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QUASICONFORMAL MAPPINGS AND SOLUTIONS OF THE DISPERSIONLESS KP HIERARCHY

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A $\bar{\partial}$ formalism for studying dispersionless integrable hierarchies is applied to the dispersionless KP (dKP) hierarchy. We relate this formalism to the theory of quasiconformal mappings on the plane and present some classes of explicit solutions of the dKP hierarchy.

Keywords: dispersionless hierarchies, quasiconformal mappings, $\bar{\partial}$ equations

1. Introduction

Dispersionless, or semiclassical, integrable hierarchies constitute an important part of integrable system theory. They are basic in various approaches to solving problems arising in physics and applied mathematics (see, e.g., [1]–[8]). These hierarchies are related to some classical problems in conformal map theory [9]. We recently proposed a $\bar{\partial}$ method for studying dispersionless integrable hierarchies [10], [11]. This method reveals an intimate connection between these hierarchies and the theory of quasiconformal mappings [12]–[15].

Our analysis is based on the nonlinear $\bar{\partial}$ equation

$$S_{\bar{z}} = W(z, \bar{z}, S_z), \tag{1}$$

where $z \in \mathbb{C}$, $S(z, \bar{z}, \mathbf{t})$ is a complex-valued function depending on an infinite set $\mathbf{t}$ of parameters (times), $S_{\bar{z}} := \partial S/\partial \bar{z}$, $S_z := \partial S/\partial z$, and W is an appropriate function of z, $\bar{z}$, and S_z.

Equation (1) implies that the first-order derivatives of S with respect to the parameters $\mathbf{t}$ satisfy the family of Beltrami equations

$$f_{\bar{z}} = \mu(z, \mathbf{t}) f_z, \tag{2}$$

where

$$\mu := W'(z, \bar{z}, S_z), \qquad W' = W_\xi(z, \bar{z}, \xi). \tag{3}$$

This provides a link between the $\bar{\partial}$ method and the theory of quasiconformal mappings, which relies on the Beltrami equation. We note that Eq. (1), in turn, is also well known from the theory of quasiconformal mappings (see, e.g., [16]).

Our method for finding solutions of (1) follows the classic schemes for solving first-order PDEs of the Hamilton–Jacobi type. Our objective in this paper is to illustrate our approach by presenting some explicit exact solutions of the dispersionless KP (dKP) hierarchy. We also prove that the simplest among these solutions (Example 1 in Sec. 4) cannot be obtained by standard methods based on the hodograph transformation technique [2].

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2. The dKP hierarchy

The dKP hierarchy is the *classical version* of the Lax-pair equations of the standard KP theory [1]–[7],

$$\frac{\partial z}{\partial t_n} = \{\Omega_n, z\}, \qquad \Omega_n(p,\mathbf{t}) := (z^n)_+, \quad n \geq 1. \tag{4}$$

Here, $z = z(p,\mathbf{t})$ is a complex function depending on a complex variable p and an infinite set $\mathbf{t} := (t_1, t_2, \dots)$ of complex parameters, which is assumed to admit a Laurent expansion of the form

$$z = p + \sum_{n\geq 1} \frac{a_n(\mathbf{t})}{p^n}, \quad p \to \infty. \tag{5}$$

We let $(z^n)_+$ denote the polynomial part of the expansion of z^n in powers of p,

$$(z)_+ = p, \qquad (z^2)_+ = p^2 + 2a_1, \qquad (z^3)_+ = p^3 + 3pa_1 + 3a_2, \qquad \text{etc.},$$

and the Poisson bracket is

$$\{F, G\} := \frac{\partial F}{\partial p}\frac{\partial G}{\partial x} - \frac{\partial F}{\partial x}\frac{\partial G}{\partial p}, \qquad x := t_1.$$

The compatibility conditions for (4) have the form

$$\frac{\partial \Omega_m}{\partial t_n} - \frac{\partial \Omega_n}{\partial t_m} + \{\Omega_m, \Omega_n\} = 0, \quad m \neq n. \tag{6}$$

