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Geometrical Description of the Weakly Efficient Solution Set for Multicriteria Location Problems

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Abstract. Multicriteria location problems have attracted much attention in the last years within the field of Location Analysis. The central task in their analysis relies on the description of the whole set of non-dominated solutions with respect to the different criteria. Solutions to this problem in particular situations are known. In this paper we characterize the solution set of the general convex multicriteria location problem in two dimensional spaces. These tools allows us to describe in the same way the solution set of several classical location models as well as many other new problems for which no previous solution was known.

1. Introduction

One of the problems which has not been satisfactorily solved to date is how to deal with uncertainty. A large number of problems that we find in the real-world present elements which escape the control of the decision maker. Often, the parameters used to describe a problem are obtained from estimations, from measurements that can be wrong, or from observations that do not fit to the real situations. At times, the problem involves more than one decision maker and each one of them may have a different criterion. In other situations, it is impossible to choose only one scenario where the problem can be formulated, or it is not clear which is the best criterion to optimize, etc.

Recently, a new field of Operations Research called Scenario Analysis has emerged. This analysis allows us to deal with uncertainty when it is originated, because there are different instances of the problem which are likely to occur. Therefore, there exists the necessity of finding a good solution for different criteria (scenarios) rather than for only one, which leads us to multicriteria problems. Under this perspective, the solution concept is the set of non-dominated (efficient) solutions.

The development of Multiobjective Programming makes possible the analysis of these problems, mainly by trying to build the non-dominated solution set of a vector optimization problem. The Scenario Analysis has been already applied to a wide range of models in Operations Research, for instance: regression analysis [1], games [10], robust solutions [14], inventory problems [20].

In location analysis the complete characterization of the non-dominated solution set in multicriteria problems has been addressed repeatedly, although it is only partially solved. The problem can be written, in great generality, as

$$\text{v-min}\big(F_1(x), F_2(x), \dots, F_k(x)\big),$$

where v-min stands for vector minimization and $F_1(\cdot), \dots, F_k(\cdot)$ represent the different criteria.

The problem where each function $F_i(\cdot)$ is the distance to a fixed point a_i, measured with the same norm, is called Point-Objective location problem and was dealt firstly by Wendell and Hurter [26] for the l_2-norm. Later, Durier [7] solved this problem for polyhedral gauges. Other references devoted to study modifications of the point-objective location models are [2,3,5,9,12,16,18,19,27], among others.

The case where the $F_i(\cdot)$ for $i = 1, \dots, k$ are weighted sums of the distances measured with the l_1-norm (Weber problem with the Manhattan norm) was solved by Hamacher and Nickel [11]. The problem where the functions $F_i(\cdot)$, for $i = 1, \dots, k$, are weighted sum of distances measured with any norm was solved by Puerto and Fernández [21,22]. In addition, the quadratic bicriteria location model has been solved by Ohsawa [17].

Finally, the case where there exists a regional demand and the functions $F_i(\cdot)$ for $i = 1, \dots, k$ are inf-distance functions (that is, the distance to the closest point of each demand set) was solved by Rodríguez-Chía [24]. Hence, although the problem has been studied many times and by different authors, the solutions are only known for particular cases. In our paper, we give the first geometrical solution for this kind of problems, where the considered objective functions are only required to be convex and inf-compact. Despite of the generality of the considered problem, the results obtained in this paper are easy to understand and the proofs basically rely on Convex Analysis tools. We will provide examples that illustrate the results, and we will relate these results with the existing ones, showing that they are particular cases of our analysis.

