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Operators with hypercyclic Cesàro means

by

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Abstract. An operator T on a Banach space $\mathcal{B}$ is said to be hypercyclic if there exists a vector x such that the orbit $\{T^n x\}_{n\geq 1}$ is dense in $\mathcal{B}$. Hypercyclicity is a strong kind of cyclicity which requires that the linear span of the orbit is dense in $\mathcal{B}$. If the arithmetic means of the orbit of x are dense in $\mathcal{B}$ then the operator T is said to be Cesàro-hypercyclic. Apparently Cesàro-hypercyclicity is a strong version of hypercyclicity. We prove that an operator is Cesàro-hypercyclic if and only if there exists a vector $x \in \mathcal{B}$ such that the orbit $\{n^{-1}T^n x\}_{n\geq 1}$ is dense in $\mathcal{B}$. This allows us to characterize the unilateral and bilateral weighted shifts whose arithmetic means are hypercyclic. As a consequence we show that there are hypercyclic operators which are not Cesàro-hypercyclic, and more surprisingly, there are non-hypercyclic operators for which the Cesàro means of some orbit are dense. However, we show that both classes, the class of hypercyclic operators and the class of Cesàro-hypercyclic operators, have the same norm-closure spectral characterization.

1. Introduction. Let T be a bounded linear operator on a complex Banach space $\mathcal{B}$. The motivation for this work comes from some questions related to ergodic theory (see [Du], [LZ], [MZ], [Sw] for instance). The uniform ergodic theory deals with the asymptotic behavior of the arithmetic means

$$M_n(T) = \frac{I + T + \ldots + T^{n-1}}{n}$$

in the operator norm (uniform) topology, as n tends to infinity. N. Dunford in 1943 (see [Du]) discussed the connections between the spectrum of T and convergence of sequences of functions $Q_n(T)$ of T. Specifically, he obtained the following basic result in uniform ergodic theory:

THEOREM (Dunford). *The sequence $M_n(T)$ uniformly converges if and only if*

(a) $\lim n^{-1}\|T^n\| = 0$, *and*

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(b) *the point* 1 *is at most a simple pole of the resolvent* $R_\lambda(T) = (T-\lambda I)^{-1}$.

Many interesting and equivalent geometric conditions for the convergence of $M_n(T)$ are also obtained in [MZ] and [LZ].

In the present work we consider the natural question (posed by J. Zemánek to the author) of connections between T and the "maximal divergence" of $M_n(T)$. The maximal divergence is understood to be the existence of a vector x in $\mathcal{B}$ such that the orbit $\{M_n(T)x\}_{n\geq 1}$ is dense in $\mathcal{B}$. Let us call such operators *Cesàro-hypercyclic* and the vector for which the last condition is satisfied, a *Cesàro-hypercyclic vector.*

An operator T is *hypercyclic* if there exists a vector x such that the orbit $\{T^n x\}$ is dense in $\mathcal{B}$. In this case the vector x is called *hypercyclic* for T. If there exists a vector x for which the set $\{\lambda T^n x : n \in \mathbb{N}, \lambda \in \mathbb{C}\}$ is dense, then the operator T and the vector x are called *supercyclic.* If there exists a sequence $\{\lambda_n\}$ for which the set $\{\lambda_n T^n x\}$ is dense we will say that T is *supercyclic for the sequence* $\{\lambda_n\}$. Finally we can extend the notion to sequences of linear operators: the sequence $\{T_n\}$ is hypercyclic if there exists x such that $\{T_n x\}$ is dense.

Our first result states that an operator T is Cesàro-hypercyclic if and only if there exists a vector x such that $\{n^{-1}T^n x\}$ is dense, that is, T is supercyclic for the sequence $\{1/n\}$. This result is similar to Dunford's Theorem in the hypercyclic setting. Observe that it implies that Cesàro hypercyclicity is a special kind of supercyclicity.

Since the Cesàro means are more regular, in general one may think that Cesàro hypercyclicity is a stronger condition than hypercyclicity. In Section 3 we characterize the unilateral and bilateral weighted shifts which are Cesàro-hypercyclic. As a consequence we show that for the case of unilateral shift, Cesàro hypercyclicity is indeed a stronger condition than hypercyclicity. Surprisingly, for the bilateral weighted shift case, there are still examples that have no vectors with dense orbit but do have dense Cesàro orbit.

Thus hypercyclicity does not imply Cesàro hypercyclicity and vice versa. In this connection, in Section 4 we show that any separable Banach space admits a Cesàro-hypercyclic operator, thus obtaining the Cesàro version of Ansari–Bernal's result (see [An], [Be]) for hypercyclicity. Finally, although both classes are quite different we show (see Section 5) that their norm closures have the same spectral characterization.

Before going further the author would like to thank the staff of the Institute of Mathematics of the Polish Academy of Sciences, specially Professor J. Zemánek, for their hospitality during the author's stay in Warsaw in February 1998.

2. Hypercyclic arithmetic means. This section is devoted to proving an analogue to Dunford's result. That is, an operator T on a Banach space is Cesàro-hypercyclic if and only if the sequence T^n/n is hypercyclic. Hence, Cesàro hypercyclicity is a special kind of supercyclicity. To prove this result we need to show some properties relating to the spectrum of Cesàro-hypercyclic operators.