Two interesting examples of nonlinear equations of the dKP hierarchy are the following:

1. For $n = 2$, Eq. (4) leads to the Benney moment equations

$$\frac{\partial a_{n+1}}{\partial t} + \frac{\partial a_{n+2}}{\partial x} + n a_n \frac{\partial a_1}{\partial x} = 0, \qquad t := -2t_2. \tag{7}$$

2. Compatibility equations (6) for $n = 2$ and $m = 3$ imply the dKP equation (the Zabolotskaya–Khokhlov equation)

$$\left(u_t - \frac{3}{2} u u_x\right)_x = \frac{3}{4} u_{yy}, \qquad u := 2a_1, \qquad t := t_3, \qquad y := t_2. \tag{8}$$

It follows from (6) (see [6]) that for any solution $z = z(p,\mathbf{t})$ of the dKP hierarchy, there exists an associated function $S = S(z,\mathbf{t})$ such that

$$\frac{\partial S(z,\mathbf{t})}{\partial t_n} = \Omega_n\big(p(z,\mathbf{t}),\mathbf{t}\big), \quad n \geq 1, \tag{9}$$

where $p = p(z,\mathbf{t})$ is obtained by inverting the solution $z = z(p,\mathbf{t})$. Without loss of generality, we assume that S admits a Laurent expansion

$$S(z,\mathbf{t}) = \sum_{n\geq 1} z^n t_n + \sum_{n\geq 1} \frac{S_n(\mathbf{t})}{z^n}, \quad z \in \Gamma, \tag{10}$$

in a circle $\Gamma = \{z : |z| = r\}$. We note that if we set $n = 1$ in (9) and use (10), then $p = p(z, \mathbf{t})$ becomes

$$p = \frac{\partial S(z, \mathbf{t})}{\partial x} = z + \sum_{n\geq 1} \frac{b_n(\mathbf{t})}{z^n}, \qquad b_n := \frac{\partial S_n}{\partial x}. \tag{11}$$

Conversely, given a function $S = S(z, \mathbf{t})$ that satisfies (9) and (10), it can be proved that the function $z = z(p, \mathbf{t})$ inverse to the function $p = p(z, \mathbf{t})$ in (11) determines a solution of the dKP hierarchy [4].

Hereafter, we call functions satisfying conditions (9) and (10) the *S-functions* of the dKP hierarchy. They are related to the τ-functions [4] because

$$S(z, \mathbf{t}) = \sum_{n\geq 1} z^n t_n + \sum_{n\geq 1} \frac{1}{nz^n} \frac{\partial \log \tau(\mathbf{t})}{\partial t_n}, \quad z \in \Gamma.$$

We note that system (9) for a dKP S-function is a set of compatible Hamilton–Jacobi-type equations

$$\frac{\partial S}{\partial t_n} = \Omega_n \left(\frac{\partial S}{\partial x}, \mathbf{t} \right), \quad n \geq 2, \tag{12}$$

which represents the *semiclassical limit* of the linear system for the wave function of the standard KP hierarchy.

Several methods for constructing solutions of the dKP hierarchy through S-functions have been devised [2], [4], [6]. Here, we use the $\bar{\partial}$ method proposed in [10], [11], in which S-functions are described via solutions of $\bar{\partial}$ equations (1). We recall that the symmetries of (1) in this approach (first-order variations $f := \delta S$) are described by the family of Beltrami equations (2), (3). In particular, this property implies that all the first-order derivatives $\partial S/\partial t_n$ of a solution of (1) satisfy (2).

We consider a local solution f of a Beltrami equation. For f having a nonzero Jacobian at a certain point z_0, all smooth local solutions F in the vicinity of z_0 are analytic functions of f: $F = F(f)$. This property suggests that under appropriate conditions, solutions S of $\bar{\partial}$ equation (1) that admit expansions of form (10) satisfy conditions (12) as well. Therefore, they provide S-functions for the dKP hierarchy. In what follows, we propose a rigorous basis for this scheme.

