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PROLONGATIONS OF VECTOR FIELDS AND THE INVARIANTS-BY-DERIVATION PROPERTY

C. Muriel* and J. L. Romero*

For any given vector field X defined on some open set $M \subset \mathbb{R}^2$, we characterize the prolongations X_n^* of X to the nth jet space $M^{(n)}$, $n \geq 1$, such that a complete system of invariants for X_n^* can be obtained by derivation of lower-order invariants. This leads to characterizations of C^∞-symmetries and to new procedures for reducing the order of an ordinary differential equation.

Keywords: C^∞-symmetry, differential invariants, reductions of ordinary differential equations

1. Introduction

One of the most-used methods for reducing the order of a given ordinary differential equation

$$\Delta(x, u^{(n)}) = 0 \tag{1}$$

is based on the existence of Lie symmetries (see [1]). The Lie symmetries of Eq. (1) are vector fields

$$X = \xi(x,u)\frac{\partial}{\partial x} + \eta(x,u)\frac{\partial}{\partial u} \tag{2}$$

defined on some open subset $M \subset \mathbb{R}^2$ such that the usual prolongation of X to the nth jet space $M^{(n)}$

$$X^{(n)} = \xi(x,u)\frac{\partial}{\partial x} + \eta(x,u)\frac{\partial}{\partial u} + \sum_{i=1}^{n} \eta^{(i)}(x,u^{(i)})\frac{\partial}{\partial u_i},$$

where $\eta^{(i)} = D_x(\eta^{(i-1)}) - D_x(\xi)u_i$, satisfies the invariance condition

$$X^{(n)}\big(\Delta(x,u^{(n)})\big) = 0, \qquad \Delta(x,u^{(n)}) = 0. \tag{3}$$

The Lie method of reduction for (1) associated with a given Lie symmetry X is based on the existence of a complete system of functionally independent invariants for $X^{(n)}$ that can be obtained by successive derivations of invariants of $X^{(1)}$. If $y(x,u)$ is an invariant of X and $w(x,u,u_1)$ is an invariant of $X^{(1)}$, then $w_1 = (D_x w)/(D_x y)$ is an invariant of $X^{(2)}$, and so on. The set $\{y, w, w_1, \ldots, w_{n-1}\}$ is thus a complete system of invariants of $X^{(n)}$. By invariance condition (4), original equation (1) can be written in terms of these invariants as an $(n-1)$th-order reduced equation,

$$\tilde{\Delta}(y, w^{(n-1)}) = 0.$$

*Departamento de Matemáticas, Universidad de Cádiz, Cádiz, Spain, e-mail: concepcion.muriel@uca.es.

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It is well known that there are order reductions for ordinary differential equations that do not come from the existence of Lie symmetries. Some examples can be seen in [2] and [3].

In this paper we characterize the different ways of prolonging a vector field that lead to a similar reduction process. The equations in the mentioned examples can be reduced by this new method.

In Sec. 2, we obtain several characterizations of the prolongations

$$X_n^* = \xi(x,u)\partial x + \eta(x,u)\partial u + \sum_{i=1}^{n} \eta_i^*(x,u^{(i)})\partial u_i$$

for which a complete system of invariants can be calculated by derivation of lower-order invariants.

In a natural way, these new prolongations lead to the concept of C^∞-symmetries [3]. These symmetries are characterized in Sec. 3 and the corresponding method of reduction is described in Theorem 4.

In Sec. 4, we also prove that a large class of reduction processes are special cases of our method. Finally, this reduction method is applied to an equation that lacks Lie symmetries.

2. Main results

We let X be vector field (2) on $M \subset \mathbb{R}^2$ and let

$$X_n^* = \xi(x,u)\frac{\partial}{\partial x} + \eta(x,u)\frac{\partial}{\partial u} + \sum_{i=1}^{n} \eta_i^*(x,u^{(i)})\frac{\partial}{\partial u_i} \tag{4}$$

be an arbitrary prolongation of X to $M^{(n)}$. For $k = 1,\ldots,n$, X_k^* denotes the corresponding projection of X_n^* to the kth jet space $M^{(k)}$.

Definition 1. Let $n \in \mathbb{N}$. We say that X_n^* has the ID (invariants-by-derivation) property of the nth order if whenever $f = f(x,u)$ is an invariant of X and $g_0 = g_0(x,u,u_1)$ is an invariant of X_1^*, the function $g_k = D_x g_{k-1}/(D_x f)$ is an invariant of X_{k+1}^* for $k = 1,\ldots,n-1$ and the set $\{f, g_0, \ldots, g_{n-1}\}$ constitutes a complete system of functionally independent invariants of X_n^*.

