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Continuous Optimization

Generalized invex monotonicity

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Abstract

In this paper the generalized invex monotone functions are defined as an extension of monotone functions. A series of sufficient and necessary conditions are also given that relate the generalized invexity of the function θ with the generalized invex monotonicity of its gradient function $\nabla\theta$. This new class of functions will be important in order to characterize the solutions of the Variational-like Inequality Problem and Mathematical Programming Problem.

Keywords: Convex programming; Mathematical programming; Invex functions; Generalized monotonicity

1. Introduction

The study of convex and similar functions and generalized convexity, has been of great importance in recent years, as can be seen in [1,2,12] or Schaible [15–18].

The Mathematical Programming Problem is defined as

$$\text{(MP)} \quad \min \quad \theta(x) \quad \text{s.t.} \quad x \in X,$$

where $\theta : X \subseteq \mathbb{R}^n \to \mathbb{R}$.

The generalized convexity plays an important role in the search for optimal conditions for solutions to the Mathematical Programming Problem (MP).

It is important to be able to fix the convexity of the model (MP), which is to say that, $X \subseteq \mathbb{R}^n$ is a convex set and the objective function θ is a convex function on X, obtaining important results in this case, like for example:

1. The solution set is convex.
2. A local minimum is a global minimum.
3. A solution of the Karush–Kuhn–Tucker conditions is a minimum.
4. If θ is strictly convex then a minimum (if it exists) is unique.

Some of these properties are shared by kinds of functions that are more general than the convex functions, something which has given rise to the study of generalized convexity and later to that of generalized monotonicity.

Just as convex functions are characterized by a monotone gradient, different kinds of generalized convex functions give rise to gradient maps with

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certain generalized monotonicity properties which are inherited from generalized convexity of the underlying function.

In [9] it was proved that the generalized convexity of θ was equivalent to the generalized monotonicity of its gradient function $\nabla\theta$, as we shall see later, in Theorem 3.3.

The role of generalized monotonicity of the operator in Variational Inequality Problems (VIPs) corresponds to the role of generalized convexity of the objective function in Mathematical Programming Problems.

Given a subset $\mathscr{A} \subseteq \mathbb{R}^n$ and a function $F : \mathscr{A} \subseteq \mathbb{R}^n \to \mathbb{R}^n$, the classical Variational Inequality Problem (VIP), is to find a vector $\bar{x} \in \mathscr{A}$, such that

$$(y - \bar{x})^{\mathrm{t}} F(\bar{x}) \geqslant 0 \quad \forall y \in \mathscr{A}.$$

In the Variational Inequality Problem (VIP) that includes the Mathematical Programming Problems (MP) the monotonicity of $F = \nabla\theta$, ensures significant results, such as:

1. if F is a monotone function then the set of solutions is convex,
2. if F is a strictly monotone (SM) function then a solution (if it exists) is unique,
3. if F is a strongly monotone (SGM) function then a solution exists and it is unique.

Moreover, from a computational point of view, the monotonicity is also fundamental, since if F is monotone, many algorithms converge towards the solution. From this, the importance of the study of generalized monotonicity is evident. In this paper, just as in the case of convexity, our aim is to find more general functions that prove the fulfillment of some of those properties.

In Section 2 we will define the new concepts of strongly invex functions and strongly pseudo invex functions, that together with previous definitions of pseudo invex (PIX) functions or quasi invex (QIX) functions, enable us to establish the relationships between the generalized invexity of θ and the invex monotonicity of $\nabla\theta$, following Karamardian and Schaible's model [9], for the convex case.

In [14] the concept of invex monotone function appeared, there called η-monotone, as a generalization of the concept of monotone function. In Section 3, we will extend the concept of invex monotone function and define the pseudo invex monotone (PIM) function, quasi invex monotone (QIM) function, and so on.

In Section 4, using the new definitions of invex and generalized invex monotone functions, we will connect both concepts and obtain sufficient and necessary conditions of generalized invex monotonicity.

Variational inequalities arise in models for a wide class of mathematical, physical, regional, economic, engineering, optimization and control, transportation, elasticity and applied sciences, etc.; see, for example, [3,6,11] and the references therein.

In Section 5, we will apply generalized invex monotonicity to calculate the solutions to the Variational-Like Inequality Problem (VLIP), a more general problem than the classical Variational Inequality Problem.

Given a closed and convex set $D \subseteq \mathbb{R}^n$ and two continuous functions $F : D \to \mathbb{R}^n$ and $\eta : D \times D \to \mathbb{R}^n$, the Variational-Like Inequality Problem (VLIP) is to find $\bar{x} \in D$, such that

$$\eta(x, \bar{x})^{\mathrm{t}} F(\bar{x}) \geqslant 0 \quad \forall x \in D.$$

If $\eta(x, \bar{x}) = (x - \bar{x})$ then (VLIP) = (VIP).

Specifically, we will prove that it is possible to establish the existence of the solution to the Variational-like Inequality Problem, assuming F to be PIM, a weaker condition than those assumed to date. We will be able to identify the solutions to the Variational-Like Inequality Problems with the solutions of the Mathematical Programming Problems, under conditions of invexity.

Finally, we will reach the solutions to the Mathematical Programming Problems, through those of the Variational-Like Inequality Problems, using the generalized invex monotonicity of the function F.

2. New definitions of invex functions

Invex functions were introduced by Hanson [5], as a generalization of differentiable convex functions. Let $\Gamma \subseteq \mathbb{R}^n$ be an open set.

Definition 2.1. The function $\theta : \Gamma \subseteq \mathbb{R}^n \to \mathbb{R}$ differentiable is invex (IX) if, $\exists \eta : \Gamma \times \Gamma \to \mathbb{R}^n$ such that $\forall x, y \in \mathbb{R}^n$,

$$\theta(y) - \theta(x) \geqslant \eta(y,x)^{\mathrm{t}} \nabla \theta(x).$$

These functions are more general than the convex and pseudo convex ones. The type of invex function is equivalent to the type of function whose stationary points are global minima. Therefore, if θ has no stationary points, then θ is invex.

