Integrability of Equations Admitting the Nonsolvable Symmetry Algebra *so*(3,R)

By C. Muriel and J. L. Romero

If an ordinary differential equation admits the nonsolvable Lie algebra $so(3,\mathbb{R})$, and we use any of its generators to reduce the order, the reduced equation does not inherit the remaining symmetries. We prove here how the lost symmetries can be recovered as C^{∞}-symmetries of the reduced equation. If the order of the last reduced equation is higher than one, these C^{∞}-symmetries can be used to obtain new order reductions. As a consequence, a classification of the third-order equations that admit $so(3,\mathbb{R})$ as symmetry algebra is given and a step-by-step method to solve the equations is presented.

1. Introduction

One of the most utilized methods to find exact solutions of ordinary differential equations is based on the concept of Lie group of transformations that have been widely studied in the literature ([1]–[3]) etc.). If an nth-order differential equation admits a k-dimensional Lie algebra, $\mathcal{G}$ as symmetry algebra, then its general solution can be obtained by means of the general solution of an $(n-k)$th-order reduced equation and the solution of a kth-order auxiliary equation. In the particular case when $\mathcal{G}$ is solvable, the process of integration can be achieved through k successive quadratures, from the general solution of

Address for correspondence: C. Muriel, Departamento de Matemáticas, Universidad de Cádiz, P.O. Box 40, 11510 Puerto Real, Cádiz, Spain. E-mail: concepcion.muriel@uca.es

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the corresponding reduced equation. Nevertheless, if $\mathcal{G}$ is nonsolvable, this step-by-step method of reduction is no longer applicable because, at some stage of the reduction process, at least one of the generators of $\mathcal{G}$ is lost as Lie symmetry.

Some order reduction processes for equations whose symmetry algebra is nonsolvable have recently been studied. This is the case of the Chazy equation [4], studied by Clarkson and Olver [5, 6]. The symmetry group associated to the Chazy equation is the most involved of the three known actions of $SL(2,\mathbb{C})$ on two-dimensional complex spaces. Clarkson and Olver [6] described a connection between these three actions via the standard prolongation process, and use this to interrelate their differential invariants. It allows them to construct fundamental differential invariants, of the most complicated action, from the invariants of the basic unimodular action. If the original equation is written in terms of these fundamental invariants, its order is reduced by three.

Apart from Lie symmetries, the concept of exponential vector field appears in [1]. Although they are not well-defined vector fields, they can be used to obtain order reductions of ordinary differential equations. These exponential vector fields raised many studies on hidden symmetries by several authors (see [7], and the references cited therein). Type I hidden symmetries are the symmetries that are lost by reduction processes and their importance yields in the fact that they can also be used to reduce the order of differential equations for which the classical Lie method is not applicable. The main problem with these symmetries is that there has been no general method for determining them and only several *ad hoc* methods have been worked out ([7]).

In [8] a new class of symmetries (C^{∞}-symmetries) has been introduced. This class contains Lie symmetries and has associated an algorithm to reduce the order of an ordinary differential equation which is more general than that corresponding to Lie symmetries. Many of the known order reduction processes that are not a consequence of the existence of Lie symmetries are a consequence of the invariance of the equation under C^{∞}-symmetries. We have also found some ordinary differential equations whose Lie symmetries are trivial, have no obvious order reductions, but can completely be integrated by using this new class of symmetries.

Type I hidden symmetries can be recovered as C^{∞}-symmetries of the reduced equations and, in particular, the C^{∞}-symmetries that are a consequence of the invariance of the equation under exponential vector fields can be calculated through a well-defined algorithm. While hidden symmetries are manifested as nonlocal symmetries of the reduced equations, whose coordinate functions depend on the integrals of the dependent and independent variables, the C^{∞}-symmetries of an equation are well-defined vector fields on the space of the variables of the equation.

In this article, we consider nth-order differential equations that admit the nonsolvable symmetry algebra $so(3,\mathbb{R})$, associated to the action of the rotation group. It can always be chosen a base of generators $\{X_1, X_2, X_3\}$ of the Lie algebra $so(3,\mathbb{R})$ such that the corresponding Lie brackets are given by

$$\begin{aligned}[X_1, X_2] &= X_3,\\ [X_1, X_3] &= -X_2,\\ [X_2, X_3] &= X_1.\end{aligned} \tag{1}$$

If we use any of the generators X_i to reduce the order of the equation, the remaining are lost as Lie symmetries of the equation are obtained at the first step of the reduction. We prove here that the lost symmetries can be recovered as C^∞-symmetries for the reduced equation. Any of these two C^∞-symmetries can be used to reduce the order again and the unused C^∞-symmetry can be recovered as a C^∞-symmetry of the equation obtained at the second stage of the reduction. As a consequence, the order of the original equation can successively be reduced by three, by means of a process which is somewhat similar to the usual for solvable algebras. The main advantage of this step-by-step method of reduction is that the general solution of the original equation can be recovered from the reduced one by solving three first-order differential equations.