We suppose that X_n^* has the ID property of the nth order and let $\{y = y(x,u),\ \alpha = \alpha(x,u)\}$ be such that $X(y) = 0$ and $X(\alpha) = 1$. We consider the system of local coordinates on $M^{(n)}$

$$\{y, \alpha, \alpha_1, \alpha_2, \ldots, \alpha_n\}, \tag{5}$$

where $\alpha_k = D_x\alpha_{k-1}/(D_x y)$ and $\alpha_0 = \alpha$ for $k \in \{1,\ldots,n\}$. We let

$$X_n^* = \frac{\partial}{\partial \alpha} + r_1^*(y,\alpha,\alpha_1)\frac{\partial}{\partial \alpha_1} + \cdots + r_n^*(y,\alpha,\ldots,\alpha_n)\frac{\partial}{\partial \alpha_n} \tag{6}$$

be the expression of X_n^* in terms of (5).

We next prove that if X_n^* has the ID property of the nth order, then the infinitesimals r_k^* for $k = 2,\ldots,n$ are completely determined by r_1^*.

We evaluate the Lie bracket $[X_n^*, D_y]$ in terms of coordinates (5),

$$
\begin{aligned}
&[X_n^*, D_y](y) = 0,\\
&[X_n^*, D_y](\alpha) = X_n^*(\alpha_1) = r_1^*(y, \alpha, \alpha_1),\\
&[X_n^*, D_y](\alpha_1) = X_n^*(\alpha_2) - D_y(X_n^*(\alpha_1)) = r_2^*(y, \alpha, \alpha_1, \alpha_2) - D_y\big(r_1^*(y, \alpha, \alpha_1)\big),\\
&\qquad\vdots\\
&[X_n^*, D_y](\alpha_i) = X_n^*(\alpha_{i+1}) - D_y\big(X_n^*(\alpha_i)\big) =\\
&\qquad\qquad = r_{i+1}^*(y, \alpha, \ldots, \alpha_{i+1}) - D_y\big(r_i^*(y, \alpha, \ldots, \alpha_i)\big),\\
&\qquad\vdots\\
&[X_n^*, D_y](\alpha_{n-1}) = X_n^*(\alpha_n) - D_y\big(X_n^*(\alpha_{n-1})\big) =\\
&\qquad\qquad = r_n^*(y, \alpha, \ldots, \alpha_n) - D_y\big(r_{n-1}^*(y, \alpha, \ldots, \alpha_{n-1})\big).
\end{aligned}
\tag{7}
$$

Let $w = w(y, \alpha, \alpha_1)$ be an invariant for X_n^* that is functionally independent of y. In this case, $\partial w/\partial\alpha_1 \neq 0$, and we have

$$
\frac{\partial w}{\partial \alpha} + r_1^* \frac{\partial w}{\partial \alpha_1} = 0. \tag{8}
$$

Because X_n^* has the ID property of the nth order, we also have $X_n^*(w_y) = X_n^*\big(D_y(w)\big) = [X_n^*, D_y](w) = 0$. Therefore,

$$
\frac{\partial w}{\partial \alpha} r_1^* + \big(r_2^* - D_y(r_1^*)\big) \frac{\partial w}{\partial \alpha_1} = 0. \tag{9}
$$

We observe that if $r_1^* \equiv 0$, then necessarily $X_n^* = X^{(n)}$, and X_n^* is hence the ordinary Lie prolongation of X. We can therefore assume that r_1^* is not a null function. If we multiply (8) by $-r_1^*$ and add (9) to the resulting equation, we obtain

$$
-(r_1^*)^2 \frac{\partial w}{\partial \alpha_1} + \big(r_2^* - D_y(r_1^*)\big) \frac{\partial w}{\partial \alpha_1} = 0.
$$

Because $\partial w/\partial\alpha_1 \neq 0$, we have

$$
r_2^* = (D_y + r_1^*)(r_1^*).
$$

Because X_n^* has the ID property of the nth order, it follows that $X_n^*(w_1) = 0$. Therefore,

$$
\frac{\partial w_1}{\partial \alpha} + \frac{\partial w_1}{\partial \alpha_1} r_1^* + \frac{\partial w_1}{\partial \alpha_2} r_2^* = 0. \tag{10}
$$

By the ID property, we also have $X_n^*(w_2) = X_n^*\big(D_y(w_1)\big) = [X_n^*, D_y](w_1) = 0$. Hence,

$$
\frac{\partial w_1}{\partial \alpha} r_1^* + \frac{\partial w_1}{\partial \alpha_1} (r_1^*)^2 + \frac{\partial w_1}{\partial \alpha_2} \big(r_3^* - D_y(r_2^*)\big) = 0. \tag{11}
$$