3. Quasiconformal mappings

Quasiconformal mappings are a natural and very rich extension of the concept of conformal mappings; it is convenient to recall some of their basic properties here (see, e.g., [10]–[15]).

Let $\mu = \mu(z)$ be a measurable function on a domain G of the complex plane such that $|\mu(z)| < k$ almost everywhere in G for some $0 < k < 1$. We call a function $f = f(z)$ a *quasiconformal mapping with complex dilatation* μ in G if

1. f is a homeomorphism $f : G \to G'$ and
2. f is a generalized solution of linear Beltrami equation (2) on G with locally square-integrable partial derivatives $f_{\bar{z}}$ and f_z.

The properties of solutions of Beltrami equation (2) are well known (see, e.g., [13]). Some of them are of principal importance for us. Before presenting these results, we introduce the Calderón–Zygmund operator [13]

$$(Th)(z) := \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{h(z')}{(z' - z)^2}\, dz' \wedge d\bar{z}', \tag{13}$$

where the integral is taken in the sense of the Cauchy principal value. We then have the fundamental result [13].

Theorem 1. *For any $p \geq 2$, the operator T defines a bounded operator in $L^p(\mathbb{C})$. Moreover, $\|T\|_p$ is continuous with respect to p and satisfies*

$$\lim_{p\to 2} \|T\|_p = 1. \tag{14}$$

This theorem implies that for any $0 \leq k < 1$, there exists $\delta(k) > 0$ such that $k\|T\|_p < 1$ for all $2 < p < 2 + \delta(k)$.

The next theorem describes the quasiconformal mapping property, which is important for the $\bar\partial$ method.

Theorem 2. *Let μ be a measurable function with compact support inside the circle $|z| < R$ and such that $\|\mu\|_\infty < k < 1$. Then for any $p > 2$ such that $k\|T\|_p < 1$, the only generalized solution of Beltrami equation (2) such that*

$$f(z) = O\left(\frac{1}{z}\right), \quad z \to \infty, \tag{15}$$

and $f_{\bar z}, f_z \in L^p(\mathbb{C})$ is $f \equiv 0$.

This result is the uniqueness part of the existence theorem for the so-called normal solutions of Beltrami equations [12]–[14]. Its proof relies on the operator T representing the action of $\partial_z \partial_{\bar z}^{-1}$ on $L^p(\mathbb{C})$, and under the conditions of the theorem, the Beltrami equation for f becomes the integral equation

$$\phi - \mu T\phi = 0, \qquad \phi := f_{\bar z},$$

on $L^p(\mathbb{C})$. Because $\|\mu T\|_p \leq k\|T\|_p < 1$, we then have $\phi \equiv 0$, and property (15) then gives $f \equiv 0$.

We now return to $\bar\partial$ equation (1) and assume that $W(z, \bar z, S_z)$ vanishes for all z outside a circle $\Gamma = \{z : |z| = r\}$. We assume that we find a solution $S = S(z, \bar z, \mathbf{t})$ of (1) inside the disk $D = \{z : |z| < r\}$ with a boundary value $S|_\Gamma := S(z, r^2/z, \mathbf{t})$ of form (10) and such that the set

$$\Omega := \Big\{\mathbf{t}\colon \sup_{z\in D}\big|W'(z, \bar z, S_z(z, \mathbf{t})\big| \leq k\Big\}$$

is nonempty for some k, $0 < k < 1$. In this case, we can apply Theorem 2 to Beltrami equation (2), (3). Moreover, taking into account that

$$\frac{\partial S}{\partial t_n} = z^n + \sum_{m\geq 1} \frac{\partial S_m(\mathbf{t})}{\partial t_n} z^{-m}, \quad z \in \Gamma,$$

we find that these functions can be continuously extended from D to analytic functions outside Γ. They therefore become solutions of (2), (3) in the whole complex plane. On the other hand, it is clear that

$$\frac{\partial S}{\partial t_n} - (z^n)_+ = \frac{\partial S}{\partial t_n} - \Omega_n\left(\frac{\partial S}{\partial x}, \mathbf{t}\right) = O\left(\frac{1}{z}\right), \quad z \to \infty,$$

and Theorem 2 therefore implies that (9) is satisfied. Consequently, $S = S(z, \bar z, \mathbf{t})$ determines an S-function of the dKP hierarchy for $\mathbf{t} \in \Omega$.

