}}\sum_{l\neq k}\frac{r_{s_k l}r_{ik} - r_{il}r_{s_k k}}{(p_k - p_l)^2}(F_k - F_l), \quad i > s_k,\\
&\frac{\partial r_{ik}}{\partial u_j} = \frac{\partial r_{jk}}{\partial u_i} + 2\sum_{l\neq k}\frac{r_{jk}r_{il} - r_{ik}r_{jl}}{(p_k - p_l)^2}, \quad i > j,
\end{aligned}
\tag{22}
$$

for $k = 1, \dots, N$, where for each k, $s_k \in \{1, \dots, N\}$ is such that $r_{s_k k} \neq 0$ and $r_{ik} = 0$ for $i > s_k$. System (22) is of the Bourlet type [8]. To see this, we note that $(u_1, \dots, u_{s_k - 1}, u_{s_k + 1}, u_{N-1})$ are principal variables for p_k and F_k, while u_{s_k} are parametric variables. Analogously, $(u_1, \dots, u_{i-1})$ are principal variables for r_{ik}, while $(u_i, \dots, u_N)$ are parametric variables. The compatibility condition with respect to the principal variables can be verified straightforwardly. On the other hand, because $r_{ks_k} \neq 0$ and $p_k \neq p_l$, $k, l = 1, \dots, N$,

$k \neq l$, it follows that the functions defining the system are analytic. Therefore, applying the Bourlet theorem, we conclude that in a neighborhood of an initial point $\mathbf{u}_0 = (u_1^{(0)}, \dots, u_N^{(0)})$, there is a *unique* solution $\{p_k, F_k, r_{ik}\}$ such that for the principal variables taking the initial values, the solution becomes a set of arbitrary analytic functions of the corresponding parametric variables. The general solution therefore depends on $N(N+1)$ arbitrary analytic functions of the parametric variables: $3N$ functions of one variable and N analytic functions of l variables for each $l = 2, \dots, N-1$.

2.3. Systems of hydrodynamic type. Implicit equations (20) are transformations of the hodograph type. This suggests the presence of hydrodynamic-type equations. Indeed, assuming that $z = z(p, \mathbf{u})$ is a regular function near the points p_i, we find from (11) that

$$\frac{\partial z}{\partial p}(p_i) = 0, \quad i = 1, \dots, N.$$

Consequently, Eqs. (2) imply

$$\sum_{j=1}^{N} \frac{\partial z_i}{\partial u_j} \frac{\partial u_j}{\partial t_n} = v_{in} \sum_{j=1}^{N} \frac{\partial z_i}{\partial u_j} \frac{\partial u_j}{\partial x}, \quad n \geq 1,$$

where $z_i := z(p_i, \mathbf{u}(\mathbf{t}))$. Thus, expressing $\mathbf{u}(\mathbf{t})$ in terms of the functions z_i, we find that $\mathbf{u}(\mathbf{t})$ satisfies the system of hydrodynamic-type equations

$$\begin{gathered} \frac{\partial \mathbf{u}}{\partial t_n} = A_n(\mathbf{u}) \frac{\partial \mathbf{u}}{\partial x}, \quad n = 1, \dots, N, \\ \mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}, \qquad A_n := K^{-1} D_n K, \\ D_n := \operatorname{diag}(v_{1n}, \dots, v_{Nn}), \qquad K_{ij} := \frac{\partial z_i}{\partial u_j}. \end{gathered} \tag{23}$$

We note that taking $v_{2i} = 2p_i$ into account, we obtain the Gibbons–Kodama formula [2]

$$A_n = v_n(A), \qquad A := \frac{A_2}{2} \tag{24}$$

from (23), where $v_n(p) := \partial\Omega_n/\partial p$. Finally, if we assume that the functions $\partial_x u_j$, $j = 1, \dots, N$, are independent, we can also express our rational functions R_i in terms of the Gibbons–Kodama matrix A as

$$R_i(p, \mathbf{u}) = \sum_{j=1}^{N} (A(\mathbf{u}) - p)^{-1}_{ji} \frac{\partial a_1}{\partial u_j}. \tag{25}$$

From (25), we find that

$$r_{ik} = -\frac{\partial z_k}{\partial u_i} r_k, \qquad r_k := \frac{\partial a_1}{\partial z_k}.$$

In terms of the new coordinates $\{z_i\}_{i=1}^N$, system (21) becomes

$$\begin{aligned} \frac{\partial r_i}{\partial z_j} &= 2 \frac{r_i r_j}{(p_j - p_i)^2}, \\ \frac{\partial p_i}{\partial z_j} &= \frac{r_j}{p_j - p_i}, \\ \frac{\partial F_i}{\partial z_j} &= r_j \frac{F_j - F_i}{(p_j - p_i)^2}. \end{aligned}$$