If we add Eq. (11) and the result of multiplying Eq. (10) by $-r_1^*$, we obtain

$$
\frac{\partial w_1}{\partial \alpha_2} \big(-r_1^* r_2^* + r_3^* - D_y(r_2^*)\big) = 0,
$$

and because $\partial w_1/\partial\alpha_2 \neq 0$,

$$r_3^* = r_1^* r_2^* + D_y(r_2^*).$$

Therefore, r_3^* can be written as

$$r_3^* = (D_y + r_1^*)(r_2^*).$$

By thus proceeding, we find that the infinitesimals of X_n^* in (6) are necessarily given by

$$r_{k+1}^* = (D_y + r_1^*)(r_k^*), \quad k = 1, \ldots, n-1.$$

Therefore,

$$X_n^* = \sum_{k=0}^{n} (D_y + r_1^*)^k(1)\frac{\partial}{\partial\alpha_k}. \tag{12}$$

By (7) and (12), we have also proved that X_n^* satisfies

$$[X_n^*, D_y] = r_1^* X_n^*.$$

We have proved that if X_n^* has the ID property of the nth order, then the function r_1^* determines the entire prolongation X_n^* in the variables $\{y, \alpha, \ldots, \alpha_n\}$: if $X(y) = 0$ and $X(\alpha) = 1$, then the function r_1^* is the coefficient of $\partial/\partial\alpha_1$ in the vector field X_n^*.

We now analyze the above results in terms of the coordinates $\{x, u, u_1, \ldots, u_n\}$. Because $D_x = (1/D_y x)D_y$, we have

$$[X_n^*, D_x] = \frac{1}{D_y x}[X_n^*, D_y] + X_n^*\left(\frac{1}{D_y x}\right) D_y = \frac{1}{D_y x} r_1^* X_n^* - \frac{X_n^*(D_y x)}{(D_y x)^2} D_y =$$

$$= \frac{r_1^*}{D_y x} X_n^* - \frac{r_1^* X_n^*(x) + D_y\big(X_n^*(x)\big)}{D_y x}\frac{D_y}{D_y x} = \lambda X_n^* + \mu D_x,$$

where λ denotes the function $r_1^*/(D_y x)$ in terms of the coordinates $\{x, u, u_1\}$ and $\mu = -(D_x + \lambda)\big(X_n^*(x)\big)$.

Hence, we have proved that if X is any given vector field (2) in $M \subset \mathbb{R}^2$, $X(y) = 0$, $X(\alpha) = 1$, and $r_1^* = r_1^*(y, \alpha, \alpha_1)$ is an arbitrary function, then the prolongation X_n^* of $X = \partial/\partial\alpha$ given by (12) is such that

$$[X_n^*, D_x] = \lambda X_n^* + \mu D_x \tag{13}$$

in terms of the coordinates $\{x, u, \ldots, u_n\}$.

Conversely, we now prove that if X_n^* satisfies (13) for some $\lambda \in C^\infty(M^{(1)})$ and $\mu = -(D_x + \lambda)(X_n^*(x))$, then X_n^* has the ID property. Let $f = f(x, u)$ and $g = g(x, u^{(k)}) \in C^\infty(M^{(k)})$ be two functionally independent invariants of X_n^*. We have $X_n^*\big(f(x, u)\big) = X_n^*\big(g(x, u^{(k)})\big) = 0$ and

$$X_n^*\left(\frac{D_x g}{D_x f}\right) = \frac{1}{(D_x f)^2}\big(D_x f \cdot X_n^*(D_x g) - D_x g \cdot X_n^*(D_x f)\big) =$$

$$= \frac{1}{(D_x f)^2}\big(D_x f \cdot [X_n^*, D_x](g) - D_x g \cdot [X_n^*, D_x](f)\big) =$$

$$= \frac{1}{(D_x f)^2}\big(D_x f \cdot (\mu \cdot D_x g) - D_x g \cdot (\mu \cdot D_x f)\big) = 0.$$

We have thus proved the following result.

Theorem 1. *Let X be vector field (2) defined on M, and let X_n^* be a prolongation of X to $M^{(n)}$ of form (4). Then X_n^* has the ID property of the nth order if and only if X_n^* satisfies (13) for some $\lambda \in C^\infty(M^{(1)})$ and $\mu = -(D_x + \lambda)(X_n^*(x))$.*

We now verify that Eq. (13) uniquely determines the coefficients η_i^* of X_n^* in terms of ξ, η, and λ. We first apply the first term in Eq. (13) to each of the coordinate functions of $\{x, u, u_1, \ldots, u_n\}$,

$$\begin{aligned}
&[X_n^*, D_x](x) = -D_x\big(\xi(x,u)\big),\\
&[X_n^*, D_x](u) = \eta_1^*(x, u^{(1)}) - D_x\big(\eta(x,u)\big),\\
&[X_n^*, D_x](u_1) = \eta_2^*(x, u^{(2)}) - D_x\big(\eta_1^*(x, u^{(1)})\big),\\
&\qquad\vdots\\
&[X_n^*, D_x](u_i) = \eta_{i+1}^*(x, u^{(i+1)}) - D_x\big(\eta_i^*(x, u^{(i)})\big),\\
&\qquad\vdots\\
&[X_n^*, D_x](u_{n-1}) = \eta_n^*(x, u^{(n)}) - D_x\big(\eta_{n-1}^*(x, u^{(n-1)})\big).
\end{aligned} \tag{14}$$