Given a complex number λ, let $M_n(\lambda)$ denote the arithmetic mean of the powers of λ, that is,

$$M_n(\lambda) = \frac{1 + \lambda + \lambda^2 + \ldots + \lambda^{n-1}}{n}, \qquad n = 1, 2, \ldots$$

Observe that if $|\lambda| > 1$ then $M_n(\lambda)$ diverges to ∞. On the other hand if $|\lambda| \le 1$ then $M_n(\lambda)$ is contained in the closed unit disk for any positive integer n. Therefore we can state the following result.

LEMMA 2.1. *Let λ, z_0 be complex numbers. Then the set $\{M_n(\lambda)z_0\}_{n\ge1}$ of complex numbers is not dense in $\mathbb{C}$.*

Denote by $\sigma_{\mathrm{p}}(T)$ the point spectrum of an operator T. It is well known that the range $\operatorname{ran}(T - \lambda)$ is dense if and only if $\lambda \notin \sigma_{\mathrm{p}}(T^\star)$.

PROPOSITION 2.2. *If T is Cesàro-hypercyclic then $\sigma_{\mathrm{p}}(T^\star) = \emptyset$, that is, $\operatorname{ran}(T - \lambda)$ is dense for any $\lambda \in \mathbb{C}$.*

Proof. Suppose that $\lambda \in \sigma_{\mathrm{p}}(T^\star)$. Then there exists $x^\star \in \mathcal{B}^\star \setminus \{0\}$ such that $T^\star x^\star = \lambda x^\star$. Now if x is a Cesàro-hypercyclic vector for T, then the sequence $\{M_n(T)x\}_{n\ge0}$ is dense in $\mathcal{B}$. Therefore the collection of complex numbers $\langle M_n(T)x, x^\star\rangle$ will be dense in $\mathbb{C}$. But for each n we have

$$\langle M_n(T)x, x^\star\rangle = \langle x, M_n(T^\star)x^\star\rangle = \langle x, M_n(\lambda)x^\star\rangle = \overline{M_n(\lambda)}\langle x, x^\star\rangle$$

and by Lemma 2.1 it is clear that the set of complex numbers defined by the right side of this equation, as n ranges through the positive integers, is not dense in the complex plane, which concludes the proof. ∎

Now let us regard x as a hypercyclic vector for the sequence $\{n^{-1}T^n\}$ or the sequence $\{n^{-1}(I - T^n)\}$, and observe that

$$\left\| \frac{T^n x}{n} - \frac{(I - T^n)x}{n} \right\| = n^{-1}\|x\| \tag{2.1}$$

converges to zero as $n \to \infty$. So (2.1) establishes the following:

PROPOSITION 2.3. *Let T be a bounded linear operator and $x \in \mathcal{B}$. Then the following conditions are equivalent:*

(a) *The sequence $\{n^{-1}T^n x\}$ is dense.*
(b) *The sequence $\{n^{-1}(I - T^n)x\}$ is dense.*

The following theorem characterizes the chaotic behaviour of $\{M_n(T)x\}$ in terms of a special kind of supercyclicity of T, in a useful form.

THEOREM 2.4. *Let T be a bounded linear operator on a Banach space $\mathcal{B}$. The following conditions are equivalent*:

(a) *The sequence of arithmetic means $M_n(T)$ is hypercyclic.*
(b) *The sequence $\{n^{-1}T^n\}$ is hypercyclic.*

Proof. The main idea of the proof is based on the equality

$$(T-I)\,\frac{I+T+\ldots+T^{n-1}}{n} = \frac{I-T^n}{n}. \tag{2.2}$$

Assume that T is Cesàro-hypercyclic. Then there exists a vector $x \in \mathcal{B}$ such that $\{M_n(T)x\}$ is dense in $\mathcal{B}$. Since T is Cesàro-hypercyclic, by Proposition 2.2, $\operatorname{ran}(T-I)$ is dense and therefore the image of a dense subset under $T-I$ is dense. Hence, the orbit

$$(T-I)(\{M_n(T)x\}) = \left\{\frac{I-T^n}{n}(x)\right\}$$

is dense in $\mathcal{B}$. Finally since $\{n^{-1}(I-T^n)\}$ is hypercyclic, from Proposition 2.3 we deduce that the sequence $\{n^{-1}T^n\}$ is hypercyclic.

For the converse assume that the sequence $\{n^{-1}T^n\}$ is hypercyclic. Then Proposition 2.3 along with expression (2.2) ensures the existence of a vector x in $\mathcal{B}$ such that the orbit

$$\{(T-I)M_n(T)(x)\} = \{M_n(T)((T-I)x)\}$$

is dense in $\mathcal{B}$. That is, the vector $(T-I)x$ is Cesàro-hypercyclic for T. ■

REMARK 2.5. From the proof of Theorem 2.4 it follows that any Cesàro-hypercyclic vector for T is also supercyclic for the sequence $\{n^{-1}\}$.

3. Unilateral and bilateral weighted shifts. This section deals with the relationship between the set of hypercyclic operators and the set of Cesàro-hypercyclic operators. In view of the properties of convergence (uniformization) that the Cesàro means usually enjoy, one may think that the condition of Cesàro hypercyclicity is stronger than hypercyclicity. In this section we will apply Theorem 2.4 to unilateral and bilateral weighted shifts. We show that any Cesàro-hypercyclic unilateral weighted shift is hypercyclic, while for the bilateral weighted shifts this is not the case: there are operators that are not hypercyclic but their Cesàro means are hypercyclic.