We note that according to the previous system,

$$\frac{\partial F_i}{\partial z_j}\frac{1}{F_j - F_i} = \frac{\partial p_i}{\partial z_j}\frac{1}{p_j - p_i} = \frac{1}{2}\frac{\partial \log r_i}{\partial z_j}, \quad i \neq j. \tag{26}$$

These relations provide a geometric interpretation of (21). Defining

$$\beta_{ij} := \frac{1}{\sqrt{r_i}}\frac{\partial \sqrt{r_j}}{\partial z_i} = \frac{\sqrt{r_i r_j}}{(p_i - p_j)^2} = \beta_{ji}, \quad i \neq j, \tag{27}$$

we obtain a family of parallel conjugate nets $\mathbf{x} = \mathbf{x}(\mathbf{u})$ given by the solutions of

$$\frac{\partial \mathbf{x}}{\partial z_i} = H_i \mathbf{X}_i, \tag{28}$$

where H_i and $\mathbf{X}_i$ (the respective Lamé coefficients and renormalized tangent vectors) are characterized by the equations

$$\frac{\partial H_i}{\partial z_i} = \beta_{ji} H_j \tag{29}$$

and

$$\frac{\partial \mathbf{X}_i}{\partial z_i} = \beta_{ij} \mathbf{X}_j. \tag{30}$$

Obviously, $H_i := \sqrt{r_i}$ solves (29), and system (26) then implies that $F_i H_i$ and $p_i H_i$ are also solutions of (29).

3. Examples

3.1. N=1 reductions. If only one function $u = u(\mathbf{t})$ participates in the reduction and we set $u = -a_1$, then Eq. (10) becomes the Abel equation

$$\frac{\partial p}{\partial u} = \frac{1}{p - p_1(u)}, \tag{31}$$

where p_1 is an arbitrary function (obviously, the set of compatibility conditions is empty in this case). From (11), we obtain the following recursion relation for the coefficients of the expansion of $z = z(p, u)$:

$$a_1 = -u, \qquad a_2 = -\int p_1(u)\,du,$$

$$a'_{m+2} = p_1(u) a'_{m+1} + m a_m, \quad m \geq 1,$$

where $a'_m := \partial a_m / \partial u$. We can now use this expansion and (20) to generate solutions of the equations of the dKP hierarchy. For instance, setting $t_n = 0$ for $n \geq 4$, we reduce (20) to the Kodama equation [4] for N=1 reductions of the dispersionless KP equation.

An explicit expression for the solution $z = z(p, u)$ of (11) is available only in a few cases. For instance, the simplest case corresponds to $p_1(u) \equiv 0$ (the dispersionless KdV reduction). We then obtain

$$z = (p^2 - 2u)^{1/2}.$$

On the other hand, in the case $p_1 \equiv 0$, we can solve (17) for determining S_- and obtain

$$S_-(p, u) = -\left(\int_0^{z(p,u)} F\left(\frac{1}{2}(q^2 - z(p,u)^2)\right) dq\right)_-.$$

3.2. N=2 reductions. We now consider the case $\mathbf{u} = (u, v)$ with $u = -a_1$. From (25), we obtain

$$\begin{aligned} \frac{\partial p}{\partial u} &= \frac{p - A_{22}}{(p - A_{11})(p - A_{22}) - A_{12}A_{21}}, \\ \frac{\partial p}{\partial v} &= \frac{A_{12}}{(p - A_{11})(p - A_{22}) - A_{12}A_{21}}, \end{aligned} \tag{32}$$

where $A := (A_{ij}(\mathbf{u}))$ is the 2×2 matrix function of Gibbons and Kodama [2]. The right-hand sides of (32) have simple poles at

$$A_{\pm} := \frac{1}{2}(\operatorname{tr} A \pm \sqrt{(\operatorname{tr} A)^2 - 4 \det A}\,).$$

In this case, Eq. (13) leads to the conditions

$$\partial_v A_{11} = \partial_u A_{12}, \qquad \begin{pmatrix} \partial_v \det A \\ -\partial_u (u + \det A) \end{pmatrix} = A \begin{pmatrix} \partial_v \operatorname{tr} A \\ -\partial_u \operatorname{tr} A \end{pmatrix}. \tag{33}$$