As a consequence, the former method provides the general form of the third-order differential equations that admit the Lie algebra $so(3,\mathbb{R})$ as symmetry algebra: i.e., any equation admitting $so(3,\mathbb{R})$ can be transformed, by a change of variables, into a equation of the form that we present in Section 4.1. Previously, an equivalent general form for these equations had been calculated by Mahomed and Leach [9] by direct methods (see also [10]). Besides, we provide here an algorithm to reduce the order of these equations, and we have obtained a simple general form for the first-order reduced equations that appear at the last step of the reduction. Moreover, we present a method to recover the general solution of the third-order equation by solving two first-order equations that can be solved by quadratures, because one of them is linear and the other one can be solved by simple integration.

2. Notation and preliminary results

Let us consider an nth-order ordinary differential equation

$$\Delta\big(x, u^{(n)}\big) = 0, \tag{2}$$

where $(x, u) \in M$, for some open subset $M \subset X \times U \simeq \mathbb{R}^2$. We denote by $M^{(k)}$ the corresponding k-jet space $M^{(k)} \subset X \times U^{(k)}$, for $k \in \mathbb{N}$. Their elements are $(x, u^{(k)}) = (x, u, u_1, \ldots, u_k)$, where, for $i = 1, \ldots, k$, u_i denotes the derivative of order i of u with respect to x. We assume that the implicit function theorem can be applied to equation (2), and, as a consequence, this equation can locally be written in the explicit form

$$u_n = \Psi\big(x, u^{(n-1)}\big). \tag{3}$$

The vector field

$$A_{(x,u)} = \frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial u} + \cdots + \Psi\big(x, u^{(n-1)}\big) \frac{\partial}{\partial u_{n-1}}$$

will be called the vector field associated with equation (3).

2.1. Lie symmetries and order reductions

It is well known [3] that a vector field X on M is a Lie symmetry of equation (3) if and only if there exists a function $\rho \in C^\infty(M^{(1)})$ such that

$$\big[X^{(n-1)}, A_{(x,u)}\big] = \rho A_{(x,u)},$$

where $X^{(n-1)}$ denotes the usual $(n-1)$th prolongation of the vector field X.

A Lie symmetry X can be used to reduce the order of the equation by one: we introduce a change of variables $\{y = y(x,u), \alpha = \alpha(x,u)\}$ such that the vector field X can be written as $X = \frac{\partial}{\partial \alpha}$, in some open set of variables $\{y, \alpha\}$, that will also be denoted by M. Since X is a Lie symmetry of (3), this equation can be written in terms of variables $\{y, \alpha^{(n)}\}$ of $M^{(n)}$ in the form

$$\alpha_n = \Phi(y, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}). \tag{4}$$

It can easily be checked that the vector field associated with equation (4) is the vector field

$$A_{(y,\alpha)} = \frac{1}{D_x(y(x,u))} A_{(x,u)}, \tag{5}$$

written in the new variables, where D_x denotes the total derivative operator with respect to x.

If we set $w = \alpha_1$ in (4), we obtain a reduced equation

$$w_{n-1} = \Phi(y, w, w_1, \ldots, w_{n-2}), \tag{6}$$

where (y, w) is in some open set $M_1 \subset \mathbb{R}^2$.

Let $\pi_X^{(k)} : M^{(k)} \to M_1^{(k-1)}$ be the projection $(y, \alpha, \alpha_1, \ldots, \alpha_k) \mapsto (y, w, \ldots, w_{k-1}) = (y, \alpha_1, \ldots, \alpha_k)$, for $k \in \mathbb{N}$. A vector field V on $M^{(k)}$ will be called $\pi_X^{(k)}$-projectable if

$$\big[X^{(k)}, V\big] = f X^{(k)},$$

for some function $f \in C^\infty(M^{(k)})$. This implies that V must take the following form in the variables $\{y, \alpha^{(k)}\}$:

$$V = \xi(y, \alpha_1, \ldots, \alpha_k) \frac{\partial}{\partial y} + \eta(y, \alpha, \alpha_1, \ldots, \alpha_k) \frac{\partial}{\partial \alpha} + \sum_{i=1}^{k} \eta_i(y, \alpha_1, \ldots, \alpha_k) \frac{\partial}{\partial \alpha_i}.$$

The $\pi_X^{(k)}$-projection of V on $M_1^{(k-1)}$ is the vector field

$$\left(\pi_X^{(k)}\right) * (V) = \xi(y, w, \ldots, w_{k-1})\frac{\partial}{\partial y} + \sum_{i=1}^{k} \eta_i(y, w, \ldots, w_{k-1})\frac{\partial}{\partial w_{i-1}}.$$

With this definition, it can be checked that the vector field $A_{(y,\alpha)}$ is $\pi_X^{(n-1)}$-projectable and its projection is the vector field $A_{(y,w)}$ associated with reduced equation (6).

2.2. C^∞-symmetries and order reductions

The concept of Lie symmetry for an ordinary differential equation can be generalized in several ways: conditional symmetries, Lie–Bäcklund symmetries, etc. In [8] we have introduced the concept of C^∞-symmetry. This concept is somewhat similar to the concept of Lie symmetry, but it is based on a different way to prolong vector fields:

Definition 1. (Generalized prolongation formula). *Let $X = \xi(x, u)\frac{\partial}{\partial x} + \eta(x, u)\frac{\partial}{\partial u}$ be a vector field defined on M, and let $\lambda \in C^\infty(M^{(1)})$ be an arbitrary function. The λ-prolongation of order n of X, denoted by $X^{[\lambda,(n)]}$, is the vector field defined on $M^{(n)}$ by*

$$X^{[\lambda,(n)]} = \xi(x, u)\frac{\partial}{\partial x} + \sum_{i=0}^{n}\eta^{[\lambda,(i)]}\left(x, u^{(i)}\right)\frac{\partial}{\partial u_i},$$

where $\eta^{[\lambda,(0)]}(x, u) = \eta(x, u)$ and

$$\begin{aligned}\eta^{[\lambda,(i)]}\left(x, u^{(i)}\right) &= D_x\left(\eta^{[\lambda,(i-1)]}\left(x, u^{(i-1)}\right)\right) - D_x(\xi(x, u))u_i \\ &\quad + \lambda\left(\eta^{[\lambda,(i-1)]}\left(x, u^{(i-1)}\right) - \xi(x, u)u_i\right),\end{aligned}$$

for $1 \le i \le n$.