Other authors like Kanniappan and Pandian [7], or Kaur and Gupta [10] (where invexity is designated η-convexity), weaken the concept of invexity searching for more general functions that continue verifying the properties of optimality mentioned previously, for the general convex functions.

Definition 2.2. A function $\theta : \Gamma \subseteq \mathbb{R}^n \to \mathbb{R}$ differentiable, is said to be

(a) strictly invex (SIX) if $\exists \eta : \Gamma \times \Gamma \to \mathbb{R}^n$ such that $\forall x, y \in \mathbb{R}^n$, $x \neq y$,

$$\theta(y) - \theta(x) > \eta(y,x)^{\mathrm{t}} \nabla \theta(x),$$

(b) pseudo invex (PIX) if $\exists \eta : \Gamma \times \Gamma \to \mathbb{R}^n$ such that $\forall x, y \in \mathbb{R}^n$,

$$\eta(y,x)^{\mathrm{t}} \nabla \theta(x) \geqslant 0 \quad \Rightarrow \quad \theta(y) - \theta(x) \geqslant 0,$$

(c) strictly pseudo invex (SPIX) if $\exists \eta : \Gamma \times \Gamma \to \mathbb{R}^n$ such that $\forall x, y \in \mathbb{R}^n$, $x \neq y$,

$$\eta(y,x)^{\mathrm{t}} \nabla \theta(x) \geqslant 0 \quad \Rightarrow \quad \theta(y) - \theta(x) > 0,$$

(d) quasi invex (QIX) if $\exists \eta : \Gamma \times \Gamma \to \mathbb{R}^n$ such that $\forall x, y \in \mathbb{R}^n$,

$$\theta(y) - \theta(x) \leqslant 0 \quad \Rightarrow \quad \eta(y,x)^{\mathrm{t}} \nabla \theta(x) \leqslant 0.$$

Hanson [5] proved that no distinction exists between pseudo invexity and invexity when θ is a scalar function.

Trivially, the differentiable convex functions are invex with $\eta(y,x) = y - x$ $\forall x, y \in \Gamma$, and the same occurs with the SIX, PIX, SPIX and QIX functions.

Let us generalize the concept of strongly convex (SGCX) and strongly pseudo convex (SGPCX) functions, given in [9]:

Definition 2.3. A function $\theta : \Gamma \subseteq \mathbb{R}^n \to \mathbb{R}$ differentiable, is strongly invex (SGIX) if there exists a function $\eta : \Gamma \times \Gamma \to \mathbb{R}^n$, and a scalar $\alpha > 0$, such that $\forall x, y \in \mathbb{R}^n$,

$$\theta(y) - \theta(x) \geqslant \eta(y,x)^{\mathrm{t}} \nabla \theta(x) + \alpha \|\eta(y,x)\|^2.$$

Definition 2.4. A function $\theta : \Gamma \subseteq \mathbb{R}^n \to \mathbb{R}$ differentiable, is strongly pseudo invex (SGPIX) if there exists a function $\eta : \Gamma \times \Gamma \to \mathbb{R}^n$ and a scalar $\alpha > 0$, such that $\forall x, y \in \mathbb{R}^n$,

$$\eta(y,x)^{\mathrm{t}} \nabla \theta(x) \geqslant 0 \quad \Rightarrow \quad \theta(y) \geqslant \theta(x) + \alpha \|\eta(y,x)\|^2.$$

The significance of all of these definitions lies in that, as in [9], the definitions of the generalized convexity of θ are connected to the generalized monotonicity of its gradient function $\nabla \theta$. With these new definitions we will establish the relationships between the generalized invexity of θ and the invex monotonicity of $\nabla \theta$.

For differentiable scalar functions $\theta : \Gamma \subseteq \mathbb{R}^n \to \mathbb{R}$ it is easily seen that we can establish the following relationships:

$$\begin{array}{ccccc}
\boxed{\text{(IX)}} & \Longleftrightarrow & \boxed{\text{(PIX)}} & \Rightarrow & \boxed{\text{(QIX)}} \\
\Uparrow & & \Uparrow & & \\
\boxed{\text{(SIX)}} & \Rightarrow & \boxed{\text{(SPIX)}} & & \\
\Uparrow & & \Uparrow & & \\
\boxed{\text{(SGIX)}} & \Rightarrow & \boxed{\text{(SGPIX)}} & &
\end{array}$$

3. New definitions of invex monotone functions

The concept of monotonicity was generated from the classic definition of monotonicity of a real function of a real variable $\psi : \mathbb{R} \to \mathbb{R}$.

Definition 3.1. Let us say that a function ψ is monotone if it is proved that

$$(y - x)^{\mathrm{t}} (\psi(y) - \psi(x)) \geqslant 0 \quad \forall x, y \in \mathbb{R}.$$

Karamardian [8] introduced another definition that extends the original concept of monotonicity. Let $C \subseteq \mathbb{R}^n$ be a convex open set.

Definition 3.2. $F : C \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is monotone (M) on C if $\forall x, y \in C$,

$$(y - x)^{\mathrm{t}}(F(y) - F(x)) \geqslant 0.$$

Karamardian [8] established a relationship between the generalized convexity of the function $\theta : C \subseteq \mathbb{R}^n \to \mathbb{R}$ and the concepts of monotonicity of its gradient function, $\nabla\theta : C \subseteq \mathbb{R}^n \to \mathbb{R}^n$, through the following result.

Theorem 3.1. *Let $\theta : C \subseteq \mathbb{R}^n \to \mathbb{R}$ be differentiable on an open convex set C. Then, θ is convex (CX) on $C \iff \forall x, y \in C,\ (y - x)^{\mathrm{t}}[\nabla\theta(y) - \nabla\theta(x)] \geqslant 0$.*

Then, the above theorem could be expressed thus:

Theorem 3.2. *A differentiable function θ on an open convex set $C \subseteq \mathbb{R}^n$ is convex $\iff$ $\nabla\theta$ is monotone on C.*

This theorem opens the door to the study of generalized monotonicity bound together with the study of generalized convexity. The concept of monotonicity for $\nabla\theta$, plays a role equivalent to that of the convexity of θ.