If we now apply the second term in Eq. (13), which can be written as $Y = \lambda\big(X_n^* - \xi(x,u)D_x\big) - D_x\big(\xi(x,u)\big)D_x$, to each of the coordinates of $\{x, u, u_1, \ldots, u_n\}$, we obtain

$$\begin{aligned}
&Y(x) = -D_x\big(\xi(x,u)\big),\\
&Y(u) = \lambda\big(\eta(x,u) - \xi(x,u)u_1\big) - D_x\big(\xi(x,u)\big)u_1,\\
&Y(u_1) = \lambda\big(\eta_1^*(x, u^{(1)}) - \xi(x,u)u_2\big) - D_x\big(\xi(x,u)\big)u_2,\\
&\qquad\vdots\\
&Y(u_i) = \lambda\big(\eta_i^*(x, u^{(i)}) - \xi(x,u)u_{i+1}\big) - D_x\big(\xi(x,u)\big)u_{i+1},\\
&\qquad\vdots\\
&Y(u_{n-1}) = \lambda\big(\eta_{n-1}^*(x, u^{(n-1)}) - \xi(x,u)u_n\big) - D_x\big(\xi(x,u)\big)u_n.
\end{aligned} \tag{15}$$

Equating the second terms in Eqs. (14) and (15), we obtain

$$\eta_{i+1}^*(x, u^{(i+1)}) = D_x\big(\eta_i^*(x, u^{(i)})\big) - D_x\big(\xi(x,u)\big)u_{i+1} + \lambda\big(\eta_i^*(x, u^{(i)}) - \xi(x,u)u_{i+1}\big) \tag{16}$$

for $i = 0, \ldots, n-1$. Expression (16) is the motivation for the following definition [3].

Definition 2. Let X be vector field (2) defined on M, and let $\lambda \in C^\infty(M^{(1)})$ be an arbitrary function. The λ-prolongation of X of the order n, denoted by $X^{[\lambda,(n)]}$, is the vector field on $M^{(n)}$ defined by

$$X^{[\lambda,(n)]} = \xi(x,u)\frac{\partial}{\partial x} + \sum_{i=0}^{n} \eta^{[\lambda,(i)]}(x, u^{(i)})\frac{\partial}{\partial u_i}, \tag{17}$$

where $\eta^{[\lambda,(0)]}(x,u)=\eta(x,u)$ and

$$\eta^{[\lambda,(i)]}(x,u^{(i)})=D_x\big(\eta^{[\lambda,(i-1)]}(x,u^{(i-1)})\big)-D_x\big(\xi(x,u)\big)u_i+ \\ +\lambda\big(\eta^{[\lambda,(i-1)]}(x,u^{(i-1)})-\xi(x,u)u_i\big),\quad 1\le i\le n.$$

Definition 2 and Theorem 1 allow giving different characterizations of the λ-prolongation of a given vector field X.

Theorem 2. *Let X be vector field (2) defined on M. Let X_n^* be a prolongation of X to the nth jet space $M^{(n)}$. The following conditions on X_n^* are equivalent:*

1. *X_n^* has the ID property.*
2. *$[X_n^*,D_x]=\lambda X_n^*+\mu D_x$ for some function $\lambda\in C^\infty(M^{(1)})$ and $\mu=-(D_x+\lambda)\big(X_n^*(x)\big)$.*
3. *$X_n^*=X^{[\lambda,(n)]}$ for some $\lambda\in C^\infty(M^{(1)})$.*
4. *If $Q=X(u)-X(x)u_1$ is the characteristic of the vector field X, then*

$$X_n^*=\sum_{i=0}^{n}(D_x+\lambda)^i(Q)\frac{\partial}{\partial u_i}+X(x)D_x$$

for some $\lambda\in C^\infty(M^{(1)})$.