Let us observe that, if $\lambda = 0$, the λ-prolongation of order n of X is the usual nth prolongation of X.

Definition 2. *Let $\Delta(x, u^{(n)}) = 0$ be an nth-order ordinary differential equation. We will say that a vector field X, defined on M, is a $C^\infty(M^{(1)})$-symmetry of the equation if there exists a function $\lambda \in C^\infty(M^{(1)})$ such that*

$$X^{[\lambda,(n)]}\left(\Delta\left(x, u^{(n)}\right)\right) = 0, \quad when \quad \Delta\left(x, u^{(n)}\right) = 0.$$

In this case we will also say that X is a λ-symmetry, or a C^∞-symmetry if there is no place for confusion.

Let us observe that if X is a 0-symmetry then X is a classical Lie symmetry.

In [8] it is proved that a vector field X on M is a $C^{\infty}(M^{(1)})$-symmetry of equation (3) if and only if there exist two functions, $\lambda, \rho \in C^{\infty}(M^{(1)})$, such that

$$\left[X^{[\lambda,(n-1)]}, A_{(x,u)}\right] = \lambda X^{[\lambda,(n-1)]} + \rho A_{(x,u)}.$$

We have also proved that if X is a $C^{\infty}(M^{(1)})$-symmetry then there exists a procedure to reduce the equation to an $(n-1)$th-order equation and a first-order equation:

THEOREM 1. *Let X be a λ-symmetry, with $\lambda \in C^{\infty}(M^{(1)})$, of the equation $\Delta(x, u^{(n)}) = 0$. Let $y = y(x,u)$ and $w = w(x,u,u_1)$ be two functionally independent first-order invariants of $X^{[\lambda,(n)]}$. The general solution of the equation can be obtained by solving an equation of the form $\Delta_r(y, w^{(n-1)}) = 0$ and an auxiliary equation $w = w(x,u,u_1)$.*

The equation $\Delta_r(y, w^{(n-1)}) = 0$ can be constructed as follows: if $y = y(x,u)$ and $w = w(x,u,u_1)$ are two functionally independent first-order invariants of $X^{[\lambda,(n)]}$, then the set

$$y, w, w_1 = \frac{D_x w}{D_x y}, \ldots, \quad w_{n-1} = \frac{D_x w_{n-2}}{D_x y}$$

constitutes a complete system of functionally independent invariants of $X^{[\lambda,(n)]}$ and, therefore, the equation can be written in terms of $\{y, w, w_1, \ldots, w_{n-1}\}$.

3. C^{∞}-symmetries and conservation of symmetries for the Lie algebra $so(3,\mathbb{R})$

Let us consider an nth-order differential equation

$$\Delta\big(x, u^{(n)}\big) = 0 \tag{7}$$

that admits the nonsolvable Lie algebra $so(3,\mathbb{R})$ as symmetry algebra. It can always be chosen as a base of generators $\{X_1, X_2, X_3\}$ of $so(3,\mathbb{R})$ such that the corresponding Lie brackets are given by (1).

If we use any of the generators X_i to reduce the order, the others are lost as Lie symmetries of the equation obtained at the first step of the reduction. Next, we study how the two lost symmetries can be recovered as C^{∞}-symmetries of the reduced equation. As a consequence, any of them can be used to reduce the order and the unused symmetry can be regained as a C^{∞}-symmetry of the $(n-2)$th-order equation. This last C^{∞}-symmetry leads to a new order reduction. As a result, the order of the equation can be reduced by three in three steps. It is sufficient to study the use of sequences that begin with X_1, because the study of the reduction process that begins with X_2 or X_3 is similar.

3.1. Reduction process

- FIRST STEP: Use of the Lie symmetry X_1:
  With the notations introduced in the previous section, if the Lie symmetry X_1 is used to reduce the order of equation (7), the reduced equation will be denoted by

$$\Delta_1\big(y, w^{(n-1)}\big) = 0, \tag{8}$$

where $(y, w) \in M_1$.

The Lie symmetries X_2, X_3 are lost for this reduced equation. However, they can be recovered as C^∞-symmetries. This result is proved in the following theorem.

THEOREM 2. *There exist two complex-valued functions, $f_2, f_3 \in C^\infty(M, \mathbb{C})$, such that the vector fields $f_2(X_2^{(1)} + iX_3^{(1)})$ and $f_3(X_2^{(1)} - iX_3^{(1)})$ are $\pi_{X_1}^{(1)}$-projectable. Their projections on M_1, $Y_2 = (\pi_{X_1}^{(1)}) * (f_2(X_2^{(1)} + iX_3^{(1)}))$ and $Y_3 = (\pi_{X_1}^{(1)}) * (f_3(X_2^{(1)} - iX_3^{(1)}))$, are C^∞-symmetries of equation* (8).