4. Solutions of the dKP hierarchy

To construct explicit solutions of the dKP hierarchy, we consider $\bar\partial$ equations of the form

$$S_{\bar z} = \theta\big(r - |z|\big) V(z, \bar z, S_z), \tag{16}$$

where $r > 0$, $\theta(\xi)$ is the standard Heaviside function, and V is an analytic function of z, $\bar{z}$, and S_z. Our scheme of solution is as follows:

1. We first generate solutions $S = S(z, \mathbf{a})$ of (16),

$$S_{\bar{z}} = V(z, \bar{z}, S_z), \quad |z| < r, \tag{17}$$

   depending on a set of free parameters $\mathbf{a} := (a_0, a_1, \dots)$.

2. We next select those solutions whose boundary value on $\Gamma = \{z : |z| = r\}$ is of form (10).

Equation (17) is a PDE of a Hamilton–Jacobi type, and the methods for generating its solutions are well developed. For example, if $V = V(S_z)$ depends only on S_z, then (17) implies

$$m_{\bar{z}} = V_m(m) m_z, \qquad m := S_z.$$

We can instantly solve this equation applying the methods of characteristics. The general solution of (17) is then implicitly characterized by

$$\begin{aligned} S &= V(m)\bar{z} + mz - f(m), \\ V_m \bar{z} + z &= f_m(m), \end{aligned} \tag{18}$$

where $f = f(m)$ is an arbitrary function. We note that from the second equation in (18), we have

$$f_m(m_0) = z, \qquad m_0 := m(z, \bar{z})\big|_{\bar{z}=0},$$

and $f_m(m_0)$ is therefore the function inverse to $m_0 = m_0(z)$.

To obtain explicit solutions, we need further simplifying assumptions. For instance, we consider cases where only a finite set of $N{+}1$ parameters $\mathbf{a} = (a_0, a_1, \dots, a_N)$ is considered. We then encounter the problem of selecting solutions S in which no terms $z^n t_n$ with $n > N$ appear in (10). Other types of solutions of the $\bar{\partial}$ equation would have time variables t_n, $n > N$, which are functions of $t_1, \dots, t_N$, and no solution of the dKP hierarchy would arise in that way.

We start the exploration of possible favorable cases by considering the class of $\bar{\partial}$ equations (17) of the form

$$S_{\bar{z}} = \bar{z}^s \sum_{m \geq 0}^{M} p_m(z) (S_z)^m, \quad |z| < r, \tag{19}$$

where $s \geq 0$, $M \geq 2$, the coefficients $p_m = p_m(z)$ are polynomials in z, and $p_M \not\equiv 0$. We seek a series solution of (19) of the form

$$S = \sum_{n \geq 0} c_n(z) \bar{z}^{n(s+1)}, \tag{20}$$

where c_0 is an arbitrary N-degree polynomial ($N \geq 2$),

$$c_0(z) = \sum_{n=0}^{N} a_n z^n, \quad a_N \neq 0. \tag{21}$$

Substituting (20) in (19), we obtain the recursion relation

$$c_{n+1} = \frac{1}{(n+1)(s+1)} \sum_{m \geq 0}^{M} p_m(z) \Bigg(\sum_{r_1 + \cdots + r_m = n} c'_{r_1} \cdots c'_{r_m} \Bigg), \quad n \geq 0, \tag{22}$$

which shows that all the coefficients $c_n(z)$ of (20) are polynomials. We must find their degrees for examining the form of S on the boundary $|z| = r$.