The coefficients in the expansion of $z(p, \mathbf{u})$ are determined by the recursion relations

$$\begin{aligned} &a_1 = -u, \qquad a_2 = -\int A_{11}\, du + A_{12}\, dv, \\ &a_3 = \int (\det A - u - A_{11} \operatorname{tr} A)\, du - A_{12} \operatorname{tr} A\, dv, \\ &\partial_u a_{m+2} = \operatorname{tr} A\, \partial_u a_{m+1} - \det A\, \partial_u a_m + m a_m - (m-1) A_{22} a_{m-1}, \\ &\partial_v a_{m+2} = \operatorname{tr} A\, \partial_v a_{m+1} - \det A\, \partial_v a_m + (m-1) A_{12} a_{m-1}. \end{aligned}$$

Choosing E to be

$$E := p \frac{F_+ - F_-}{A_+ - A_-} + \frac{A_+ F_- - A_- F_+}{A_+ - A_-},$$

where

$$F_{\pm}(\mathbf{u}) := \left.\frac{\partial S_-(p)}{\partial p}\right|_{A_{\pm}}, \qquad F := \frac{A_- F_+ - A_+ F_-}{A_+ - A_-}, \qquad G := \frac{F_- - F_+}{A_+ - A_-},$$

we reduce (18) to the form

$$\begin{pmatrix} -\partial_v F \\ \partial_u F \end{pmatrix} = A \begin{pmatrix} \partial_v G \\ -\partial_u G \end{pmatrix}. \tag{34}$$

Hence, if A and $F_{\pm}$ satisfy the corresponding consistency conditions and we set $t_n = 0$ for $n > 4$, then the first flows of the dKP hierarchy can be found by solving the system of equations

$$4\left(A_{\pm}^3 - 2uA_{\pm} - \int A_{11}\, du + A_{12}\, dv\right) t_4 + 3(A_{\pm}^2 - u) t_3 + 2A_{\pm} t_2 + x = -F_{\pm} \tag{35}$$

for $\mathbf{u}$. If $t_4 = 0$, these equations are equivalent to the Kodama system for N=2 reductions [4],

$$\begin{aligned} &-3(u + \det A) t_3 + x = F, \\ &3 \operatorname{tr} A\, t_3 + 2t_2 = G. \end{aligned} \tag{36}$$

A particularly interesting case arises if we impose the conditions $u = -a_1$ and $v = -a_2$, which corresponds to the choice

$$A = \begin{pmatrix} 0 & 1 \\ -V & W \end{pmatrix}, \qquad V := A_+A_-, \qquad W := A_+ + A_-.$$

We then find that (33) becomes

$$\begin{aligned} &\partial_v V + \partial_u W = 0, \\ &\partial_u V - V\partial_v W + W\partial_v V + 1 = 0. \end{aligned} \tag{37}$$

Hence, setting

$$V = \partial_u Z, \qquad W = -\partial_v Z,$$

we can formulate (37) as the Monge–Ampere equation

$$\partial_{uu} Z + \partial_u Z\, \partial_{vv} Z - \partial_v Z\, \partial_{uv} Z + 1 = 0. \tag{38}$$

Analogously, (34) can be written as

$$\begin{aligned} &F = \partial_u T, \qquad G = \partial_v T, \\ &\partial_{uu} T + V\partial_{vv} T + W\partial_{uv} T = 0. \end{aligned} \tag{39}$$

We next construct some solutions of the dKP equation. A solution of (37) and (39) is

$$W = \frac{2v}{u}, \qquad V = \frac{v^2}{u^2} + cu^2 + u, \qquad T = k_1 u + k_2 v.$$

The corresponding hodograph solutions for (3) are

$$u(x, y, t) = \frac{1}{6ct}\left(-6t + \sqrt{36t^2 + c[12t(x - k_1) - (2y - k_2)^2]}\right)$$

and

$$u(x, y, t) = \frac{12t(x - k_1) - (2y - k_2)^2}{72t^2},$$

which correspond to the respective cases $c \neq 0$ and $c = 0$.

Another solution of (37) and (39) is

$$W = \frac{2v}{u}, \qquad V = \frac{v^2}{u^2} + u, \qquad T = k\frac{v}{u}.$$

It leads to a hodograph solution of (3) implicitly defined by the algebraic equation

$$72t^2u^3 + 4(y^2 - 3tx)u^2 = k^2.$$

We obtain a solution of (3), implicitly defined by a transcendental equation, by choosing

$$W = cv + d, \qquad V = ae^{cu} + \frac{1}{c}, \qquad T = k_1 u + k_2 v, \quad a, c \neq 0.$$