Proof: Let us denote $\widetilde{X_2} = X_2 + iX_3$ and $\widetilde{X_3} = X_2 - iX_3$. It can easily be checked that

$$\big[X_1^{(k)}, \widetilde{X_2}^{(k)}\big] = -i\widetilde{X_2}^{(k)}, \quad \big[X_1^{(k)}, \widetilde{X_3}^{(k)}\big] = i\widetilde{X_3}^{(k)},$$

for $k \in \mathbb{N}$. Let $f_2, f_3 \in C^\infty(M, \mathbb{C})$ be such that

$$X_1(f_2) = if_2, \quad X_1(f_3) = -if_3.$$

We have

$$\big[X_1^{(k)}, f_2\widetilde{X_2}^{(k)}\big] = 0, \quad \big[X_1^{(k)}, f_3\widetilde{X_3}^{(k)}\big] = 0,$$

for $k \in \mathbb{N}$; therefore, the vector fields $f_2\widetilde{X_2}^{(1)}, f_3\widetilde{X_3}^{(1)}$ are $\pi_{X_1}^{(1)}$-projectable. Let us denote $Y_i = (\pi_{X_1}^{(1)}) * (f_i\widetilde{X_i}^{(1)})$, for $i = 2, 3$.

By using the prolongation formula given in Definition 1, it can be checked that $fX^{(k)} = (fX)^{[\lambda,(k)]}$, where $\lambda = \frac{-D(f)}{f}$. Hence

$$\Big(\big(\pi_{X_1}^{(1)}\big) * \big(f_i\widetilde{X_i}^{(1)}\big)\Big)^{(k)} = Y_i^{[\lambda_i,(k)]}, \quad \lambda_i = \frac{-D_y(f_i)}{f_i} \quad (i = 2, 3). \tag{9}$$

Because X_2 and X_3 are Lie symmetries of the original equation, we get

$$\big[f_i\widetilde{X_i}^{(n-1)}, A_{(y,\alpha)}\big] = \widetilde{\lambda_i}f_i\widetilde{X_i}^{(n-1)} + \widetilde{\rho_i}A_{(y,\alpha)} \quad (i = 2, 3),$$

for some functions $\widetilde{\lambda_i}, \widetilde{\rho_i}$. It can be checked, by using the identity of Jacobi for the families of vector fields $\{X_1^{(n-1)}, f_2\widetilde{X_2}^{(n-1)}, A_{(y,\alpha)}\}$ and $\{X_1^{(n-1)}, f_3\widetilde{X_3}^{(n-1)}, A_{(y,\alpha)}\}$, that the functions $\widetilde{\lambda_i}$ and $\widetilde{\rho_i}$ are $X_1^{(1)}$-invariant. We define

λ_i and ρ_i, respectively, by $(\pi_{X_1}^{(n-1)}) * (\lambda_i) = \widetilde{\lambda_i}$ and $(\pi_{X_1}^{(n-1)}) * (\rho_i) = \widetilde{\rho}_i$, for $i = 2, 3$. Therefore,

$$\begin{aligned}
\left[Y_i^{[\lambda_i,(n-1)]}, A_{(y,w)}\right] &= \left[\left(\pi_{X_1}^{(n-1)}\right) * \left(f_i \widetilde{X}_i^{(n-1)}\right), \left(\pi_{X_1}^{(n-1)}\right) * \left(A_{(y,\alpha)}\right)\right] \\
&= \left(\pi_{X_1}^{(n-1)}\right) * \left(\widetilde{\lambda_i} f_i \widetilde{X}_i^{(n-1)} + \widetilde{\rho}_i A_{(y,\alpha)}\right) \\
&= \lambda_i \left(\pi_{X_1}^{(n-1)}\right) * \left(f_i \widetilde{X}_i^{(n-1)}\right) + \rho_i \left(\pi_{X_1}^{(n-1)}\right) * \left(A_{(y,\alpha)}\right) \\
&= \lambda_i Y_i^{[\lambda_i,(n-1)]} + \rho_i A_{(y,w)}.
\end{aligned}$$

This concludes the proof. □

As a consequence of the previous theorem, any of the two C^∞-symmetries, Y_2 and Y_3, can be used to reduce the order of equation (8). We will only study here the use of the C^∞-symmetry Y_2, because the corresponding study for the vector field Y_3 is similar.