Karamardian and Schaible [9] also introduce the following definitions:

Definition 3.3. $F : C \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is said to be

(a) strictly monotone (SM) on C if $\forall x, y \in C$, $x \neq y$,

$$(y - x)^{\mathrm{t}}(F(y) - F(x)) > 0,$$

(b) strongly monotone (SGM) on C if $\forall x, y \in C$, $\exists \beta > 0$,

$$(y - x)^{\mathrm{t}}(F(y) - F(x)) \geqslant \beta\|y - x\|^2,$$

(c) pseudo monotone (PM) on C if $\forall x, y \in C$,

$$(y - x)^{\mathrm{t}}F(x) \geqslant 0 \;\Rightarrow\; (y - x)^{\mathrm{t}}F(y) \geqslant 0,$$

(d) strictly pseudo monotone (SPM) on C if $\forall x, y \in C$, $x \neq y$,

$$(y - x)^{\mathrm{t}}F(x) \geqslant 0 \;\Rightarrow\; (y - x)^{\mathrm{t}}F(y) > 0,$$

(e) strongly pseudo monotone (SGPM) on C if $\forall x, y \in C, \exists \beta > 0$,

$$(y - x)^{\mathrm{t}}F(x) \geqslant 0 \;\Rightarrow\; (y - x)^{\mathrm{t}}F(y) \geqslant \beta\|y - x\|^2,$$

(f) quasi monotone (QM) on C if $\forall x, y \in C$,

$$(y - x)^{\mathrm{t}}F(x) > 0 \;\Rightarrow\; (y - x)^{\mathrm{t}}F(y) \geqslant 0.$$

The following table of relations between the previous definitions of generalized monotonicity can be established:

$$\begin{array}{ccccc}
\boxed{(\mathrm{M})} & \Rightarrow & \boxed{(\mathrm{PM})} & \Rightarrow & \boxed{(\mathrm{QM})} \\
\Uparrow & & \Uparrow & & \\
\boxed{(\mathrm{SM})} & \Rightarrow & \boxed{(\mathrm{SPM})} & & \\
\Uparrow & & \Uparrow & & \\
\boxed{(\mathrm{SGM})} & \Rightarrow & \boxed{(\mathrm{SGPM})} & &
\end{array}$$

In the following result, given in [9], we can see how the generalized convexity of θ can characterize itself from the monotonicity of $\nabla\theta$, that is,

Theorem 3.3. *θ is convex (strictly convex, strongly convex, pseudo convex, strictly pseudo convex, quasi convex) on C, if and only if, $\nabla\theta$ is monotone (strictly monotone, strongly monotone, pseudo monotone, strictly pseudo monotone, quasi monotone) on C.*

In this section, following the line of the previous theorem, we characterize the invex functions through the conditions of generalized invex monotonicity of their gradient functions.

In [14] the concept of invex monotone function appeared, there called η-monotone. We shall extend the concept of invex monotone (IM) to others that are more general and we shall relate them to invex functions.

Definition 3.4. $F : X \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is said to be

(a) invex monotone (IM) on X if $\exists \eta : X \times X \to \mathbb{R}^n$ such that $\forall x, y \in X$,

$$\eta(y, x)^{\mathrm{t}}(F(y) - F(x)) \geqslant 0,$$

(b) pseudo invex monotone (PIM) on X if $\exists \eta : X \times X \to \mathbb{R}^n$ such that $\forall x, y \in X$,

$$\eta(y, x)^{\mathrm{t}}F(x) \geqslant 0 \;\Rightarrow\; \eta(y, x)^{\mathrm{t}}F(y) \geqslant 0,$$

(c) quasi invex monotone (QIM) on X if $\exists \eta : X \times X \to \mathbb{R}^n$ such that $\forall x, y \in X$,

$$\eta(y,x)^{t}F(x) > 0 \;\Rightarrow\; \eta(y,x)^{t}F(y) \geqslant 0,$$

(d) strictly invex monotone (SIM) on X if $\exists \eta : X \times X \to \mathbb{R}^n$ such that $\forall x, y \in X,\ x \neq y$,

$$\eta(y,x)^{t}(F(y) - F(x)) > 0,$$

(e) strictly pseudo invex monotone (SPIM) on X if $\exists \eta : X \times X \to \mathbb{R}^n$ such that $\forall x, y \in X,\ x \neq y$,

$$\eta(y,x)^{t}F(x) \geqslant 0 \;\Rightarrow\; \eta(y,x)^{t}F(y) > 0,$$

(f) strongly invex monotone (SGIM) on X if $\exists \eta : X \times X \to \mathbb{R}^n$ and $\beta > 0$ such that $\forall x, y \in X$,

$$\eta(y,x)^{t}(F(y) - F(x)) \geqslant \beta \|\eta(y,x)\|^2,$$

(g) strongly pseudo invex monotone (SGPIM) on X if $\exists \beta > 0$ and $\eta : X \times X \to \mathbb{R}^n$ such that $\forall x, y \in X$,

$$\eta(y,x)^{t}F(x) \geqslant 0 \;\Rightarrow\; \eta(y,x)^{t}F(y) \geqslant \beta \|\eta(y,x)\|^2.$$

Let us observe that a monotone function is a particular case of an invex monotone function when $\eta(y,x) = y - x\ \forall x, y$.