Table 1

(M,N)	$V(z,S_z)$	$S\|_{\bar z=0}$
$(2,2)$	$\alpha(S_z)^2+\big(\sum_{i=0}^{1}\beta_i z^i\big)S_z+\sum_{i=0}^{2}\gamma_i z^i$	$\sum_{i=0}^{2}a_i z^i$
$(3,2)$	$\alpha(S_z)^3+\big(\sum_{i=0}^{1}\beta_i z^i\big)(S_z)^2+\big(\sum_{i=0}^{2}\gamma_i z^i\big)S_z+\sum_{i=0}^{3}\eta_i z^i$	$\sum_{i=0}^{2}a_i z^i$
$(2,3)$	$\alpha(S_z)^2+\big(\sum_{i=0}^{2}\beta_i z^i\big)S_z+\sum_{i=0}^{4}\gamma_i z^i$	$\sum_{i=0}^{3}a_i z^i$

Lemma. *If the degrees of the coefficients p_m in* (19) *satisfy the conditions*

$$\deg p_m \le (M-m)(N-1), \quad m=0,1,\dots,M,$$

then

$$\deg c_n = n\big[M(N-1)-N\big]+N, \quad n\ge 0. \tag{23}$$

Proof. We apply the induction principle. It is obvious that (23) holds for $n=0$. We now suppose that it holds for $n'\le n$ and consider the terms in expression (22) for c_{n+1}. By taking into account that $r_1+\dots+r_m=n$, we obtain

$$\begin{aligned}\deg(p_m c'_{r_1}\cdots c'_{r_m}) &= \sum_{i=1}^{m}\big[r_i\big[M(N-1)-N\big]+N-1\big]+\deg p_m = \\ &= n\big[M(N-1)-N\big]+m(N-1)+\deg p_m \le \\ &\le n\big[M(N-1)-N\big]+M(N-1)=(n+1)\big[M(N-1)-N\big]+N.\end{aligned}$$

Moreover, because p_M is a nonzero constant, it is clear that the corresponding terms in (22) satisfy the condition

$$\deg(p_M c'_{r_1}\cdots c'_{r_M}) = n\big[M(N-1)-N\big]+M(N-1)=(n+1)\big[M(N-1)-N\big]+N.$$

We therefore conclude that (23) holds for c_{n+1} as well, which proves the lemma.

Assuming that series (20) converges for some $r\ge 0$, we find that under the assumption of the above lemma, the continuous extension of S to the boundary $|z|=r$ is

$$S=\sum_{n\ge 0} r^{2n(s+1)}\frac{c_n(z)}{z^{n(s+1)}}, \qquad \frac{c_n(z)}{z^{n(s+1)}}=O(z^{d_n}), \tag{24}$$

where

$$d_n = n\big[M(N-1)-N-s-1\big]+N. \tag{25}$$

Because the solution corresponding to (21) depends on $N+1$ free parameters $(a_0,\dots,a_N)$, only those cases for which $d_n\le N$ for all $n\ge 0$ are of interest. It is obvious from (25) that this is possible only if

$$N\le \frac{M+s+1}{M-1}. \tag{26}$$

For example, if we set $s=0$, this means that we have only the three possibilities listed in Table 1.

Example 1. The simplest case in the class $(2,2)$ corresponds to

$$S_{\bar{z}} = \theta\bigl(1-|z|\bigr)(S_z)^2, \tag{27}$$

where $S|_{\bar{z}=0}$ is a quadratic polynomial. This yields

$$S = \begin{cases} \dfrac{1}{2}\dfrac{(z-b)^2}{a-2\bar{z}} - c, & |z| \le 1, \\ \dfrac{1}{2}\dfrac{z(z-b)^2}{az-2} - c, & |z| \ge 1. \end{cases}$$

The regularity of S inside the unit circle requires $|a| > 2$.