Let $\{e_n\}_{n\geq 0}$ be the canonical basis of $\ell^2(\mathbb{Z}^+)$. If $\{w_n\}_{n\geq 1}$ is a bounded sequence in $\mathbb{C}\setminus\{0\}$, then the *unilateral backward weighted shift* $T:\ell^2\to\ell^2$ is defined by

$$Te_n = w_n e_{n-1}, \quad n\geq 1, \quad Te_0 = 0.$$

Since the properties of hypercyclicity and Cesàro hypercyclicity are invariant under similarity, we can suppose that the weights are positive (see [Sh, Prop. 1]). Analogously $\ell^2(\mathbb{Z})$ denotes the Hilbert space of bilateral sequences which are 2-summable, and let $\{e_n\}_{n\in\mathbb{Z}}$ be the canonical basis of $\ell^2(\mathbb{Z})$. If $\{w_n\}_{n\in\mathbb{Z}}$ is a bounded sequence in $\mathbb{C}\setminus\{0\}$ the *bilateral weighted shift* is defined by

$$Te_n = w_n e_{n-1}.$$

As in the case of the unilateral shift we can suppose that the weights are positive.

The hypercyclicity of unilateral and bilateral shifts has been studied in several works (see [Gr], [MS], [Ro], [Sa1–3]). The basic tool to check if an operator is hypercyclic is the *Hypercyclicity Criterion*. This criterion is a sufficient condition for hypercyclicity that was discovered by Carol Kitai in her 1982 unpublished thesis (see [Ki]). It was rediscovered later by Gethner and Shapiro (see [GeS]).

HYPERCYCLICITY CRITERION. *Let $\{T_n\}$ be a sequence of bounded operators on a separable Banach space $\mathcal{B}$. Suppose that there exists a strictly increasing sequence $\{n_k\}$ of positive integers for which there are*

(a) *a dense subset $X\subset\mathcal{B}$ such that $\|T_{n_k}x\|\to 0$ for every $x\in X$;*

(b) *a dense subset $Y\subset\mathcal{B}$ and a sequence of mappings $S_k : Y\to Y$ such that $T_{n_k}S_k =$ identity on Y, and $\|S_k y\|\to 0$ for every $y\in Y$.*

Then the sequence $\{T_{n_k}\}$ is hypercyclic, that is, there exists a vector $x\in\mathcal{B}$ such that $\{T_{n_k}x\}$ is dense in $\mathcal{B}$.

Observe that if a sequence $\{T_n\}$ of operators satisfies the criterion for a sequence $\{n_k\}$ then it satisfies it for any subsequence $\{r_k\}\subset\{n_k\}$. Hence if a sequence $\{T_n\}$ satisfies the criterion for $\{n_k\}$ then $\{T_{r_k}\}$ is hypercyclic for any subsequence $\{r_k\}\subset\{n_k\}$; when this phenomenon happens the sequence $\{T_n\}$ is said to be *hereditarily hypercyclic*. The Hypercyclicity Criterion and hereditary hypercyclicity are equivalent for the sequence of iterates $T_n = T^n$ of a single operator (see [Bès]).

From [Sa2, Theorem 2.8] it is not difficult to deduce that a unilateral weighted backward shift is hypercyclic if and only if there is an increasing sequence $\{n_k\}$ of positive integers such that

$$\lim_{k\to\infty}\prod_{i=1}^{n_k} w_{i+q} = \infty \tag{3.1}$$

for each non-negative integer q. Moreover in [LM2] it is shown that any hypercyclic unilateral shift T satisfies the Hypercyclicity Criterion (that is, the sequence $T_n = T^n$ satisfies it).

PROPOSITION 3.1. *Let T be a unilateral weighted backward shift with weight sequence $\{w_n\}_{n\geq 1}$. Then T has hypercyclic Cesàro means if and only if there exists an increasing sequence $\{n_k\}$ of positive integers such that*

$$\lim_{k\to\infty} \frac{\prod_{i=1}^{n_k} w_{i+q}}{n_k} = \infty$$

for each non-negative integer q.

Proof. Assume that T is Cesàro-hypercyclic. Let q be a non-negative integer and $z = \sum_{j=0}^{q} e_j$. Given $\varepsilon > 0$, since T is Cesàro-hypercyclic, there exist a vector $x = \sum_{j=1}^{\infty} x_j e_j$ and n large enough $(n > 2q)$ such that

$$\|x - z\| < \varepsilon \tag{3.2}$$

and

$$\left\| \frac{T^n}{n} x - z \right\| < \varepsilon. \tag{3.3}$$

Condition (3.2) follows from the fact that the set of hypercyclic vectors for a hypercyclic sequence is a dense G_δ set (see [GoS, Theorem 1.2]). From (3.2) we see that $|x_j| < \varepsilon$ for $j > q$ and $|x_j - 1| < \varepsilon$ for $0 \leq j \leq q$.