Proof. The equivalence of conditions 1 and 2 is given by Theorem 1. In the paragraphs preceding Definition 2, we proved that conditions 2 and 3 are also equivalent. We now verify that $3\Leftrightarrow 4$. Clearly, Eq. (17) can be written as

$$\begin{aligned}\eta^{[\lambda,(i)]}(x,u^{(i)})&=(D_x+\lambda)\big(\eta^{[\lambda,(i-1)]}(x,u^{(i-1)})\big)-(D_x+\lambda)\big(\xi(x,u)\big)u_i=\\&=(D_x+\lambda)\big(\eta^{[\lambda,(i-1)]}(x,u^{(i-1)})\big)-(D_x+\lambda)\big(\xi(x,u)u_i\big)+\xi(x,u)u_{i+1}=\\&=(D_x+\lambda)\big(\eta^{[\lambda,(i-1)]}(x,u^{(i-1)})-\xi(x,u)u_i\big)+\xi(x,u)u_{i+1}\end{aligned}$$

for $1\le i\le n$. For $i-1$, the previous formula becomes

$$\eta^{[\lambda,(i-1)]}(x,u^{(i-1)})-\xi(x,u)u_i=(D_x+\lambda)\big(\eta^{[\lambda,(i-2)]}(x,u^{(i-2)})-\xi(x,u)u_{i-1}\big).$$

Hence, by recurrence,

$$\begin{aligned}\eta^{[\lambda,(i)]}(x,u^{(i)})&=(D_x+\lambda)\big(\eta^{[\lambda,(i-1)]}(x,u^{(i-1)})-\xi(x,u)u_i\big)+\xi(x,u)u_{i+1}=\\&=(D_x+\lambda)^2\big(\eta^{[\lambda,(i-2)]}(x,u^{(i-2)})-\xi(x,u)u_{i-1}\big)+\xi(x,u)u_{i+1}=\cdots=\\&=(D_x+\lambda)^i\big(\eta^{[\lambda,(0)]}(x,u)-\xi(x,u)u_1\big)+\xi(x,u)u_{i+1}=\\&=(D_x+\lambda)^i(Q)+\xi(x,u)u_{i+1}.\end{aligned}$$

This proves the equivalence of conditions 3 and 4.

3. Applications to order reductions of ordinary differential equations

As we have seen in the previous section, the λ-prolongation of a vector field X produces a vector field $X^{[\lambda,(n)]}$ on $M^{(n)}$ that has the ID property of the nth order. Because that property is the basis of the Lie method for reducing the order of ordinary differential equations, it is natural to expect a useful role of λ-prolongations in obtaining reduction processes. We consider the vector fields that admit a λ-prolongation for some $\lambda \in C^\infty(M^{(1)})$ that leaves the equation invariant [3].

Definition 3. Let $\Delta(x, u^{(n)}) = 0$ be an nth-order ordinary differential equation with $(x, u) \in M \subset \mathbb{R}^2$. A vector field X defined on M is a $C^\infty(M^{(1)})$-*symmetry* of the equation if there exists a function $\lambda \in C^\infty(M^{(1)})$ such that

$$X^{[\lambda,(n)]}\big(\Delta(x, u^{(n)})\big) = 0, \qquad \Delta(x, u^{(n)}) = 0.$$

In this case, we also say that X is a λ-symmetry of the given equation.

Our next result presents a characterization of $C^\infty(M^{(1)})$-symmetries based on the second characterization of the λ-prolongation given in Theorem 2.

We suppose that the equation $\Delta(x, u^{(n)}) = 0$ can be written locally in the explicit form

$$u_n = F(x, u^{(n-1)}). \tag{18}$$

With this equation, we associate the vector field defined on $M^{(n-1)}$

$$A = \frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial u} + \cdots + F(x, u^{(n-1)}) \frac{\partial}{\partial u_{n-1}}.$$

Theorem 3. *If a vector field X is a $C^\infty(M^{(1)})$-symmetry of Eq.* (18) *for some $\lambda \in C^\infty(M^{(1)})$, then*

$$[X^{[\lambda,(n-1)]}, A] = \lambda \cdot X^{[\lambda,(n-1)]} + \mu \cdot A \tag{19}$$

for $\mu = -(D_x + \lambda)\big(X(x)\big) \in C^\infty\big(M^{(1)}\big)$. Conversely, if

$$X_n^* = \xi(x, u) \frac{\partial}{\partial x} + \sum_{i=0}^{n-1} \eta_i^*(x, u^{(i)}) \frac{\partial}{\partial u_i}$$

is a vector field defined on $M^{(n-1)}$ such that $[X_n^*, A] = \lambda \cdot X_n^* + \mu \cdot A$ for some $\lambda, \mu \in C^\infty(M^{(1)})$, then

$$X = \xi(x, u) \frac{\partial}{\partial x} + \eta_0^*(x, u) \frac{\partial}{\partial u}$$

is a λ-symmetry of Eq. (18)*, and $X_n^* = X^{[\lambda,(n-1)]}$.*

Proof. We observe that $D_x(x) = A(x)$ and $D_x(u_i) = A(u_i)$ for $i = 0, \ldots, n-2$. Theorem 2 proves that the values of the two terms in (19) coincide on the set $\{x, u, \ldots, u_{n-2}\}$. On the other hand,