- SECOND STEP: Use of the C^∞-symmetry Y_2:
  We choose a system of coordinates $\{z = z(y, w), \beta = \beta(y, w)\}$ such that Y_2 can be written as $\frac{\partial}{\partial \beta}$. Let $\mu = \mu(z, \beta, \beta_z)$ be an invariant of ${Y_2}^{[\lambda_2,(1)]}$ such that it is functionally independent of z. Thus, we obtain an order reduction for equation (8) by using Y_2 which can be written in explicit form as

$$\mu_{n-2} = \Phi\left(z, \mu^{(n-3)}\right), \tag{10}$$

where $(z, \mu) \in M_2$, for some open set M_2 of a two-dimensional manifold on $\mathbb{C} \times \mathbb{C}$. Let us denote by $\pi_{Y_2}^{[\lambda_2,(k)]} : M_1^{(k)} \to M_2^{(k-1)}$ the map defined by $(z, \beta, \mu, \mu_1, \cdots, \mu_{k-1}) \mapsto (z, \mu, \mu_1, \cdots, \mu_{k-1})$. The vector field associated with equation (10) is

$$A_{(z,\mu)} = \left(\pi_{Y_2}^{[\lambda_2,(n-2)]}\right) * \left(A_{(z,\beta,\mu)}\right),$$

where $A_{(z,\beta,\mu)}$ denotes the vector field

$$A_{(z,\beta,\mu)} = \frac{\partial}{\partial z} + \beta_z \frac{\partial}{\partial \beta} + \mu_1 \frac{\partial}{\partial \mu} + \cdots + \Phi\left(z, \mu^{(n-3)}\right) \frac{\partial}{\partial \mu_{n-3}}.$$

THEOREM 3. *There exists a function $g_3 \in C^\infty(M_1, \mathbb{C})$ such that $g_3 Y_3^{[l_3,(1)]}$ is $\pi_{Y_2}^{[\lambda_2,(1)]}$-projectable and its projection $(\pi_{Y_2}^{[\lambda_2,(1)]}) * (g_3 Y_3)$ is a C^∞-symmetry of equation* (10).

Proof: It can be checked that

$$\left[Y_2^{[\lambda_2,(k)]}, Y_3^{[\lambda_3,(k)]}\right] = h_2 Y_2^{[\lambda_2,(k)]} + h_3 Y_3^{[\lambda_3,(k)]}$$

for some functions $h_2, h_3 \in C^\infty(M_1, \mathbb{C})$. Let $g_3 \in C^\infty(M_1, \mathbb{C})$ be a function such that $Y_2(g_3) = -h_3 g_3$. Then

$$\left[Y_2^{[\lambda_2,(k)]}, g_3 Y_3^{[\lambda_3,(k)]}\right] = g_3 h_2 Y_2^{[\lambda_2,(k)]},$$

for $k \in \mathbb{N}$. Therefore, $g_3 Y_3^{[\lambda_3,(1)]}$ is $\pi_{Y_2}^{[\lambda_2,(1)]}$-projectable.

Since Y_3 is a C^∞-symmetry of equation (8), there exist two functions $\widetilde{\lambda}_3, \widetilde{\rho}_3$ such that

$$\left[g_3 Y_3^{[\lambda_3,(n-2)]}, A_{(z,\beta,\mu)}\right] = \widetilde{\lambda}_3 g_3 Y_3^{[\lambda_3,(n-2)]} + \widetilde{\rho}_3 A_{(z,\beta,\mu)}. \tag{11}$$

By the identity of Jacobi for the vector fields $\{Y_2^{[\lambda_2,(n-2)]}, g_3 Y_3^{[\lambda_3,(n-2)]}, A_{(z,\beta,\mu)}\}$, it can be checked that the functions $\widetilde{\lambda}_3$ and $\widetilde{\rho}_3$ are $Y_2^{[\lambda_2,(n-2)]}$-invariant. Hence, if λ_3' and ρ_3 are such that $(\pi_{Y_2}^{[\lambda_2,(n-2)]}) * (\lambda_3') = \widetilde{\lambda}_3$ and $(\pi_{Y_2}^{[\lambda_2,(n-2)]}) * (\rho_3) = \widetilde{\rho}_3$, we finally obtain

$$\begin{aligned}&\left[\left(\pi_{Y_2}^{[\lambda_2,(n-2)]}\right) * \left(g_3 Y_3^{[\lambda_3,(n-2)]}\right), A_{(z,\mu)}\right]\\ &\quad = \lambda_3'\left(\pi_{Y_2}^{[\lambda_2,(n-2)]}\right) * \left(g_3 Y_3^{[\lambda_3,(n-2)]}\right) + \rho_3 A_{(z,\mu)}.\end{aligned}$$

Therefore $(\pi_{Y_2}^{[\lambda_2,(1)]}) * (g_3 Y_3^{[\lambda_3,(1)]})$ is a λ_3'-symmetry of equation (10) that will be denoted by Z_3.

- THIRD STEP: Use of the C^∞-symmetry Z_3:
  The $(n-2)$th-order equation obtained at the second stage, once we have used the Lie symmetry X_1 and the C^∞-symmetry Y_2, admits the vector field Z_3 as a C^∞-symmetry. Clearly Z_3 can be used to reduce again the order of the equation.

3.2. Recovery of solutions

The method of reduction by using C^∞-symmetries allows us to reduce the order of any equation admitting $so(3, \mathbb{R})$ as symmetry algebra by three successive one-order reductions. As a consequence, the general solution of the original equation can be obtained, through the reduced one, by solving three auxiliary first-order equations.

If the order of the original equation is three, after two order reductions, a first-order differential equation is obtained. At this last step of the reduction, the unused Lie symmetry that has been lost can be recovered as a C^∞-symmetry. In the following section, we will show how the method of the C^∞-symmetries can be used to give a general form of the first-order equations that appear in the last step of the reduction process. The corresponding two first-order auxiliary equations will also be given, and we will show the way they can be solved.