From (3.3), it follows that

$$\left| \frac{\prod_{i=1}^{n} w_{i+s}}{n} x_{s+n} - 1 \right| < \varepsilon$$

for $0 \leq s \leq q$. Therefore, if we take into account that $|x_{s+n}| < \varepsilon$ $(n > 2q)$, it follows that

$$\left| \frac{\prod_{i=1}^{n} w_{i+s}}{n} \right| > \frac{1-\varepsilon}{\varepsilon}$$

for $0 \leq s \leq q$, and this proves the necessity.

Conversely, suppose that there exists an increasing sequence $\{n_k\}$ of positive integers such that $\lim_{k\to\infty} n_k^{-1} \prod_{i=1}^{n_k} w_{i+q} = \infty$ for each non-negative integer q. It is sufficient to show that the sequence $T_k = T^{n_k}/n_k$ satisfies the Hypercyclicity Criterion. Indeed, take $X = Y = \text{linear span}\{e_n\}_{n\geq 0}$. Since $T^n e_k = 0$ for n large, we have $T_k \to 0$ pointwise on X. Define the sequence of linear mappings S_k as

$$S_k e_q = n_k \Big(\prod_{i=1}^{n_k} w_{i+q} \Big)^{-1} e_{q+n_k}.$$

Observe that $T_k S_k$ is the identity on Y and since $n_k (\prod_{i=1}^{n_k} w_i)^{-1} \to 0$ for each q, we have $S_k \to 0$ pointwise on Y. Hence $n^{-1}T^n$ satisfies the Hypercyclicity Criterion and therefore T is Cesàro-hypercyclic, which yields the desired result. ∎

REMARK 3.2. The proof above actually gives more, namely, every unilateral weighted backward shift with hypercyclic Cesàro means is hereditarily Cesàro-hypercyclic, that is, there exists an increasing sequence $\{n_k\}$ of positive integers such that for any subsequence $\{r_k\} \subset \{n_k\}$, the sequence $M_{r_k}(T)$ is hypercyclic.

Proposition 3.1 together with the result of Salas (see condition (3.1) in this section) yield the following corollary.

COROLLARY 3.3. *Every Cesàro-hypercyclic unilateral weighted shift is hypercyclic.*

From [Sa2, Theorem 2.1] we know that a bilateral weighted backward shift is hypercyclic if and only if there exists a sequence $\{n_k\}$ of positive integers such that for any integer q,

$$\lim_{k\to\infty} \prod_{i=1}^{n_k} w_{q+i} = \infty \quad \text{and} \quad \lim_{k\to\infty} \prod_{i=0}^{n_k-1} w_{q-i} = 0. \tag{3.4}$$

In order to characterize when a bilateral backward shift is Cesàro-hypercyclic we follow the techniques used in [Sa1,2] and [MS]. The details are left to the reader.

PROPOSITION 3.4. *Let T be a bilateral weighted shift with weight sequence $\{w_n\}_{n\in\mathbb{Z}}$. Then T is Cesàro-hypercyclic if and only if there exists an increasing sequence $\{n_k\}$ of positive integers such that for any integer q,*

$$\lim_{k\to\infty} \frac{\prod_{i=1}^{n_k} w_{i+q}}{n_k} = \infty \quad \textit{and} \quad \lim_{k\to\infty} \frac{\prod_{i=0}^{n_k-1} w_{q-i}}{n_k} = 0.$$

REMARK 3.5. As in Remark 3.2, Proposition 3.4 can be strengthened as follows: every Cesàro-hypercyclic bilateral weighted shift is hereditarily Cesàro-hypercyclic.

EXAMPLE 3.6. *The bilateral backward shift T defined by the weight sequence*

$$w_n = \begin{cases} 1 & \textit{if } n \le 0, \\ 2 & \textit{if } n \ge 1, \end{cases}$$

is not hypercyclic, but it is Cesàro-hypercyclic.

Proof. Observe that $\prod_{i=0}^{n-1} w_{q-i}$ is constant, therefore by Salas' theorem (see condition (3.4) in this section) T is not hypercyclic. On the other hand observe that

$$\frac{\prod_{i=0}^{n-1} w_{q-i}}{n} = \frac{C}{n} \quad \text{and} \quad \frac{\prod_{i=1}^{n} w_{q+i}}{n} = \begin{cases} 2^n/n & \text{if } q > 0, \\ 2^{n+q}/n & \text{if } q \le 0. \end{cases}$$

Therefore by Proposition 3.4 the operator T is Cesàro-hypercyclic. ∎

4. Existence of Cesàro-hypercyclic operators on separable Banach spaces. In [An] and [Be], S. Ansari and L. Bernal, solving independently a question posed by Rolewicz (see [Ro]), showed that for any separable Banach space $\mathcal{B}$ there exists a hypercyclic operator T on $\mathcal{B}$. In the present section we will show that any separable Banach space admits a Cesàro-hypercyclic operator.

The crucial fact in this section is to show the following result:

THEOREM 4.1. *Let T be a unilateral weighted backward shift defined on $\ell^2(\mathbb{Z}^+)$, with a positive weight sequence $\{w_n\}_{n\geq 1}$, and let r be a positive integer. Then the operator sequence*

$$\frac{(I+T)^n}{n^r}$$

satisfies the Hypercyclicity Criterion.

Theorem 4.1 contains Theorem 3.3 of [Sa2] and Proposition 4.3 of [LM1]. The proof of Theorem 4.1 is easier than that of Proposition 4.3 of [LM1] thanks to the following version of the Hypercyclicity Criterion which appears in [GoS] and [Le, Theorem 2].