Now we shall see some examples of the earlier definitions:

Example 3.1. $F(x) = x^2$ is strictly invex monotone (SIM) over the set $X = \{x \in \mathbb{R},\ x \geqslant 0\}$, with respect to $\eta(y,x) = e^y - e^x$, since it is verified that, $\exists \eta : X \times X \to \mathbb{R}^n$, such that $\forall x, y \in X$, $x \neq y$,

$$\eta(y,x)^{t}(F(y) - F(x)) = (e^y - e^x)(y^2 - x^2) > 0.$$

Example 3.2.

$$F(x) = \begin{cases} 0 & \text{if } x \leqslant 0, \\ x & \text{if } x > 0, \end{cases}$$

is pseudo invex monotone (PIM), with respect to $\eta(y,x) = e^y - e^x$ over the set $X = \mathbb{R}$, since it is verified that $\exists \eta : X \times X \to \mathbb{R}^n$ such that $\forall x, y \in X$,

$$\eta(y,x)^{t}F(x) \geqslant 0 \;\Rightarrow\; \eta(y,x)^{t}F(y) \geqslant 0.$$

Example 3.3.

$$F(x) = \begin{cases} -x & \text{if } x \leqslant 0, \\ 0 & \text{if } x > 0, \end{cases}$$

is quasi invex monotone (QIM) with respect to $\eta(y,x) = e^y - e^x$ over the set $X = \mathbb{R}$, since it is verified that $\exists \eta : X \times X \to \mathbb{R}^n$ such that $\forall x, y \in X$,

$$\eta(y,x)^{t}F(x) > 0 \;\Rightarrow\; \eta(y,x)^{t}F(y) \geqslant 0.$$

In accordance with the earlier definitions, the following table of relations between the different concepts of generalized invex monotonicity is established:

(IM)	$\Rightarrow$	(PIM)	$\Rightarrow$	(QIM)
$\Uparrow$		$\Uparrow$		
(SIM)	$\Rightarrow$	(SPIM)		
$\Uparrow$		$\Uparrow$		
(SGIM)	$\Rightarrow$	(SGPIM)		

4. Relationships between the generalized invexity of θ and the generalized invex monotonicity of $\nabla\theta$

In this section, we extend the relationships between the generalized convexity of θ and the generalized monotonicity of $J = \nabla\theta$ defining a new type of generalized invex monotone function. We will connect the generalized invexity of θ to the generalized invex monotonicity of its gradient function $\nabla\theta$ through sufficient and necessary conditions. The following concept will be of importance in these proofs:

Definition 4.1. Let us say that the function $\eta : X \times X \to \mathbb{R}^n$ is a skew function if $\eta(x,y) + \eta(y,x) = 0\ \forall x, y \in X \subseteq \mathbb{R}^n$.

Let Γ be an open subset of $\mathbb{R}^n$.

Theorem 4.1. *If the function $\theta : \Gamma \subseteq \mathbb{R}^n \to \mathbb{R}$ is invex (IX) on Γ with respect to function $\eta : \Gamma \times \Gamma \to \mathbb{R}^n$ skew, then $\nabla\theta : \mathbb{R}^n \to \mathbb{R}^n$ is invex monotone (IM), with respect to the same η.*

Proof. Suppose that θ is invex on Γ, then $\exists \eta(x,y) \in \mathbb{R}^n$ such that $\forall x, y \in \mathbb{R}^n$,

$$\theta(x) - \theta(y) - \eta(x,y)^{t}\nabla\theta(y) \geqslant 0$$

by changing x for y,

$$\theta(y) - \theta(x) - \eta(y,x)^{t}\nabla\theta(x) \geqslant 0$$

because of the skewness of η, and adding these two inequalities, one has

$\eta(x,y)^t(\nabla\theta(x) - \nabla\theta(y)) \geqslant 0.$

Therefore $\nabla\theta$ is an invex monotone (IM) function with respect to the same η. □

In the following example we see the necessity of the hypothesis that η be skew.

Example 4.1. Let $\theta : [0, \pi/2) \to \mathbb{R}$, such that $\theta(x) = x + \operatorname{sen} x$. The function θ is invex (IX) on $[0, \pi/2)$, with respect to $\eta(y,x) = (\operatorname{sen} y - \operatorname{sen} x)/(1 + \cos x)$, because, $\forall x, y \in \mathbb{R}^n$, it is verified that

$$y + \operatorname{sen} y - x - \operatorname{sen} x - \left(\frac{\operatorname{sen} y - \operatorname{sen} x}{1 + \cos x}\right)(1 + \cos x) \geqslant 0.$$

As η is not skew, we can see that $\nabla\theta$ is not invex monotone (IM), since

$$\left(\frac{\operatorname{sen} y - \operatorname{sen} x}{1 + \cos x}\right)(1 + \cos y - 1 - \cos x) < 0.$$

In a similar way to Theorem 4.1 it can be proved that

Theorem 4.2. *If the function $\theta : \Gamma \subseteq \mathbb{R}^n \to \mathbb{R}$ is strictly invex (SIX) on Γ, with respect to η skew, then $\nabla\theta : \mathbb{R}^n \to \mathbb{R}^n$ is strictly invex monotone (SIM) with respect to the same η.*

The proof of this theorem is the same as that of Theorem 4.1 changing $\geqslant$ for $>$.

Theorem 4.3. *If the function $\theta : \Gamma \subseteq \mathbb{R}^n \to \mathbb{R}$ is strongly invex (SGIX) on Γ, with respect to η skew, then $\nabla\theta : \mathbb{R}^n \to \mathbb{R}^n$ is strongly invex monotone (SGIM) with respect to the same η.*

Proof. Suppose that θ is strongly invex (SGIX) on Γ, then there exists a vectorial function $\eta : \Gamma \times \Gamma \to \mathbb{R}^n$ and a scalar $\alpha > 0$, such that $\forall x, y \in \Gamma$,

$$\theta(y) - \theta(x) \geqslant \eta(y,x)^t\nabla\theta(x) + \alpha\|\eta(y,x)\|^2$$

by changing x for y, $\exists\beta > 0$, such that

$$\theta(x) - \theta(y) \geqslant \eta(x,y)^t\nabla\theta(y) + \beta\|\eta(x,y)\|^2$$

due to the skewness of η and adding these two inequalities one has

$$\eta(x,y)^t(\nabla\theta(x) - \nabla\theta(y)) \geqslant (\alpha + \beta)\|\eta(y,x)\|^2.$$

Therefore $\nabla\theta$ is strongly invex monotone (SGIM) with respect to the same η. □

As the invexity and pseudo invexity coincide for scalar functions, we can prove that

Corollary 4.1. *If the function $\theta : \Gamma \subseteq \mathbb{R}^n \to \mathbb{R}$ is invex (IX) with respect to η skew, then $\nabla\theta : \mathbb{R}^n \to \mathbb{R}^n$ is pseudo invex monotone (PIM) on Γ with respect to the same η.*

Proof. From Theorem 4.1 and as invex monotonicity (IM) implies pseudo invex monotonicity (PIM), the theorem is proved. □

Now, we will establish the relationship that exists between the strictly pseudo invexity of θ and the strictly PIX monotonicity of $\nabla\theta$.