In this case, u is determined from the equation

$$-3\left(u + ae^{cu} + \frac{1}{c}\right)t + (x - k_1) = 0.$$

3.3. N=3 reductions. In this case, we assume that $\mathbf{u} = (u, v, w)$ are given by the first coefficients of the expansion of $p = p(z, \mathbf{u})$

$$p = z + \frac{u}{z} + \frac{v}{z^2} + \frac{w}{z^3} + O\biggl(\frac{1}{z^4}\biggr).$$

The reduction is then defined by the system

$$\begin{aligned} \frac{\partial p}{\partial u} &= \frac{p^2 - Vp + R + u}{p^3 - Vp^2 + Rp + H}, \\ \frac{\partial p}{\partial v} &= \frac{p - V}{p^3 - Vp^2 + Rp + H}, \\ \frac{\partial p}{\partial w} &= \frac{1}{p^3 - Vp^2 + Rp + H}, \end{aligned} \tag{40}$$

where $p^3 - Vp^2 + Rp + H$ have three simple roots. Compatibility conditions (13) can be formulated as

$$\begin{aligned} &V_v = -R_w, && V_u = H_w + uV_w, \\ &R_v = -H_w + RV_w - VR_w, && R_u = VH_w - HV_w + uR_w - 2, \\ &H_v = 1 - VH_w + HV_w, && H_u = -V + RH_w - HR_w + uH_w. \end{aligned} \tag{41}$$

If we now choose $S = S_+(p)$ and $t_n = 0$ for $n > 4$ and take Eqs. (15) and $a_1 = -b_1 = -u$ and $a_2 = -b_2 = -v$ into account, we obtain

$$\frac{\partial S_+(p)}{\partial p} = 4t_4(p^3 - Vp^2 + Rp - H) = x + 2py + 3(p^2 - u)t + 4(p^3 - 2up - v)t_4. \tag{42}$$

From this equation, it is clear that we can obtain solutions of the first two members of the dKP hierarchy by solving the system

$$V = -\frac{3t}{4t_4}, \qquad R = \frac{y}{2t_4} - 2u, \qquad H = v - \frac{x - 3tu}{4t_4}. \tag{43}$$

For instance, trying a function V of the form $V = V(u, v)$ in (41), we find the solution of the first two members of the dKP hierarchy implicitly determined by the transcendental equation

$$k_1^3 x - 2k_1^2 k_2 y + 3k_1 k_2^2 t + 4(k_1^2 k_3 + 3k_1 k_2 - k_2^3)t_4 + (12k_1^2 k_2 t_4 - 3k_1^3 t)u + 4k_1^3 k_5 t_4 e^{k_1 u} = 0$$

and

$$v = -\frac{k_3}{k_1} - \frac{3t}{4k_1 t_4} - \frac{k_2}{k_1}u.$$

In the particular case $k_5 = 0$, we find

$$u(x, y, t, t_4) = \frac{k_1^3 x - 2k_1^2 k_2 y + 3k_1 k_2^2 t + 4(k_1^2 k_3 + 3k_1 k_2 - k_2^3)t_4}{3k_1^2(k_1 t - 4k_2 t_4)}.$$

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REFERENCES

1. I. M. Krichever, *Comm. Pure Appl. Math.*, **47**, 437 (1992); *Comm. Math. Phys.*, **143**, 415 (1992).
2. Y. Kodama and J. Gibbons, *Phys. Lett. A*, **135**, 167 (1989).
3. J. Gibbons and S. P. Tsarev, *Phys. Lett. A*, **211**, 19 (1996); **258**, 263 (1999).
4. Y. Kodama, *Progr. Theor. Phys. Suppl.*, **94**, 184 (1988); *Phys. Lett. A*, **129**, 223 (1988).
5. T. Takasaki and T. Takebe, *Int. J. Mod. Phys. A*, **7** Suppl. 1B, 889 (1992); *Rev. Math. Phys.*, **7**, 743 (1995).
6. B. A. Kupershmidt, *J. Phys. A*, **23**, 871 (1990).
7. V. V. Geogjaev, "The quasiclassical limit of the inverse scattering problem method," in: *Singular Limits of Dispersive Waves* (Proc. NATO Adv. Res. Workshop, Lyons, France, July 8–12, 1991, NATO ASI Ser. B 320, N. M. Ercolani et al., eds.), Plenum, New York (1994), p. 53; Y. Kodama and J. Gibbons, "Solving dispersionless Lax equations," in: Ibid., p. 64; Y. Kodama, *Phys. Lett. A*, **147**, 477 (1990).
8. L. Bianchi, *Lezioni di Geometria Differenziale* (3rd ed.), Zanichelli, Bologna (1924).