THEOREM A. *Let $\{T_n\}$ be a sequence of commuting bounded operators. The sequence $\{T_n\}$ satisfies the Hypercyclicity Criterion if and only if for any two non-void open sets $\mathcal{U}$, $\mathcal{V}$ and any open neighborhood $\mathcal{W}$ of the origin, there exists a positive integer n such that*

$$T_n(\mathcal{U})\cap\mathcal{W}\neq\emptyset \quad \text{and} \quad T_n(\mathcal{W})\cap\mathcal{V}\neq\emptyset.$$

The proof of Theorem 4.1 uses techniques of [Sa2] to prove that $I+T$ is hypercyclic whenever T is a weighted backward shift with positive weights. We will also need a lemma whose proof is a suitable modification of Lemma 3.2 in [Sa2].

LEMMA B. *Fix a positive integer r. Let $C_n = (c_{i,j}(n))$ be the $2^k\times 2^k$ matrix with*

$$c_{i,j}(n) = \frac{1}{n^r}\binom{n}{2^k+j-i}.$$

Let $B_n = (b_i(n))$ be a column vector such that $b_i(n)$ is a rational function in n of degree at most 2^k-i-r, where $i=1,\dots,2^k$. Then for n large enough we have $\det C_n\neq 0$ and there exists a solution $X_n=(x_i(n))$ of the equation $B_n = C_nX_n$ and the entries $x_i(n)$ satisfy $|x_i(n)|\leq P/n^i$, where P is a constant.

Now we have all the ingredients to establish the proof of Theorem 4.1.

Proof of Theorem 4.1. In accordance with Theorem A, let $\mathcal{U}, \mathcal{V}$ be two non-void open sets and $\mathcal{W}$ an open neighborhood of the origin. We can suppose that there exists a dense set in $\ell^2(\mathbb{Z}^+)$ having the form $\mathcal{D} = \{z_j = \sum_{i=0}^{2^j-1} z_{i,j}e_i : j \in \mathbb{Z}^+\}$.

Since $\mathcal{D}$ is dense and $\mathcal{U}$ is open, we can choose $z_k \in \mathcal{U} \cap \mathcal{D}$ for some positive integer k, that is, $z_k = \sum_{i=0}^{2^k-1} z_{i,k}e_i$. We add to the vector z_k a suitable vector $x = \sum_{i=1}^{2^k} x_i e_{2^k+i-1}$ to be determined afterwards, with the same length as z_k, and we try to solve the following system:

$$\left\langle \frac{(I+T)^n}{n^r}\Big(\sum_{j=0}^{2^k-1} z_{j,k}e_j + \sum_{i=1}^{2^k} x_i e_{2^k+i-1}\Big), e_p \right\rangle = 0 \tag{4.1}$$

with $p = 0, \ldots, 2^k - 1$. Recall that

$$T^m e_s = \begin{cases} \prod_{l=s-m+1}^{s} w_l e_{s-m} & \text{if } m \le s, \\ 0 & \text{otherwise,} \end{cases}$$

and observe that the matrix of the system (4.1) is the $2^k \times 2^k$ matrix $D = (d_{i,j})$ with entries

$$d_{i,j} = \frac{1}{n^r}\binom{n}{2^k + j - i} \prod_{s=i}^{2^k+j-1} w_s.$$

Then

$$\det D = \Big(\prod_{j=-2^k+1}^{2^k-1} (w_{2^k+j})^{2^k-|j|}\Big) \det C$$

where $C = c_{i,j}$ and $c_{i,j} = n^{-r}\binom{n}{2^k+j-i}$. By applying Lemma B, it follows that if n is large enough, the system (4.1) is solvable, and the solution satisfies $|x_i| \le P/n^i$ for some constant P. Since

$$\frac{(I+T)^n}{n^r}\Big(\sum_{j=0}^{2^k-1} z_{j,k}e_j + \sum_{i=1}^{2^k} x_i e_{2^k+i-1}\Big) = \frac{1}{n^r}\sum_{p=1}^{2^k}\Big(\sum_{j=0}^{p-1}\binom{n}{j}T^j\Big)x_p e_{2^k+p-1},$$

it follows that

$$\begin{aligned}\left\|\frac{(I+T)^n}{n^r}\Big(\sum_{j=0}^{2^k-1} z_{j,k}e_j + \sum_{i=1}^{2^k} x_i e_{2^k+i-1}\Big)\right\| &\le \frac{1}{n^r}\sum_{p=1}^{2^k}\Big(\sum_{j=0}^{p-1}\binom{n}{j}\|T\|^j\Big)\|x_p\| \\ &\le \frac{L}{n^{1+r}}\end{aligned}$$

where L is another constant. That is, there exists a positive integer n_0 such that if $n \ge n_0$ then there exists a vector $x(n)$ (with small norm) such that

$$z_k + x(n) \in \mathcal{U} \quad \text{and} \quad \frac{(I+T)^n}{n^r}(z_k + x(n)) \in \mathcal{W}, \tag{4.2}$$

that is,

$$\frac{(I+T)^n}{n^r}(\mathcal{U}) \cap \mathcal{W} \neq \emptyset \quad \text{if } n \geq n_0.$$