On the boundary $|z| = 1$, we have

$$S = \frac{1}{2a}z^2 + \left(\frac{1}{a^2} - \frac{b}{a}\right) z + \frac{2}{a^3} + \frac{b^2}{2a} - \frac{2b}{a^2} - c + O\left(\frac{1}{z}\right).$$

Therefore, to fit the required form of an S-function of the dKP hierarchy, we must identify

$$x = \frac{1}{a^2} - \frac{b}{a}, \qquad t_2 = \frac{1}{2a}, \qquad c = \frac{2}{a^3} + \frac{b^2}{2a} - \frac{2b}{a^2}.$$

On the other hand, the complex dilatation for the corresponding Beltrami equation (1), (2) is

$$\mu(z,\bar{z}) := 2\theta\bigl(1-|z|\bigr)\frac{z-b}{a-2\bar{z}}. \tag{28}$$

We then obtain the bound

$$|\mu(z,\bar{z})| < 2\frac{|b|+1}{|a|-2}, \quad z \in \mathbb{C}.$$

Thus, for any $0 < k < 1$, we have $|\mu(z)| \le k$ provided $k|a| > 2\bigl(|b| + k + 1\bigr)$. Hence, there is a nonempty domain in the space of parameters in which Beltrami equation (1), (2) satisfies the conditions assumed in our discussion in Sec. 3. It follows that

$$p := S_x = \frac{z^2 - 4t_2 z + 2(x + 4t_2^2)}{z - 4t_2}, \quad |z| = 1,$$

and we obtain the solution of the t_2 flow (Benney flow) of the dKP hierarchy

$$z = \frac{p}{2} + 2t_2 + \sqrt{\left(\frac{p}{2} - 2t_2\right)^2 - 2x - 8t_2^2}\,. \tag{29}$$

We note that this solution depends on the time parameters via two functions $u_1 = 2t_2$ and $u_2 = -2x - 8t_2^2$; indeed, we can rewrite it in the form

$$z = \frac{p}{2} + u_1 + \sqrt{\left(\frac{p}{2} - u_1\right)^2 + u_2}\,.$$

If this solution could be obtained by the hodograph methods [2], it would correspond to such a reduction of the dKP hierarchy in which the functions $\mathbf{u} = (u_1, u_2)$ satisfy a diagonalizable hydrodynamic-type system with the Riemann invariants provided by zeros of the function $\partial z(p,\mathbf{u})/\partial p$. But this function has no zeros for $u_2 \ne 0$ because

$$\frac{\partial z(p,\mathbf{u})}{\partial p} = \frac{p/2 - u_1 + \sqrt{(p/2-u_1)^2 + u_2}}{2\sqrt{(p/2-u_1)^2 + u_2}}.$$

Therefore, we conclude that solution (29) of the dKP hierarchy cannot be obtained with the hodograph technique approach.

Example 2. The $\bar{\partial}$ equation

$$S_{\bar{z}} = \theta\big(1 - |z|\big)(S_z)^3, \tag{30}$$

where $S|_{\bar{z}=0}$ is a quadratic polynomial, is an example of the $(3,2)$ case. The corresponding S-function is

$$S = \bar{z}m^3 + (z-b)m - \frac{a}{2}m^2 - c, \quad |z| < 1,$$

where

$$m = \frac{1}{6\bar{z}}\big(a - \sqrt{a^2 - 12(z-b)\bar{z}}\,\big).$$

The defining relations for the dKP parameters (x, t_2) are

$$x = \frac{b}{6}\big(\sqrt{a^2-12} - a\big), \qquad t_2 = \frac{1}{108}\big(18a - a^3 + (a^2-12)^{3/2}\big). \tag{31}$$

The corresponding solution of the t_2 flow of the dKP hierarchy is

$$z = a\Big(a - \sqrt{a^2-12}\Big)\frac{p}{12} + \frac{3x}{\sqrt{a^2-12}} + \frac{1}{12}\left[\left(a\big(a - \sqrt{a^2-12}\,\big)p + \frac{36x}{\sqrt{a^2-12}}\right)^2 - \right.$$
$$\left. - 12\left(\big(a - \sqrt{a^2-12}\,\big)p + \frac{3\big(a + \sqrt{a^2-12}\,\big)x}{\sqrt{a^2-12}}\right)^2\right]^{1/2},$$

where $a = a(t_2)$ follows from (31).