One of the applications of the solution concept proposed in this paper is that it can be interpreted as a global sensitivity analysis of location problems. Assume that $F_i(\cdot)$, $i = 1, \dots, k$, is the distance function to the ith demand entity. A decision-maker wants to minimize the total weighted distance to the new facility to be located. However, the importance (weighted) given to each demand entity is unknown. Thus, the problem to be considered is

$$\min\left\{\sum_{i=1}^{k} w_i F_i(x)\colon\ x \in \mathbb{R}^2,\ w_i \geqslant 0,\ i = 1, \dots, k\right\}. \tag{1}$$

The set of optimal locations of problem (1) for any choice of weights $(w_1, \dots, w_k)$ coincides with the set of weak-Pareto solutions of

$$\operatorname*{v-min}_{x\in\mathbb{R}^2}\big(F_1(x), \dots, F_k(x)\big). \tag{2}$$

This solution set is the one that we characterize geometrically in this paper. Thus, our solution gives the region of variation of the solutions of any optimization problem of the form given by (1). Notice that any partial information known about the weights $(w_1, \dots, w_k)$ can be incorporated into problem (1). This simply means that in problem (2) we must consider transformations of the original objective functions. (See [4] and [15] for further details.)

The paper is organized in four sections. In the second section we present the model and the notation used throughout. Section 3 contains the main results of the paper. It reduces the characterization of the non-dominated solution set of the general k-criteria problem to the three-criteria case for which a geometrical construction is given. For the sake of readability, the proofs of several technical lemmas are deferred to the appendix. Finally, section 4 is devoted to the concluding remark.

2. The model

We consider a finite set of convex, inf-compact functions $F_1(\cdot), \dots, F_k(\cdot)$, defined on a bidimensional space X, which represent different criteria or scenarios. Recall that a real function $f(\cdot)$ is said to be inf-compact if its lower level sets $\{x \in X\colon f(x) \leqslant \rho\}$ are compact for any $\rho \in \mathbb{R}$. Our goal is to find the set of points $x \in X$ such that there is no $y \in X$ that improves the value of $F_i(x)$ for all $i = 1, \dots, k$. Therefore, the formulation of the problem is:

$$\operatorname*{v-min}_{x \in X} \big(F_1(x), \dots, F_k(x)\big). \tag{3}$$

We consider as solution set for this problem

$$\begin{aligned} \mathrm{WE}(F_1, \dots, F_k) := \big\{ x \in X\colon \text{ for each } y \in X, y \neq x, \text{ exists } F_i\, (i = 1, \dots, k) \\ \text{such that } F_i(x) \leqslant F_i(y) \big\}. \end{aligned}$$

Usually, this set is called set of weakly efficient points.

In order to improve the readability of the paper we use the following notation. The level set of the function $F(\cdot)$ for a value $\rho \in \mathbb{R}$ is given by

$$L_{\leqslant}(F, \rho) := \big\{ x \in X\colon F(x) \leqslant \rho \big\}$$

and the strict level set is

$$L_{<}(F, \rho) := \big\{ x \in X\colon F(x) < \rho \big\}.$$

It should be noted that these two families of sets are convex when the function $F(\cdot)$ is convex.

In the same way, we define the complement of the strict level set for a value $\rho \in \mathbb{R}$ as

$$L_{\geqslant}(F, \rho) := \big\{ x \in X\colon F(x) \geqslant \rho \big\},$$

and the level curve for a value $\rho \in \mathbb{R}$ as

$$L_{=}(F, \rho) := \{x \in X:\ F(x) = \rho\}.$$

For a convex inf-compact function $F_i(\cdot)$ we will use the notation

$$\mathcal{X}^*(F_i) := \operatorname*{argmin}_{x \in X} F_i(x). \tag{4}$$

It should be noted that this set is convex by the convexity of $F_i(\cdot)$.

The tangent cone $T_B(x)$ to the set B at point x is:

$$T_B(x) := \overline{\text{cone}}(B - x),$$

where for any set S, $\overline{S}$ stands for the topological closure of S.