On the other hand let $z_k = \sum_{j=0}^{2^k-1} z_{j,k} e_j$ be a vector in $\mathcal{W} \cap \mathcal{D}$ and let $y_l = \sum_{j=0}^{2^l-1} y_{j,l} e_j$ be a vector in $\mathcal{V} \cap \mathcal{D}$. We can suppose without loss of generality that $k \geq l$. Again we add to z_k one vector with the same "length" as z_k and we consider the linear system of equations

$$\left\langle \frac{(I+T)^n}{n^r}\left(\sum_{j=0}^{2^k-1} z_{j,k} e_j + \sum_{i=1}^{2^k} x_i e_{2^k+i-1}\right) - y_l, e_p \right\rangle = 0 \tag{4.3}$$

with $p = 0, \ldots, 2^k - 1$. Observe that the coefficient matrix of the system (4.3) is the $2^k \times 2^k$ matrix $D = (d_{i,j})$ with entries

$$d_{i,j} = \frac{1}{n^r}\binom{n}{2^k + j - i} \prod_{s=i}^{2^k+j-1} w_s,$$

and the free term of (4.3) is $B_n = (b_i(n))$, where $b_i(n)$ is a rational function in n of degree at most $2^k - i - r$. Therefore applying again Lemma B it follows that if n is large enough, the system (4.3) has a solution, and this solution satisfies $|x_i| \leq P/n^i$ for some constant P. Since

$$\frac{(I+T)^n}{n^r}\left(\sum_{j=0}^{2^k-1} z_{j,k} e_j + \sum_{i=1}^{2^k} x_i e_{2^k+i-1}\right) - y_l = \frac{1}{n^r}\sum_{p=1}^{2^k}\left(\sum_{j=0}^{p-1}\binom{n}{j} T^j\right) x_p e_{2^k+p-1},$$

it follows that

$$\left\|\frac{(I+T)^n}{n^r}\left(\sum_{j=0}^{2^k-1} z_{j,k} e_j + \sum_{i=1}^{2^k} x_i e_{2^k+i-1}\right) - y_l\right\| \leq \frac{1}{n^r}\sum_{p=1}^{2^k}\left(\sum_{j=0}^{p-1}\binom{n}{j}\|T\|^j\right)\|x_p\| \leq \frac{L}{n^{1+r}}$$

where L is a new constant. Hence, if n is large enough there exists a vector $x(n)$ (with small norm) such that

$$z_k + x(n) \in \mathcal{W} \quad \text{and} \quad \frac{(I+T)^n}{n^r}(z_k + x(n)) \in \mathcal{V}.$$

Therefore by (4.2) if n is large enough, we have

$$\frac{(I+T)^n}{n^r}(\mathcal{U}) \cap \mathcal{W} \neq \emptyset \quad \text{and} \quad \frac{(I+T)^n}{n^r}(\mathcal{W}) \cap \mathcal{V} \neq \emptyset.$$

As a consequence of the latter, by Theorem A, $(I+T)^n/n^r$ satisfies the Hypercyclicity Criterion, and the proof is complete. ■

REMARK 4.2. (a) The proof of Theorem 4.1 can be adapted for $\ell^p(\mathbb{Z}^+)$ with $1 \leq p < \infty$.

(b) Observe that Theorem 4.1 implies in particular that the operator $I + T$, where T is any unilateral weighted backward shift, is Cesàro-hypercyclic.

Let $\mathcal{B}$ be a separable complex Banach space and let $\{x_n\}$, $\{x_n^\star\}$ be two sequences in $\mathcal{B}$ and $\mathcal{B}^\star$ respectively. Recall that $\{x_n, x_n^\star\}$ is a *bounded biorthogonal system* if both sequences are bounded, $x_n^\star(x_m) = \delta_{n,m}$ and the linear span of $\{x_n\}$ is dense in $\mathcal{B}$. As application of the Hahn–Banach theorem one can show that any separable Banach space admits a bounded biorthogonal system (see [LT, Theorem 1.4.f]). Therefore in $\mathcal{B}$ we can consider the subspace

$$X_1 = \Big\{ \sum_{n=0}^{\infty} a_n x_n : a_n \in \mathbb{C},\ \sum_{n=0}^{\infty} |a_n| < \infty \Big\},$$

which is a Banach space endowed with the norm

$$\Big\| \sum_{n=0}^{\infty} a_n x_n \Big\|_1 = \sum_{n=0}^{\infty} |a_n|.$$

In fact X_1 is isomorphic to $\ell^1(\mathbb{Z}^+)$ by the isomorphism $J : \ell^1 \to X_1$ defined by $Je_n = x_n$, where $\{e_n\}$ is the standard unit vector basis of $\ell^1(\mathbb{Z}^+)$. Let $T_n : \ell^1 \to \ell^1$ be a sequence of bounded linear operators. A sequence of operators $\widehat{T}_n : \mathcal{B} \to \mathcal{B}$ is said to be a *quasi-extension* of $\{T_n\}$ if $\widehat{T}_n|X_1 = JT_nJ^{-1}$.