4. General method to solve a third-order equation admitting the nonsolvable symmetry algebra $so(3,\mathbb{R})$

Let

$$\Delta\left(x, u^{(3)}\right) = 0 \tag{12}$$

be an arbitrary third-order equation admitting $so(3,\mathbb{R})$ as symmetry algebra.

The action of the rotation group $so(3,\mathbb{R})$ on a two-dimensional real manifold can be modeled by the transformation group generated by the following vector fields:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x}, \quad X_2 = \cos(x)\cot(u)\frac{\partial}{\partial x} + \sin(x)\frac{\partial}{\partial u}, \\ X_3 &= -\sin(x)\cot(u)\frac{\partial}{\partial x} + \cos(x)\frac{\partial}{\partial u}. \end{aligned} \tag{13}$$

By means of a change of variables in equation (12), we can assume that the symmetry algebra is generated by the vector fields $\{X_1, X_2, X_3\}$ given in (13).

Let us study in coordinates the reduction process described in the previous section.

- FIRST STEP: USE OF X_1:
  Let $\{y = u, \alpha = x\}$ be the change of variables such that the vector field X_1 takes the form $X_1 = \frac{\partial}{\partial\alpha}$, and we consider the corresponding system of local coordinates $\{y, \alpha^{(3)}\}$ in $M^{(3)}$. The vector fields X_2 and X_3 in (13) are then expressed as follows:

$$\begin{aligned} X_2 &= \sin(\alpha)\frac{\partial}{\partial y} + \cos(\alpha)\cot(y)\frac{\partial}{\partial\alpha}, \\ X_3 &= \cos(\alpha)\frac{\partial}{\partial y} - (\sin(\alpha)\cot(y))\frac{\partial}{\partial\alpha}. \end{aligned}$$

  By Theorem 2, there exist two functions f_2, f_3 such that the vector fields $f_2(X_2^{(1)} + iX_3^{(1)})$ and $f_3(X_2^{(1)} - iX_3^{(1)})$ are $\pi_{X_1}^{(1)}$-projectable. It can be checked that

$$\begin{aligned} \widetilde{X_2} = X_2^{(1)} + iX_3^{(1)} = e^{-i\alpha}\Big(&-i\frac{\partial}{\partial y} + \cot(y)\frac{\partial}{\partial\alpha} \\ &+ (-\alpha_y{}^2 - i\alpha_y\cot(y) - \csc(y)^2)\frac{\partial}{\partial\alpha_y}\Big) \end{aligned}$$

  and

$$\begin{aligned} \widetilde{X_3} = X_2^{(1)} - iX_3^{(1)} = e^{i\alpha}\Big(&i\frac{\partial}{\partial y} + \cot(y)\frac{\partial}{\partial\alpha} \\ &+ \left(-\alpha_y{}^2 + i\alpha_y\cot(y) - \csc(y)^2\right)\frac{\partial}{\partial\alpha_y}\Big). \end{aligned}$$

Therefore, the vector fields $f_2(X_2^{(1)} + iX_3^{(1)})$ and $f_3(X_2^{(1)} - iX_3^{(1)})$ are $\pi_{X_1}^{(1)}$-projectable for the functions

$$f_2 = e^{i\alpha},\, f_3 = e^{-i\alpha}.$$

We denote $w = \alpha_y = \frac{1}{u_x}$ and equation (12) can be expressed as a second-order differential equation

$$\Delta_1\big(y, w^{(2)}\big) = 0, \tag{14}$$

for $(y, w) \in M_1$ which corresponds to equation (8).

By Theorem 2, the vector fields $Y_2 = (\pi_{X_1}^{(1)}) * (f_2(X_2^{(1)} + iX_3^{(1)}))$ and $Y_3 = (\pi_{X_1}^{(1)}) * (f_3(X_2^{(1)} - iX_3^{(1)}))$ are C^∞-symmetries of equation (8). It can be checked that these C^∞-symmetries and the corresponding functions l_2, l_3 are:

$$Y_2 = -i\frac{\partial}{\partial y} + (-w^2 - iw\cot(y) - \csc(y)^2)\frac{\partial}{\partial w}, \quad \lambda_2 = -iw$$

$$Y_3 = i\frac{\partial}{\partial y} + (-w^2 + iw\cot(y) - \csc(y)^2)\frac{\partial}{\partial w}, \quad \lambda_3 = iw.$$

Let us observe that $Y_3 = \bar{Y}_2, \lambda_3 = \bar{\lambda}_2$.

- SECOND STEP: USE OF Y_2:
  In the system of coordinates

$$\left\{z = \ln\left(\cot\left(\frac{y}{2}\right)\right) - i\arctan(w\sin(y)),\, \beta = -iy\right\} \tag{15}$$

the vector field Y_2 can be written as $\frac{\partial}{\partial\beta}$. It can be checked that

$$\mu = \frac{\beta_z \sinh(\beta)}{2\left(\cosh(z) - \cosh(\beta)\sinh(z)\right)} \tag{16}$$

is a first-order invariant of $Y_2^{[\lambda_2,(1)]}$. Thus, we obtain an order reduction for equation (14) by using Y_2:

$$\Delta_2\big(z, \mu^{(1)}\big) = 0, \tag{17}$$

where $(z, \mu) \in M_2$.