$$\begin{aligned}
[X^{[\lambda,(n-1)]}, A](u_{n-1}) &= X^{[\lambda,(n-1)]}\big(A(u_{n-1})\big) - A\big(X^{[\lambda,(n-1)]}(u_{n-1})\big) = \\
&= X^{[\lambda,(n-1)]}\big(F(x, u^{(n-1)})\big) - A\big(\eta^{[\lambda,(n-1)]}(x, u^{(n-1)})\big), \\
X^{[\lambda,(n)]}(u_n) &= D_x\big(\eta^{[\lambda,(n-1)]}(x, u^{(n-1)})\big) - D_x\big(\xi(x,u)\big) u_n + \\
&\quad + \lambda\big(\eta^{[\lambda,(n-1)]}(x, u^{(n-1)}) - \xi(x,u) u_n\big).
\end{aligned} \tag{20}$$

Because X is a λ-symmetry,

$$X^{[\lambda,(n)]}(u_n) = X^{[\lambda,(n-1)]}\big(F(x,u^{(n-1)})\big), \qquad u_n = F(x,u^{(n-1)}).$$

Hence, the second equation in (20) says that

$$\begin{aligned} X^{[\lambda,(n-1)]}\big(F(x,u^{(n-1)})\big) &= A\big(\eta^{[\lambda,(n-1)]}(x,u^{(n-1)})\big) - A\big(\xi(x,u)\big)F(x,u^{(n-1)}) + \\ &\quad + \lambda\big(\eta^{[\lambda,(n-1)]}(x,u^{(n-1)}) - \xi(x,u)F(x,u^{(n-1)})\big) = \\ &= A\big(\eta^{[\lambda,(n-1)]}(x,u^{(n-1)})\big) + \lambda\big(\eta^{[\lambda,(n-1)]}(x,u^{(n-1)}) + \mu F(x,u^{(n-1)})\big). \end{aligned}$$

Therefore,

$$\begin{aligned} [X^{[\lambda,(n-1)]}, A](u_{n-1}) &= \lambda\big(\eta^{[\lambda,(n-1)]}(x,u^{(n-1)}) + \mu F(x,u^{(n-1)})\big) = \\ &= \lambda X^{[\lambda,(n-1)]}(u_{n-1}) + \mu F(x,u^{(n-1)}) = (\lambda \cdot X^{[\lambda,(n-1)]} + \mu \cdot A)(u_{n-1}), \end{aligned}$$

and we obtain (19).

We now prove the converse assertion. If we apply both terms in $[X_n^*, A] = \lambda \cdot X_n^* + \mu \cdot A$ to x, we find that necessarily $\mu = -(D_x + \lambda)\big(\xi(x,u)\big)$. Hence, by Theorem 2, we have $X_n^* = X^{[\lambda,(n-1)]}$. Because $[X_n^*, A](u_{n-1}) = \lambda X_n^*(u_{n-1}) + \mu A(u_{n-1})$, we obtain

$$\begin{aligned} X_n^*\big(F(x,u^{(n-1)})\big) &= A\big(\eta^{[\lambda,(n-1)]}(x,u^{(n-1)})\big) + \lambda\eta^{[\lambda,(n-1)]}(x,u^{(n-1)}) - \\ &\quad - \big(D_x(\xi(x,u)) + \lambda\xi(x,u)\big) \cdot F(x,u^{(n-1)}). \end{aligned} \tag{21}$$

To verify that X satisfies

$$X^{[\lambda,(n)]}\big(u_n - F(x,u^{(n-1)})\big) = 0, \qquad u_n = F(x,u^{(n-1)}), \tag{22}$$

we evaluate

$$\begin{aligned} X^{[\lambda,(n)]}\big(u_n - F(x,u^{(n-1)})\big) &= D_x\big(\eta^{[\lambda,(n-1)]}(x,u^{(n-1)})\big) - u_n D_x\big(\xi(x,u)\big) + \\ &\quad + \lambda\big(\eta^{[\lambda,(n-1)]}(x,u^{(n-1)}) - \xi(x,u)u_n\big) - X^{[\lambda,(n)]}\big(F(x,u^{(n-1)})\big) \end{aligned}$$

for $u_n = F(x,u^{(n-1)})$. By (21), we obtain (22). Therefore, X is a λ-symmetry of the equation.

Our next objective is to show how a $C^\infty(M^{(1)})$-symmetry can be used to reduce the order of an ordinary differential equation.