Finally we need to show that dense subsets in X_1 are dense in $\mathcal{B}$. For this, it is sufficient to show that the identity map $(X_1, \|\cdot\|_1) \to (\mathcal{B}, \|\cdot\|)$ is continuous. But this follows easily because of the boundedness of the sequence $\{x_n\}$:

$$\Big\| \sum_{n=0}^{\infty} a_n x_n \Big\| \leq \sum_{n=0}^{\infty} |a_n| \cdot \|x_n\| \leq M \sum_{n=0}^{\infty} |a_n| = M\|x\|_1.$$

THEOREM 4.3. *Every separable Banach space admits a Cesàro-hypercyclic operator.*

Proof. Let $\{t_n\}$ be any sequence of positive numbers with $\sum_{n=1}^{\infty} t_n < 1$, and let $\widehat{T}(x) = \sum_{n=0}^{\infty} t_{n+1} x_{n+1}^\star(x) x_n$. Since the sequence $\{x_n^\star\}$ is bounded, the operator $\widehat{T}$ is bounded. Now we consider the operator $\widehat{I} + \widehat{T}$ on $\mathcal{B}$. Observe that it is a quasi-extension of the operator $I + T$, where T is the weighted backward shift defined in ℓ^1 by $Te_0 = 0$ and $Te_n = t_n e_{n-1}$.

By Theorem 4.1 and Remark 4.2(a), $I + T$ is Cesàro-hypercyclic on ℓ^1. On the other hand let $J : \ell^1 \to X_1$ be the natural isomorphism. Since $(\widehat{I} + \widehat{T})|X_1 = J(I+T)J^{-1}$ it follows that if x is a Cesàro-hypercyclic vector for $I + T$, then Jx is a Cesàro-hypercyclic vector for $(\widehat{I} + \widehat{T})|X_1$. But if the orbit $\{(\frac{I+\widehat{T}}{n})^n(x)\}$ is dense in X_1 for some x, then it is also dense in $\mathcal{B}$. Therefore $I+\widehat{T}$ is a Cesàro-hypercyclic operator on $\mathcal{B}$. ∎

5. The norm closure of the class of Cesàro-hypercyclic operators. In this section we provide a spectral description of the closure of the class of all Cesàro-hypercyclic operators on Hilbert spaces.

Let $\mathcal{H}$ be a separable Hilbert space and denote by $HC(\mathcal{H})$ the class of all hypercyclic operators and by $CH(\mathcal{H})$ the class of all Cesàro-hypercyclic operators. We already know that the two classes are different. However, we will show that their closures in the operator norm topology coincide. In [He1] Herrero provides a spectral description of the norm closure, $\overline{HC}(\mathcal{H})$, of the class $HC(\mathcal{H})$. The class of all Cesàro-hypercyclic operators $CH(\mathcal{H})$ is invariant under similarity, and therefore it can be analyzed with the approximation machinery developed in [He1, 3]. By means of Theorem 2.4, we will show that the techniques used in [He2] can be applied to obtain an analogous result for the class $CH(\mathcal{H})$.

The *spectrum* of an operator T is the set $\sigma(T)=\{\lambda\in\mathbb{C}: T-\lambda I$ is not invertible$\}$. If $\mathcal{K}(\mathcal{H})$ denotes the ideal of all compact operators acting on $\mathcal{H}$, then the *Calkin algebra* is the quotient space $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. If $T\in\mathcal{L}(\mathcal{H})$, the canonical projection of T onto $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ will be denoted by $\widetilde{T}$. The *essential spectrum* of T is $\sigma_{\mathrm{e}}(T)=\sigma(\widetilde{T})$. Let $T^\star$ denote the adjoint of T. Recall that $T\in\mathcal{L}(\mathcal{H})$ is called *semi-Fredholm* if $\operatorname{ran} T$ is closed and the index $\operatorname{Ind}(T)=\dim(\operatorname{Ker}(T))-\dim(\operatorname{Ker}(T^\star))$ is finite. The set $\varrho_{\text{s-F}}(T)$ denotes the *semi-Fredholm domain* of T, that is, the set of all complex numbers λ such that $T-\lambda$ is semi-Fredholm. Finally, we denote by $\sigma_{\mathrm{W}}(T)$ the *Weyl spectrum* of T (that is, of complex numbers λ such that $T-\lambda$ is not a semi-Fredholm operator of index 0) and by $\sigma_0(T)$ the set of all *normal eigenvalues* of T (that is, of isolated points of $\sigma(T)$ which are not in $\sigma_{\mathrm{e}}(T)$). The spectral description of $\overline{CH}(\mathcal{H})$ is the following:

THEOREM 5.1. *The class $\overline{CH}(\mathcal{H})$ consists of those operators $T\in\mathcal{L}(\mathcal{H})$ satisfying the conditions:*

(1) $\sigma_{\mathrm{W}}(T)\cup\partial\mathbb{D}$ *is connected;*
(2) $\sigma_0(T)=\emptyset$; *and*
(3) $\operatorname{Ind}(T-\lambda)\ge 0$ *for all* $\lambda\in\varrho_{\text{s-F}}(T)$.

An essential step in the proof of Theorem 5.1 is the following result.