Theorem 4.4. *If the function $\theta : \Gamma \subseteq \mathbb{R}^n \to \mathbb{R}$ is strictly pseudo invex (SPIX) on Γ, with respect to η skew, then $\nabla\theta : \mathbb{R}^n \to \mathbb{R}^n$ is strictly pseudo invex monotone (SPIM) on Γ with respect to the same η.*

Proof. Let $\theta(x)$ be strictly pseudo invex (SPIX). Then $\exists\eta(x,y) \in \mathbb{R}^n$ such that $\forall x, y \in \Gamma,\ x \neq y$,

$$\eta(x,y)^t\nabla\theta(y) \geqslant 0 \Rightarrow \theta(y) < \theta(x).$$

We want to show that $\exists\eta(x,y) \in \mathbb{R}^n$ such that $\forall x, y \in \Gamma,\ x \neq y$,

$$\eta(x,y)^t\nabla\theta(y) \geqslant 0 \Rightarrow \eta(x,y)^t\nabla\theta(x) > 0.$$

To reduce this to the absurd, suppose that

$$\eta(x,y)^t\nabla\theta(x) \leqslant 0,$$

as $\eta(y,x) + \eta(x,y) = 0$, then

$$\eta(y,x)^t\nabla\theta(x) \geqslant 0.$$

As θ is (SPIX), we would have $\theta(x) < \theta(y)$, which is a contradiction. □

We will reach quasi invex monotonicity starting from quasi invexity.

Theorem 4.5. *If the function $\theta : \Gamma \subseteq \mathbb{R}^n \to \mathbb{R}$ is quasi invex (QIX) on Γ with respect to η skew, then $\nabla\theta : \mathbb{R}^n \to \mathbb{R}^n$ is quasi invex monotone (QIM) on Γ with respect to the same η.*

Proof. We want to show that $\exists\eta(x,y)\in\mathbb{R}^n$ such that $\forall x,y\in\Gamma$,

$$\eta(x,y)^{\mathrm{t}}\nabla\theta(y)>0 \;\Rightarrow\; \eta(x,y)^{\mathrm{t}}\nabla\theta(x)\geqslant 0.$$

Let θ be QIX, then $\exists\eta(x,y)\in\mathbb{R}^n$ such that $\forall x,y\in\Gamma$,

$$\theta(x)\leqslant\theta(y) \;\Rightarrow\; \eta(x,y)^{\mathrm{t}}\nabla\theta(y)\leqslant 0.$$

Let $x,y\in\Gamma$, be such that $\eta(x,y)^{\mathrm{t}}\nabla\theta(y)>0$ then $\theta(x)>\theta(y)$.

As θ is (QIX) one has

$$\eta(y,x)^{\mathrm{t}}\nabla\theta(x)\leqslant 0.$$

Due to the skewness of η then $\eta(x,y)^{\mathrm{t}}\nabla\theta(x)\geqslant 0$. □

Just as sufficient conditions are important, so are necessary conditions. In principle, necessary conditions are not generally true, so we must establish certain premises to be able to fix those conditions.

Theorem 4.6. *Let C be an open convex subset of* $\mathbb{R}^n$. *Suppose that*:
1. $\nabla\theta:\mathbb{R}^n\to\mathbb{R}^n$ *is strictly pseudo invex monotone* (*SPIM*) *with respect to* $\eta(y,x)>0\quad\forall x,y\in C$,
2. η *is linear function in the first argument and skew*,

then, $\theta:C\subseteq\mathbb{R}^n\to\mathbb{R}$ *is SPIX on C, with respect to* η.

Proof. As $\nabla\theta:C\subseteq\mathbb{R}^n\to\mathbb{R}^n$ is strictly pseudo invex monotone (SPIM) with respect to η on C, $\exists\eta:C\times C\to\mathbb{R}^n$ such that $\forall x,y\in C$,

$$\eta(y,x)^{\mathrm{t}}\nabla\theta(x)\geqslant 0 \;\Rightarrow\; \eta(y,x)^{\mathrm{t}}\nabla\theta(y)>0.$$

Let $x,y\in C$ we define $x(\lambda)=x+\lambda(y-x)\in C$, for $0<\lambda<1$.

If

$$\eta(x(\lambda),x)^{\mathrm{t}}\nabla\theta(x)\geqslant 0,$$

it is implied that

$$\eta(x(\lambda),x)^{\mathrm{t}}\nabla\theta(x(\lambda))>0.$$

As η is skew and η being linear in the first argument

$$\eta(x(\lambda),x)^{\mathrm{t}}\nabla\theta(x(\lambda))=(1-\lambda)\eta(x,x)^{\mathrm{t}}\nabla\theta(x(\lambda))+\lambda\eta(y,x)^{\mathrm{t}}\nabla\theta(x(\lambda)),$$

then

$$\eta(x(\lambda),x)^{\mathrm{t}}\nabla\theta(x(\lambda))=\lambda\eta(y,x)^{\mathrm{t}}\nabla\theta(x(\lambda))>0,$$

then

$$\nabla\theta(x+\lambda(y-x))>0.$$

So if we integrate the earlier expression between 0 and 1, we would have

$$\int_0^1\nabla\theta(x+\lambda(y-x))\,\mathrm{d}\lambda>0,$$

then

$$\theta(y)-\theta(x)>0.$$

So θ is (SPIX) with respect to η on C. □

In the same way it can be proved that

Theorem 4.7. *Let C be an open convex subset of* $\mathbb{R}^n$. *Suppose that*:
1. $\nabla\theta:\mathbb{R}^n\to\mathbb{R}^n$ *is pseudo invex monotone* (*PIM*) *with respect to* $\eta(y,x)>0\;\forall x,y\in C$,
2. η *is linear in the first argument and skew*,

then, $\theta:C\subseteq\mathbb{R}^n\to\mathbb{R}$ *is PIX or invex* (*PIX*) $=$ (*IX*), *on C with respect to* η.