In terms of $\{z, \beta, \mu\}$ we have

$$Y_3^{[l_3,(1)]} = 2\,\mathrm{csch}(\beta)\frac{\partial}{\partial z} - \frac{\partial}{\partial\beta} - 4\,\mathrm{csch}(\beta)\,\mu^2\sinh(z)\frac{\partial}{\partial\mu}.$$

By Theorem 3, there exists a function g_3 such that $g_3 Y_3^{[l_3,(1)]}$ is $\pi_{Y_2}^{[\lambda_2,(1)]}$-projectable. In this case we have that, for $g_3 = \sinh(\beta)$, the projection

$$Z_3 = \left(\pi_{Y_2}^{[\lambda_2,(1)]}\right) * \left(g_3 Y_3^{[l_3,(1)]}\right) = 2\frac{\partial}{\partial z} - 4\mu^2 \sinh\,(z)\frac{\partial}{\partial \mu}$$

is a C^∞-symmetry of equation (17) for the function

$$l_3' = 2\mu \sinh(z).$$

- THIRD STEP: USE OF Z_3:
After the second stage of the reduction process, we have obtained the first-order differential equation (17) that admits Z_3 as λ_3'-symmetry. To determine the most general and simple form of first-order differential equations that admit Z_3 as λ_3'-symmetry, we use the change of variables $\{s = -\frac{1}{\mu} + 2\cosh(z), r = z\}$ where Z_3 is simply expressed as $Z_3 = 2\frac{\partial}{\partial r}$. In coordinates $\{s, r, r_s\}$ equation (17) can be written in an explicit form as

$$r_s = F(s, r).$$

This equation admits $Z_3 = 2\frac{\partial}{\partial r}$ as $\widetilde{\lambda}_3$-symmetry, for the function $\widetilde{\lambda}_3 = \frac{-2r_s \sinh(r)}{s-2\cosh(r)}$.

By Definition 1,

$$Z_3^{[\widetilde{\lambda}_3,(1)]} = 2\frac{\partial}{\partial r} - \frac{4r_s \sinh(r)}{s - 2\cosh(r)}\frac{\partial}{\partial r_s}.$$

If we impose

$$Z_3^{[\widetilde{\lambda}_3,(1)]}(r_s - F(r, s)) = 0 \text{ when } r_s = F(s, r),$$

we obtain that

$$F(s, r) = C(s)(-s + 2\ \cosh(r))$$

for some arbitrary function C depending on s. Therefore, we have obtained the following general form for the first-order differential equation obtained at the last stage of the reduction process

$$r_s = C(s)\,(-s + 2\ \cosh(r))\,. \tag{18}$$

If, in this third stage, we use coordinates $\{r, s, s_r\}$ instead of $\{s, r, r_s\}$, the general form we would obtain is

$$s_r = \frac{B(s)}{(-s + 2\ \cosh(r))}, \tag{19}$$

where B is an arbitrary function depending on s.

4.1. Classification of equations

The step-by-step method of reduction based on the existence of C^∞-symmetries, for third-order equations (12), leads to first-order differential equations of the form (18)–(19).

As a consequence, we can give a complete classification of the third-order ordinary differential equations that admit the nonsolvable Lie algebra $so(3,\mathbb{R})$ as symmetry algebra. If we write the first-order reduced equation (18) (resp. (19)) in terms of the original system of coordinates, we obtain the general form of third-order differential equations that admit the symmetry algebra generated by the vector fields (13):

$$u_3 = \csc(u)\left(u_1^2 + \sin(u)^2\right)^2 \widetilde{C}(s) + u_1\left(-1 + u_1^2 - 3u_2\cot(u) - \frac{3\left(u_1^2 + u_1^4 - u_2^2 - u_2\sin(2u)\right)}{u_1^2 + \sin(u)^2}\right),$$

where

$$s = \frac{-2i\left(-u_2\sin(u) + \cos(u)\left(2u_1^2 + \sin(u)^2\right)\right)}{\left(u_1^2 + \sin(u)^2\right)^{\frac{3}{2}}}$$

and $\widetilde{C}(s) = \frac{i}{4C(s)}$ (resp. $\widetilde{C}(s) = \frac{i}{4}B(s)$).

4.2. Recovery of solutions

Let us consider the first-order reduced equation (18). Because $\{s = -\frac{1}{\mu} + 2\cosh(z), r = z\}$, equation (18) in terms of $\{z, \mu, \mu_z\}$ is

$$\left(\frac{\mu_z}{\mu^2} + 2\sinh(z)\right) C\left(-\frac{1}{\mu} + 2\cosh(z)\right) = \mu. \tag{20}$$

Similarly, equation (19) in terms of $\{z, \mu, \mu_z\}$ is

$$\frac{\mu_z}{\mu^2} + 2\sinh(z) = B\left(-\frac{1}{\mu} + 2\cosh(z)\right)\mu. \tag{21}$$

If $\mu = H_0(z, C_1)$ solves equation (20) or (21), by (16), we get

$$\beta_z \sinh(\beta) = 2H_0(z, C_1)(\cosh(z) - \cosh(\beta)\sinh(z)).$$

We set $\widetilde{\beta} = \cosh(\beta)$. Clearly $\widetilde{\beta}$ must satisfy the first-order linear equation

$$\widetilde{\beta}_z = 2H_0(z, C_1)(\cosh(z) - \widetilde{\beta}\sinh(z)). \tag{22}$$