Theorem 4. *Let X be a λ-symmetry, where $\lambda \in C^\infty(M^{(1)})$, of the equation $\Delta(x,u^{(n)}) = 0$. Let $y = y(x,u)$ and $w = w(x,u,u_1)$ be two functionally independent invariants of $X^{[\lambda,(n)]}$. The general solution of the equation can be obtained by solving an equation of the form $\Delta_r(y,w^{(n-1)}) = 0$ and an auxiliary equation $w = w(x,u,u_1)$.*

Proof. Let $y = y(x,u)$ and $w = w(x,u,u_1)$ be two functionally independent invariants of $X^{[\lambda,(n)]}$ such that w depends on u_1. By Theorem 1, $w_1 = D_x w/(D_x y)$ is an invariant for $X^{[\lambda,(n)]}$, which is functionally independent of y and w because w_1 depends on u_2. From w_1 and y, we construct a third-order invariant for $X^{[\lambda,(n)]}$ by derivation, and so on. Therefore, the set $\{y, w, w_1, \dots, w_{n-1}\}$ is a complete set of functionally independent invariants for $X^{[\lambda,(n)]}$. Because X is a $C^\infty(M^{(1)})$-symmetry of the equation by hypothesis, this can be written in terms of $\{y, w, w_1, \dots, w_{n-1}\}$. The resulting equation is a $(n-1)$th-order equation of the form

$$\Delta_r(y, w^{(n-1)}) = 0. \tag{23}$$

We can recover the general solution of the original equation from the general solution of (23) and the corresponding first-order auxiliary equation

$$w = w(x, u, u_1).$$

4. C^∞-symmetries and order reductions

In this section, we show that many of the known reduction processes for ordinary differential equations can be obtained via the above method as a consequence of the existence of C^∞-symmetries of the given equations.

Theorem 5. *Let*

$$\Delta_1(x, u^{(n)}) = 0 \tag{24}$$

be an nth-order ordinary differential equation. If there exists a transformation

$$\begin{aligned} y &= y(x, u), \\ w &= w(x, u, u_1), \end{aligned} \tag{25}$$

where $\partial w/\partial u_1 \neq 0$, such that (24) *can be written in terms of the variables (y, w) as*

$$\Delta_2(y, w^{(n-1)}) = 0, \tag{26}$$

then there exists a C^∞-symmetry X of Eq. (24) *such that* (26) *is the corresponding reduced equation.*

Proof. Let $\alpha \in C^\infty(M)$ be such that the functions y and α are functionally independent. We set $\alpha_1 = D_x\alpha/(D_x y) \in C^\infty(M^{(1)})$ and consider the local coordinates (y, α, α_1) on $M^{(1)}$. We determine a vector field of the form

$$X = \xi(y, \alpha)\frac{\partial}{\partial y} + \eta(y, \alpha)\frac{\partial}{\partial \alpha}$$

and a function $\lambda(y, \alpha, \alpha_1) \in C^\infty(M^{(1)})$ such that X is a λ-symmetry of the equation and the functions y and w are invariants of $X^{[\lambda,(1)]}$.

We set $\xi = 0$ and $\eta = 1$ and determine λ from the condition $X^{[\lambda,(1)]}(w) = 0$. Because

$$X^{[\lambda,(1)]} = \frac{\partial}{\partial \alpha} + \lambda\frac{\partial}{\partial \alpha_1}$$

by Definition 2, we deduce that $\lambda = -(\partial w/\partial \alpha)/(\partial w/\partial \alpha_1)$.

We prove that the vector field $X = \partial/\partial\alpha$ is a λ-symmetry of the equation for the function $\lambda = -(\partial w/\partial \alpha)/(\partial w/\partial \alpha_1)$. We set $w_i = d^{(i)}w/dy^{(i)}$ for $1 \leq i \leq n-1$. It is clear that the set $\{y, \alpha, w, \dots, w_{n-1}\}$ is a system of coordinates in $M^{(n)}$. From the construction of X and λ, it follows that $\{y, w, \dots, w_{n-1}\}$ are invariants for the vector field $X^{[\lambda,(n)]}$, and therefore $X^{[\lambda,(n)]} = \partial/\partial\alpha$ in terms of the new local coordinates. Because Eq. (24) can be written in terms of these local coordinates as Eq. (26) by hypothesis, we obtain

$$X^{[\lambda,(n)]}\big(\Delta_2(y, w^{(n-1)})\big) = \frac{\partial}{\partial \alpha}\big(\Delta_2(y, w^{(n-1)})\big) = 0.$$

This proves that X is a λ-symmetry of the equation.

To verify that (26) is the reduced equation that corresponds to the λ-symmetry by Theorem 4, it suffices to observe that the reduced equation can be obtained by writing the equation in terms of the complete system $\{y, w, \ldots, w_{n-1}\}$ of invariants of $X^{[\lambda,(n)]}$.