PROPOSITION 5.2. *Assume that T is a Cesàro-hypercyclic operator. Then*

(i) $\sigma_{\mathrm{p}}(T^\star)=\emptyset$;
(ii) $\operatorname{Ind}(T-\lambda)\ge 0$ *for all* $\lambda\in\varrho_{\text{s-F}}(T)$;

(iii) $\sigma_{\mathrm{W}}(T) = \sigma(T)$; *and*
(iv) $\sigma(T) \cup \partial\mathbb{D}$ *is a connected set.*

Proposition 5.2 states that $CH(\mathcal{H}) \subset \{T \in \mathcal{L}(\mathcal{H}) : T \text{ satisfies } (1)\text{–}(3)\}$. On the other hand using the continuity properties of the Riesz–Dunford functional calculus and the stability properties of the semi-Fredholm operators (see [He1, Chapter 1], [Ka, Chapter 4]), we deduce that the class of operators satisfying (1)–(3) of Theorem 5.1 is a closed set in the operator norm topology. Therefore Proposition 5.2 shows that

$$\overline{CH}(\mathcal{H}) \subset \{T \in \mathcal{L}(\mathcal{H}) : T \text{ satisfies } (1)\text{–}(3)\}. \tag{5.1}$$

Proof of Proposition 5.2. Observe that (i) was proved in Proposition 2.2. On the other hand using the basic properties of semi-Fredholm operators (the reader is referred to the classical book of Kato [Ka, Chapter IV]) we see that (ii) and (iii) are consequences of (i).

As in [Ki], note that if an operator T is Cesàro-hypercyclic and it is the direct sum of two operators $T_1 \oplus T_2 = T$ acting on $H_1 \oplus H_2$, then each compression T_i is Cesàro-hypercyclic on H_i, $i = 1, 2$. If $\sigma(T)$ includes a connected component σ that is contained in $\mathbb{D}$ and if H_1 and H_2 are the Riesz spectral invariant subspaces of T associated with σ and $\sigma(T) \setminus \sigma$ with $\sigma(T|H_1) = \sigma$ and $\sigma(T|H_2) = \sigma(T)\setminus\sigma$, it follows easily that $\|(T|H_1)^n x\| \to 0$ as $n \to \infty$, in particular $\|n^{-1}(T|H_1)^n x\| \to 0$ for each $x \in H_1$. And this contradicts the Cesàro hypercyclicity of $T|H_1$.

Now let us prove that $\sigma(T)$ cannot include a closed subset $\sigma \subset \mathbb{C} \setminus \overline{\mathbb{D}}$. Since a sequence $\{T_n\}$ of invertible operators is hypercyclic if and only if the sequence $\{T_n^{-1}\}$ of the corresponding inverses is also hypercyclic (see [GoS, p. 234]), for every operator T that is Cesàro-hypercyclic and invertible, the sequence $\{nT^{-n}\}$ is hypercyclic. Suppose that $\sigma(T)$ includes a closed subset $\sigma \subset \mathbb{C} \setminus \overline{\mathbb{D}}$. Let T_1 be the operator associated to σ in the Riesz spectral decomposition theorem and H_1 be the invariant subspace corresponding to T_1. Observe that T_1 is invertible and $\sigma(T_1^{-1}) \subset \mathbb{D}$ so $\|T_1^{-n}x\| \to 0$ exponentially as $n \to \infty$, for each $x \in H_1$. Therefore $\|nT_1^{-n}x\| \to 0$ for each $x \in H_1$, which contradicts the fact that T_1 is Cesàro-hypercyclic on H_1. ∎

In order to prove the reverse inclusion in (5.1), it is necessary to construct some models. That is, given an operator T satisfying (1)–(3), we must construct a Cesàro-hypercyclic T_ε such that $\|T - T_\varepsilon\| < \varepsilon$. Herrero showed that given an operator satisfying (1)–(3), there exists a hypercyclic operator T_ε such that $\|T - T_\varepsilon\| < \varepsilon$. Following the proof of D. A. Herrero (see [He2, Proposition 2.4 and Theorem 2.1]) and taking into account Theorem 2.4 one can show that the model constructed therein is also a Cesàro-hypercyclic operator. Hence, this proves the reverse inclusion in (5.1) and thus the proof of Theorem 5.1 is finished.

6. Final remarks

1. *Infinite-dimensional subspaces of Cesàro-hypercyclic vectors.* In [LM2] it is shown that the existence of an infinite-dimensional subspace of hypercyclic vectors for an operator T basically depends on the essential spectrum of T. For the Cesàro-hypercyclic case an analogous result can be obtained with some rearrangements in the proofs.

THEOREM. *Let T a bounded operator on a separable Banach space $\mathcal{B}$.*

(a) *If $M_n(T)$ satisfies the Hypercyclicity Criterion and the essential spectrum of T intersects the closed unit disk, then T has an infinite-dimensional closed subspace whose non-zero elements are Cesàro-hypercyclic for T.*

(b) *If the essential spectrum of T does not intersect the closed unit disk then all closed subspaces of Cesàro-hypercyclic vectors for T have a finite dimension.*

2. In an analogous way we can define the notion of Cesàro supercyclicity. An operator T on a separable Banach space $\mathcal{B}$ is *Cesàro-supercyclic* if there exists a vector x such that the set $\{\lambda M_n(T)x : n \in \mathbb{N}, \lambda \in \mathbb{C}\}$ is dense in $\mathcal{B}$. Using the techniques of Section 2 and some ideas which appear in [MS] one can prove that Cesàro supercyclicity is equivalent to supercyclicity.

3. Finally, as we can see in Dunford's theorem, condition (b) is not used along the work. It would be interesting to know the role that condition (b) plays in the hypercyclic setting.

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