Let $\widetilde{\beta} = H_1(z, C_1, C_2)$ be the general solution of the previous equation. By (15)

$$\cos(y) = H_1\left(\ln\left(\cot\left(\frac{y}{2}\right)\right) - i\arctan(w\sin(y)), C_1, C_2\right) \tag{23}$$

is the general solution of equation (14). When w in (23) is locally expressed, in terms of y, as $w = H_2(y, C_1, C_2)$, the general solution of the original equation is obtained from

$$\alpha_y = H_2(y, C_1, C_2)$$

by simple quadrature with respect to the variable y:

$$\alpha = \int H_2(y, C_1, C_2)\, dy + C_3 = H_3(y, C_1, C_2, C_3).$$

In terms of the original variables $\{x, u\}$ the general solution of equation (12) is given by

$$x = H_3(u, C_1, C_2, C_3).$$

5. An example

Let us consider the third-order differential equation

$$u_3 = u_1 \left(-1 + u_1^2 - 3u_2 \cot(u) - \frac{3\left(u_1^2 + u_1^4 - u_2^2 - u_2 \sin(2u)\right)}{u_1^2 + \sin(u)^2} \right) \tag{24}$$

that admits the Lie algebra $so(3,\mathbb{R})$ as symmetry algebra. The corresponding first-order reduced equation (21) is given by

$$\frac{\mu_z}{\mu^2} + 2\sinh(z) = 0.$$

The general solution of this equation is $\mu = H_0(z, c_1) = \frac{-1}{-2\cosh(z) + c_1}$. The linear first-order equation corresponding to equation (22) is

$$\widetilde{\beta}_z + \frac{2\sinh(z)}{-2\cosh(z) + c_1}\widetilde{\beta} = \frac{2\cosh(z)}{-2\cosh(z) + c_1}.$$

This equation can easily be integrated and its general solution is

$$\widetilde{\beta} = \frac{-c_2 + 2\sinh(z)}{-c_1 + 2\cosh(z)}.$$

By (15), we have

$$\cos(y) = -\frac{-2iw + 2\cot(y) - c_2\sqrt{1 + w^2 \sin(y)^2}}{2iw\cos(y) - 2\csc(y) + c_1\sqrt{1 + w^2\sin(y)^2}}$$

and w can locally be expressed as

$$w = \pm \frac{i\,(c_2 - c_1 \cos(y)) \csc(y)}{\sqrt{(c_2 - c_1 \cos(y))^2 + 4\,\sin(y)^2}}.$$

To obtain real solutions of the equation, we choose $c_1 = iC_1$ and $c_2 = iC_2$, where C_1 and C_2 are arbitrary real constants. Thus

$$w = \pm \frac{(C_2 - C_1 \cos(y)) \csc(y)}{\sqrt{-\,(C_2 - C_1 \cos(y))^2 + 4\,\sin(y)^2}}$$

is real, when variable y is defined on some open set of $\mathbb{R}$ such that $-(C_2 - C_1 \cos(y))^2 + 4\,\sin(y)^2 > 0$. Because $w = \alpha_y$, by integration with respect to y we get

$$\alpha = \pm\frac{1}{2}\arctan\left(\frac{2(-C_1 + C_2\cos(y))\sqrt{-(C_2 - C_1\cos(y))^2 + 4\,\sin(y)^2}}{(-C_1 + C_2\cos(y))^2 + (-C_2 + C_1\cos(y))^2 - 4\,\sin(y)^2}\right) - C_3. \tag{25}$$

When α and y in (25) are replaced by x and u, respectively, we obtain the general solution of equation (24). This general solution can be expressed, in implicit form, as follows:

$$\tan(2x + C_3) = \pm\frac{2(-C_1 + C_2\cos(u))\sqrt{-(C_2 - C_1\cos(u))^2 + 4\,\sin(u)^2}}{(-C_1 + C_2\cos(u))^2 + (-C_2 + C_1\cos(u))^2 - 4\,\sin(u)^2}. \tag{26}$$

Some particular solutions can easily be expressed. For instance, when $C_2 = 0, C_1 = 2$ we get the following one-parameter family of solutions of the original equation (24):

$$u = \pm\frac{1}{2}\arccos(-\tan(C_3 + x))^2.$$

6. Conclusions

When the classical Lie method is used to reduce the order of any ordinary differential equation admitting the three-dimensional nonsolvable Lie algebra $so(3,\mathbb{R})$ as symmetry algebra, then at least one of its generators is lost in the reduction process.

Nevertheless, in this article we have proved that the method of reduction by using C^∞-symmetries can be applied to carry out three successive one-order reductions, if the order of the original equation is $n > 3$.

If $n = 3$, the Lie method of reduction is not applicable, because the corresponding third-order auxiliary equation is equivalent to the original equation. We have proved here that, in this case, after two successive one-order reductions, the first-order reduced equation conserves the unused symmetry as a C^∞-symmetry. This let us give a simple general form for the first-order equations achieved at the end of the reduction procedure. As an additional result, we get a complete classification of the third-order equations that admit $so(3,\mathbb{R})$.

The main consequence of this step-by-step method of reduction is that the general solution of the original equation can be obtained from the reduced one by solving three first-order differential equations. Two of these equations can be solved by quadratures, because one of them is linear and the other can directly be solved by integration. In particular, if $n = 3$, the general solution of the original equation can be obtained from the first-order reduced equation by two quadratures.

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Universidad de Cádiz, Spain

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