Example. We consider the second-order differential equation

$$uu_{xx} - 2u_x^2 + u^2 u_x x - u^4 x^2 - u^3 - u^2 = 0. \tag{27}$$

This equation has no Lie symmetries, which is proved in the appendix. The Lie reduction method therefore cannot be used to reduce its order. But it can be shown that by the transformation $w = u_x/u - xu$, Eq. (27) becomes

$$w_x = w^2 + 1. \tag{28}$$

By Theorem 5, this reduction corresponds to the existence of a λ-symmetry of the equation. We calculate a vector field X and a function λ such that $X(x) = 0$ and $X^{[\lambda,(1)]}(w) = 0$. Then $\xi = 0$ and by Definition 2,

$$\eta\left(-\frac{u_x}{u^2} - x\right) + \big(D_x(\eta) + \lambda\eta\big)\left(\frac{1}{u}\right) = 0. \tag{29}$$

It is clear that if we choose $\eta = 1$, then $\lambda = ux + u_x/u$ satisfies (29). By Theorem 5, $X = \partial/\partial u$ is a λ-symmetry of Eq. (27) for $\lambda = ux + u_x/u$. This can also be explicitly proved by verifying the equation in Definition 2. In this case, we have

$$X^{[\lambda,(2)]} = \frac{\partial}{\partial u} + \left(ux + \frac{u_x}{u}\right)\frac{\partial}{\partial u_x} + \left(u + 3u_x x + u^2 x^2 + \frac{u_{xx}}{u}\right)\frac{\partial}{\partial u_{xx}}.$$

If we let Δ denote the left-hand side of (27), we can verify that

$$X^{[\lambda,(2)]}(\Delta) = \frac{2}{u}\Delta,$$

i.e., $X = \partial/\partial u$ is a λ-symmetry of Eq. (27). By construction, x and $w = u_x/u - xu$ are invariants of $X^{[\lambda,(1)]}$, and Eq. (28) is the reduced equation that corresponds to X. It is also clear that Eq. (28) can be integrated in quadratures.

5. Conclusions

The classical Lie method for reducing the order of ordinary differential equations with a Lie symmetry X is based on the existence of a complete system of invariants for the prolongation $X^{(n)}$ that can be calculated by derivation of lower-order invariants.

We have characterized the prolongations of vector fields with that property. The invariance of the equation under any of these prolongations leads to the concept of the C^∞-symmetry. As proved in Theorem 4, an algorithm for reducing the order of the equation is associated with these new symmetries. Many of the known reduction processes for ordinary differential equations can be obtained via the above method as a consequence of the existence of C^∞-symmetries of the given equations.

Appendix

We here prove that Eq. (27) has no Lie symmetries. A vector field

$$X = p(x,u)\frac{\partial}{\partial x} + r(x,u)\frac{\partial}{\partial u}$$

is a Lie symmetry of Eq. (27) if the infinitesimals p and r satisfy the determining system

$$\text{e1}: \quad 2p_u + up_{uu} = 0,$$

$$\text{e2}: \quad 2p_u u^3 x + r_{uu}u^2 - 2p_{ux}u^2 - 2r_u u + 2r = 0,$$

$$\text{e3}: \quad -3p_u u^4 x^2 + p_x u^2 x + rux - 3p_u u^3 - 3p_u u^2 + pu^2 + 2r_{ux}u - p_{xx}u - 4r_x = 0,$$

$$\text{e4}: \quad (r_u - 2p_x)u^3x^2 - (3rux + 2pu^2 - r_x)ux + (r_u - 2p_x)u^2 + (r_u - 2r - 2p_x)u + r_{xx} - r = 0.$$

The first equation shows that p must be given by

$$p(x,u) = p_1(x) + p_2(x)u^{-1}.$$

After substituting this value in equality e2, we have

$$-2p_2ux + r_{uu}u^2 - 2r_u u + 2r + 2{p_2}' = 0.$$

Integrating with respect to u, we obtain

$$r(x,u) = -2p_2u(\log u + 1)x + r_1u^2 + r_2u - p_2',$$

where r_1 and r_2 are functions of x. Then equality e3 becomes

$$-2u\log u(p_2ux^2 - 2p_2'x - 2p_2) + (p_2x^2 + r_1ux + r_2x + p_1'x + p_1)u^2 - (2r_2' - 4p_2 + p_1'')u + 3(p_2'' + p_2) = 0.$$

We deduce that $p_2 = r_1 = 0$ and

$$r_2 = -p_1' - \frac{p_1}{x}.$$

Equality e4 becomes

$$(2p_1'x^2 - 2p_1'x^3 - p_1''x^4)u + (-p_1'''x^3 - 2p_1'x^3 - p_1''x^2 + 2p_1'x - 2p_1) = 0,$$

and we deduce that $p_1 = c_1x + c_2/x^2$ for $c_1, c_2 \in \mathbb{R}$ such that

$$c_1x^5 + 2c_2x^2 + 6c_2 = 0.$$

Therefore, $c_1 = c_2 = 0$. This proves that Eq. (27) has no Lie symmetries.

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