Example 3. The class $(2,3)$ is the most interesting because it provides solutions of the dKP hierarchy depending on the parameters (x, t_2, t_3). We consider

$$S_{\bar{z}} = \theta\big(1 - |z|\big)(S_z)^2 \tag{32}$$

with a cubic polynomial $S|_{\bar{z}=0}$. Taking $m_0 := S_z|_{\bar{z}} = az^2 + bz + c$ and using (18), we obtain

$$f(m) = -\frac{b}{2a}m + \frac{1}{12a^2}(4am + b^2 - 4ac)^{3/2} + d = -\frac{b}{2a}m + \frac{1}{12a^2}(4a\bar{z}m + 2az + b)^3 + d \tag{33}$$

and

$$m = \frac{1}{8\bar{z}^2}\left[\frac{1}{a} - 4\left(z + \frac{b}{2a}\right)\bar{z} - \sqrt{\frac{4}{a}\left(\frac{b^2}{a} - 4c\right)\bar{z}^2 - \frac{8}{a}\left(z + \frac{b}{2a}\right)\bar{z} + \frac{1}{a^2}}\,\right]. \tag{34}$$

Hence, we have

$$S = \left(z + \frac{b}{2a}\right)m + \bar{z}m^2 - \frac{1}{12a^2}(4a\bar{z}m + 2az + b)^3 - d, \quad |z| < 1. \tag{35}$$

It is then clear that for the function S to be continuous we must require [11]

$$\frac{4}{a}\left(\frac{b^2}{a} - 4c\right)\bar{z}^2 - \frac{8}{a}\left(z + \frac{b}{2a}\right)\bar{z} + \frac{1}{a^2} \neq 0, \quad |z| < 1.$$

We also note that S is regular at the origin because $\lim_{z\to 0} m = c$.

We outline the calculation of $u = -2\partial S_1/\partial x$. We need to compute the first few terms in the expansion of S for $|z| = 1$. For this, we use the identity

$$S_z = m - \left(\frac{m}{z}\right)^2, \quad |z| = 1. \tag{36}$$

Then, expanding m for $|z| = 1$ and setting

$$S_z = 3t_3 z^2 + 2t_2 z + x - \frac{S_1}{z^2} + \dots, \quad |z| = 1,$$

in (36), we obtain

$$3t_3 = -\frac{3}{4} + \frac{3}{8a} + \frac{1}{32a^2}\big((1-8a)^{3/2} - 1\big), \qquad 2t_2 = \frac{b}{8a^2}\big(1 - 4a - \sqrt{1-8a}\,\big),$$
$$x = -\frac{b^2}{8a^2}\left(1 + \frac{4a-1}{2\sqrt{1-8a}}\right) + \frac{c}{4a}\big(1 - \sqrt{1-8a}\,\big),$$

and

$$S_1 = \frac{\big(2b^2 + (1-8a)c\big)^2}{(1-8a)^{5/2}}. \tag{37}$$

These expressions imply that $u = -2\partial S_1/\partial x$ is ($t := t_3$, $y := t_2$)

$$u = \frac{4\big(5 - 12t + 4\sqrt{1-12t}\,\big)^2\big((-1+12t)x - 4y^2\big)}{3(1+4t)\big(1 - 12t + 2\sqrt{1-12t}\,\big)^3}, \tag{38}$$

which satisfies (8). We note that according to [8], solution (38) belongs to the class of solutions of the dKP equation that yield Einstein–Weyl structures conformal to Einstein metrics.

More general cases corresponding to (19) with $s \neq 0$ and z dependence in its right-hand side will be considered elsewhere.

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