Proof. Suppose that $\nabla\theta:C\subseteq\mathbb{R}^n\to\mathbb{R}^n$ is PIM with respect to η on C, that is, $\exists\eta:C\times C\to\mathbb{R}^n$ skew, such that $\forall x,y\in C$,

$$\eta(y,x)^{\mathrm{t}}\nabla\theta(x)\geqslant 0 \;\Rightarrow\; \eta(y,x)^{\mathrm{t}}\nabla\theta(y)\geqslant 0.$$

As C is a convex set then $x(\lambda)=x+\lambda(y-x)\in C\;\forall\lambda\in[0,1]$.

If

$$\eta(x(\lambda),x)^{\mathrm{t}}\nabla\theta(x)\geqslant 0,$$

as $\nabla\theta$ is (PIM) implies that

$$\eta(x(\lambda),x)^{\mathrm{t}}\nabla\theta(x(\lambda))\geqslant 0\quad\forall\lambda\in[0,1].$$

As η is skew and linear in the first argument

$$\eta(x(\lambda),x)^{\mathrm{t}}\nabla\theta(x(\lambda))=(1-\lambda)\eta(x,x)^{\mathrm{t}}\nabla\theta(x(\lambda))+\lambda\eta(y,x)^{\mathrm{t}}\nabla\theta(x(\lambda)),$$

then

$$\eta(x(\lambda),x)^{\mathrm{t}}\nabla\theta(x(\lambda))=\lambda\eta(y,x)^{\mathrm{t}}\nabla\theta(x(\lambda))\geqslant 0.$$

As $\lambda\in[0,1]$ and $\eta(y,x)>0\;\forall x,y\in C$ then

$$\nabla\theta(x+\lambda(y-x)) \geqslant 0 \quad \forall\lambda \in [0,1].$$

If we integrate the last expression between 0 and 1, we have

$$\int_0^1 \nabla\theta(x+\lambda(y-x))\,\mathrm{d}\lambda \geqslant 0.$$

So

$$\theta(y)-\theta(x) \geqslant 0,$$

θ is (PIX) = (IX) with respect to η on C. □

As the invex monotonicity (IM) implies the pseudo invex monotonicity (PIM), we have:

Corollary 4.2. *Let C be an open convex subset of $\mathbb{R}^n$. Suppose that*:
1. *$\nabla\theta : \mathbb{R}^n \to \mathbb{R}^n$ is invex monotone (IM) with respect to $\eta(y,x) > 0$ $\forall x,y \in C$,*
2. *η is linear in the first argument and skew,*

then $\theta : C \subseteq \mathbb{R}^n \to \mathbb{R}$ is invex (IX) on C with respect to η.

5. Application to the resolution of the Variational-Like Inequality Problem and the Mathematical Programming Problem

In this section, we will characterize the solutions to the Variational-Like Inequality Problem (VLIP) through the generalized invex monotonicity of the functions that define the problem.

Subsequently, our objective will be to identify the solutions to the Mathematical Programming Problems (MP), with those of the Variational-Like Inequality Problem (VLIP).

In [13], the existence of solutions to the Variational-Like Inequality Problem (VLIP) is proved, based on the continuity of the function $F : M \to \mathbb{R}^n$, where M is a convex and compact subset of $\mathbb{R}^n$. The condition that F be continuous is excessive. So, we can reduce it to F being hemicontinuous, that is, continuous on any linear segment of M.

Definition 5.1. F is called hemicontinuous if $\forall u,v \in D$ the function $t \to v^{\mathrm{t}}F(u+tv)$ is continuous at 0^+, with $0 \leqslant t \leqslant 1$.

In [19], the existence of solutions to the Variational-Like Inequality Problem (VLIP), are studied, assuming F to be invex monotone (IM).

In this section, we will prove that it is possible to establish the existence of solutions to a Variational-Like Inequality Problem (VLIP), under weaker hypotheses than those assumed in other work. In order to do this we need some prior definitions and lemmas.

In the first place, let us remember the concept of KKM-function:

Definition 5.2. A function $V : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is called KKM-function, if for every finite subset $\{u_1, u_2, \dots, u_n\}$ of $\mathbb{R}^n$, their convex hull

$$\mathrm{conv}(\{u_1,u_2,\dots,u_n\}) \subset \bigcup_{i=1}^{n} V(u_i).$$

Lemma 5.1 [4]. *Let a nonempty subset $\mathscr{A} \subset \mathbb{R}^n$ and a KKM-function $V : \mathscr{A} \to 2^{\mathbb{R}^n}$. If $V(u)$ is a compact set $\forall u \in \mathscr{A}$, then*

$$\bigcap_{u\in\mathscr{A}} V(u) \neq \emptyset.$$

The following lemma will subsequently be used to prove the existence of solutions to a (VLIP) problem.

Lemma 5.2. *Let C be a nonempty convex set in $\mathbb{R}^n$. Suppose that*:
1. *$F : C \to \mathbb{R}^n$ is pseudo invex monotone (PIM) with respect to η and a hemicontinuous function on C,*
2. *$\eta : C \times C \to \mathbb{R}^n$ is a skew function,*
3. *η is linear in the first argument.*

Then $u \in C$ satisfies

$$\eta(v,u)^{\mathrm{t}}F(u) \geqslant 0 \quad \forall v \in C \tag{1}$$

if and only if it satisfies

$$\eta(v,u)^{\mathrm{t}}F(v) \geqslant 0 \quad \forall v \in C. \tag{2}$$

Proof. ($\Rightarrow$) Let $u \in C$ be a solution of (1). Since F is (PIM) with respect to η, for every $v \in C$, we have

$$\eta(v,u)^{\mathrm{t}}F(u) \geqslant 0 \;\Rightarrow\; \eta(v,u)^{\mathrm{t}}F(v) \geqslant 0.$$

($\Leftarrow$) Let $v,u \in C$ and we consider $w = tv + (1-t)u \in C$, with $0 < t < 1$. Hence by (2),

$$\eta(tv + (1-t)u, u)^{\mathrm{t}} F(u + t(v-u)) \geqslant 0.$$

Since η is linear in the first argument and η is skew, that is, $\eta(u,u)^{\mathrm{t}}F(u) = 0,\ \forall u \in C$, we have

$$t\eta(v,u)^{\mathrm{t}} F(u + t(v-u)) \geqslant 0.$$

Dividing by t,

$$\eta(v,u)^{\mathrm{t}} F(u + t(v-u)) \geqslant 0.$$

Since F is a hemicontinuous function on C, we may allow $t \to 0^+$, we obtain

$$\eta(v,u)^{\mathrm{t}} F(u) \geqslant 0 \quad \forall v \in C. \qquad \square$$

Starting from the hypotheses of the pseudo invex monotonicity (PIM) of F and the linearity of η, we will prove the following theorem of existence:

Theorem 5.1. *Let M be a nonempty, compact and convex set of $\mathbb{R}^n$, such that*
1. *$F : M \to \mathbb{R}^n$ is PIM with respect to η and a hemicontinuous function on M,*
2. *$\eta : M \times M \to \mathbb{R}^n$ is a continuous and skew function,*
3. *η linear in the first argument.*

Then there exists $u_0 \in M$, such that

$$\eta(v, u_0)^{\mathrm{t}} F(u_0) \geqslant 0 \quad \forall v \in M.$$

Proof. Let the point-to-set function $V_1 : M \to 2^M$, such that

$$V_1(v) = \{u \in M : \eta(v,u)^{\mathrm{t}} F(u) \geqslant 0\} \quad \forall v \in M.$$

First, we prove that V_1 is a KKM-function.

Suppose that $\{v_1, v_2, \ldots, v_n\} \subset M$, $\sum_{i=1}^n \alpha_i = 1$, $\alpha_i \geqslant 0,\ i = 1, \ldots, n$, and

$$v = \sum_{i=1}^n \alpha_i v_i \notin \bigcup_{i=1}^n V_1(v_i).$$

Then we have

$$\eta(v_i, v)^{\mathrm{t}} F(v) < 0 \ \Rightarrow \ \sum_{i=1}^n \alpha_i \eta(v_i, v)^{\mathrm{t}} F(v) < 0$$

$$\forall i = 1, \ldots, n.$$

Since η is linear in the first argument,

$$\eta\left(\sum_{i=1}^n \alpha_i v_i, v\right)^{\mathrm{t}} F(v) = \sum_{i=1}^n \alpha_i \eta(v_i, v)^{\mathrm{t}} F(v) < 0,$$

then

$$\eta\left(\sum_{i=1}^n \alpha_i v_i, \sum_{i=1}^n \alpha_i v_i\right)^{\mathrm{t}} F\left(\sum_{i=1}^n \alpha_i v_i\right) < 0,$$

which is a contradiction of the assumption of skewness which demands that

$$\eta(v,v)^{\mathrm{t}} F(v) = 0 \quad \forall v \in M.$$

So we derived

$$\mathrm{conv}\,(\{v_1, v_2, \ldots, v_n\}) \subset \bigcup_{i=1}^n V_1(v_i)$$

and therefore, V_1 is a KKM-function.

Let the point-to-set function $V_2 : M \to 2^M$, such that

$$V_2(v) = \{u \in M : \eta(v,u)^{\mathrm{t}} F(v) \geqslant 0\} \quad \forall v \in M.$$

Now we show that $V_1(v) \subset V_2(v)\ \forall v \in M$.

Let $u \in V_1(v)$, that is, $\eta(v,u)^{\mathrm{t}} F(u) \geqslant 0$, by the (PIM) we have

$$\eta(v,u)^{\mathrm{t}} F(v) \geqslant 0,$$

that is, $u \in V_2(v)$.

As $V_1 \subset V_2$ and V_1 is a KKM-function then V_2 is a KKM-function.

By Lemma 5.2,

$$\bigcap_{v \in M} V_1(v) = \bigcap_{v \in M} V_2(v).$$

Moreover, $V_2(v)$, for every $v \in M$, is closed, since F and η are continuous.

As $V_2(v)$ is closed for every $v \in M$, and M is a bounded set, then $V_2(v)$ is bounded, hence $V_2(v)$ is compact. By Lemma 5.1

$$\bigcap_{v \in M} V_1(v) = \bigcap_{v \in M} V_2(v) \neq \emptyset.$$

Hence there exists an $u_0 \in M$ such that

$$\eta(v, u_0)^{\mathrm{t}} F(u_0) \geqslant 0 \quad \forall v \in M. \qquad \square$$

So then, we have been able to prove the existence of solutions to the Variational-Like Inequality Problem (VLIP), assuming the PIM of F, a weaker hypothesis than IM.

We have seen that PIM, assures us of the existence of a solution to a Variational-Like Inequality Problem (VLIP), but not the uniqueness of such a solution. To achieve this we assume SPIM.

Corollary 5.1. *Let M be a nonempty, compact and convex subset of* $\mathbb{R}^n$, *such that*

1. $F : M \to \mathbb{R}^n$ *is strictly pseudo invex monotone (SPIM) with respect to* η *and a hemicontinuous function on* M,
2. $\eta : M \times M \to \mathbb{R}^n$ *is a continuous and skew function,*
3. η *linear in the first argument.*

Then there exists unique $u_0 \in M$, *such that*

$$\eta(v, u_0)^t F(u_0) \geqslant 0 \quad \forall v \in M.$$

Proof. As (SPIM) $\Rightarrow$ (PIM) and by Theorem 5.1 we have guaranteed the existence of a solution to (VLIP). Now we prove the uniqueness.

Suppose that (VLIP) has two distinct solutions, say u_0 and u_1. Then

$$\eta(u_1, u_0)^t F(u_0) \geqslant 0, \tag{3}$$

and

$$\eta(u_0, u_1)^t F(u_1) \geqslant 0 \quad \forall u_0 \in M. \tag{4}$$

Since F is (SPIM), it follows from (3) that $\eta(u_1, u_0)^t F(u_1) > 0\ \forall u_0 \in M$. Due to the skewness of η then $\eta(u_0, u_1)^t F(u_1) < 0$, which contradicts (4). □

Next we will relate the study of the Variational-Like Inequality Problem (VLIP) to the Mathematical Programming Problem (MP), using invex sets and functions.

Definition 5.3. Let $u \in H$. Then, the set H is said to be invex at u with respect to η, if, for each $v \in H, 0 \leqslant t \leqslant 1, u + t\eta(v, u) \in H$. H is said to be an invex set with respect to η, if H is invex at each $u \in H$.

Let us consider the Mathematical Programming Problem (MP), where the set $X = H$ is an invex set. The following theorem proves that every solution to a Variational-Like Inequality Problem (VLIP) is a solution to the associated Mathematical Programming Problem (MP).

Theorem 5.2. *Let* $\theta : H \to \mathbb{R}$ *be an invex function with respect to* η, *where H be an invex set. The element* $u \in H$ *satisfies the inequality*

$$\eta(v, u)^t \nabla\theta(u) \geqslant 0 \quad \forall v \in H, \tag{5}$$

if and only if, $u \in H$ *is the minimum of the (MP) problem.*

Proof. ($\Rightarrow$) As θ is an invex function it implies that

$$\theta(v) - \theta(u) \geqslant \eta(v, u)^t \nabla\theta(u) \geqslant 0\ \forall v \in H$$

showing that $u \in H$ is the minimum of the function θ.

($\Leftarrow$) Let $u \in H$ be a minimum of the function θ. Then, for every $v \in H,\ \alpha \in (0, 1], u + \alpha\eta(v, u) \in H$,

$$\theta(u + \alpha\eta(v, u)) - \theta(u) \geqslant 0 \quad \forall v \in H.$$

Since θ is invex at $u \in H$, dividing the above inequality by α and letting $\alpha \to 0^+$, we have

$$\eta(v, u)^t \nabla\theta(u) \geqslant 0.$$

So $u \in H$ is a solution of (VLIP). □

Therefore, u is one solution to a (VLIP) problem, if and only if, u is the minimum of the Mathematical Programming Problem (MP), when θ is invex. Consequently, in invex environments, the solutions to the VLIP are equivalent to the minima of (MP).

In the following theorems, we will use the invex monotonicity to characterize solutions to the Variational-Like Inequality Problem (VLIP) and, in so doing, determine the minima of the Mathematical Programming Problem (MP).

Theorem 5.3. *Let M be a nonempty, compact and convex set of* $\mathbb{R}^n$, *such that*

1. $F : M \to \mathbb{R}^n$ *is PIM with respect to* η *and a hemicontinuous function on* M,
2. $\eta : M \times M \to \mathbb{R}^n$ *is a continuous and skew function,*
3. η *linear in the first argument.*

So then, every solution $u_0 \in \text{int}(M)$ *to the Variational-Like Inequality Problem (VLIP), will also be a solution to the associated Mathematical Programming Problem (MP).*

Proof. As F is PIM, by Theorem 5.1, $u_0 \in M$ is a solution to the (VLIP) problem. M is a convex set, so the interior of the same, $\text{int}(M)$, is convex and open. $\nabla\theta$ is (PIM) and η is linear in the first argument and skew, so by the necessary (4.7), θ is (PIX) in $\text{int}(M)$ and therefore invex (PIX) = (IX). Using Theorem 5.2, $u_0 \in \text{int}(M)$ is a solution to the (MP). □

In the above theorem, we have proved that we can reach the solutions to a MP, through those of a Variational-Like Inequality Problem, using the PIM of the function F.

Just as the (PIM) assures us of the existence of a solution, in the next theorem it is the strictly pseudo invex monotonicity (SPIM), that assures us of the uniqueness of said solution.

Theorem 5.4. *Let M be a nonempty, compact and convex set of $\mathbb{R}^n$, such that*
1. *$F : M \to \mathbb{R}^n$ is SPIM with respect to η and a hemicontinuous function on M,*
2. *$\eta : M \times M \to \mathbb{R}^n$ is a continuous and skew function,*
3. *η linear in the first argument.*

So then, if the only solution u_0 of the Variational-Like Inequality Problem (VLIP) belongs to $\text{int}(M)$, *it will also be the only solution to the Mathematical Programming Problem (MP).*

Proof. As F is strictly pseudo invex monotone (SPIM), by Corollary 5.1, $u_0 \in M$ is the only solution to the (VLIP) problem. M is a convex set, so the interior of the same, $\text{int}(M)$, is convex and open. $\nabla\theta$ is (SPIM) and η is linear in the first argument and skew, by the necessary condition (4.6), θ is (SPIX) $\Rightarrow$ (PIX) = (IX) in $\text{int}(M)$. If we suppose that $u_0 \in \text{int}(M)$, then by Theorem 5.2, u_0 is the only solution to the (MP) problem. □

6. Conclusions

In this paper we have defined the concepts of strongly invex (SGIX) and strongly pseudo invex (SGPIX) functions, as new types of generalized invex functions. The concept of generalized monotonicity has been extended to the new ones of generalized invex monotonicity. Thus, we have generalized the concept of invex monotone function and defined the functions PIM, QIM, and so on. We have managed to prove that there is a relationship between the generalized invexity of the function θ and the generalized invex monotonicity of the function $\nabla\theta$, by way of the necessary and sufficient conditions. It has been shown that in an environment of invexity we have identified the solutions to the Variational-Like Inequality Problem (VLIP) and the Mathematical Programming Problem (MP). Thanks to pseudo invex monotonicity (PIM) we have been able to prove the existence of solutions to both problems.

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