For two functions, $F_i(\cdot)$ and $F_j(\cdot)$, with $i, j \in \{1, \dots, k\}$, let

$$\begin{aligned} I_{ij}^{\leqslant}(x) &:= L_{\leqslant}(F_i, F_i(x)) \cap L_{\leqslant}(F_j, F_j(x)), \\ I_{ij}^{<}(x) &:= L_{<}(F_i, F_i(x)) \cap L_{<}(F_j, F_j(x)), \\ I_{ij}^{=}(x) &:= L_{=}(F_i, F_i(x)) \cap L_{=}(F_j, F_j(x)). \end{aligned}$$

For a general closed set $A \subset X$ we denote by $\text{Bd}(A)$ the boundary of this set, $\text{ri}(A)$ its relative interior and $\text{rBd}(A)$ the relative boundary of A. Notice that in the plane the relative interior (relative boundary) is the entire interior or the interior (interior of the boundary) with respect a line. Finally, for two points x and y we denote the segment defined by x and y as $\overline{xy}$.

3. Geometrical construction of the solution set

In this section we study problem (3). We will prove that the geometrical structure of $\text{WE}(F_1, F_2, F_3)$ is given by a kind of hull delimited by the chains of bicriteria solutions of any pair of functions F_i, F_j, $i, j = 1, 2, 3$. This result enables us to obtain the set $\text{WE}(F_1, \dots, F_k)$ by union of three-criteria solution sets already characterized. In order to do that, our first result states a useful characterization of the solution set $\text{WE}(F_1, \dots, F_k)$, which will be used later.

Theorem 3.1. It holds that:

$$x \in \text{WE}(F_1, \dots, F_k) \quad \Longleftrightarrow \quad \bigcap_{i=1}^{k} L_{<}(F_i, F_i(x)) = \emptyset.$$

Proof. If $x \notin \text{WE}(F_1, \dots, F_k)$, there exists $z \in X$ such that $F_i(z) < F_i(x)$ for each $i = 1, \dots, k$, that means,

$$z \in \bigcap_{i=1}^{k} L_{<}(F_i, F_i(x)).$$

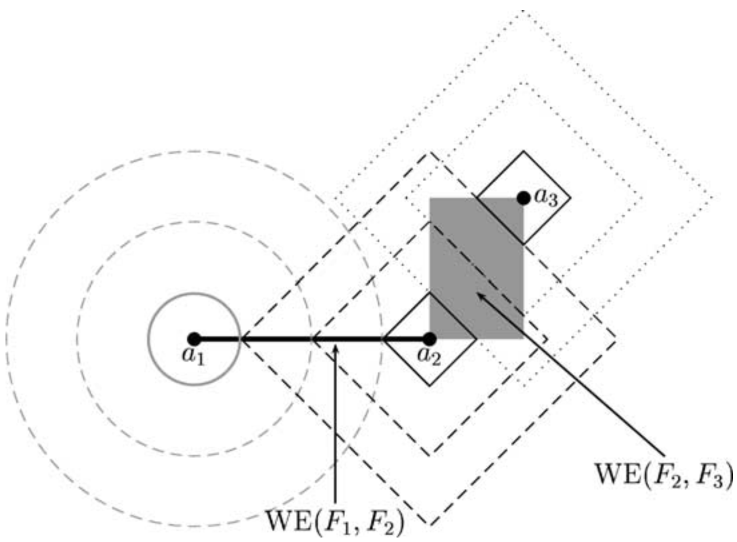

Figure 1. Illustration of example 3.1.

Hence, we obtain that

$$\bigcap_{i=1}^{k} L_<\big(F_i, F_i(x)\big) \neq \emptyset.$$

Since the implications above can be reversed the proof is concluded. □

Remark 3.1. For the case $k = 2$ the previous result states that the set WE(F_1, F_2) coincides with tangential cusps between the level curves of functions $F_1(\cdot)$ and $F_2(\cdot)$ union with $\mathcal{X}^*(F_1) \cup \mathcal{X}^*(F_2)$ (see example 3.1).

Example 3.1 (See figure 1). Let us consider the points $a_1 = (0, 0)$, $a_2 = (10, 0)$, $a_3 = (14, 6)$ and the functions $F_1(x) = \|x - a_1\|_2$, $F_2(x) = \|x - a_2\|_1$, $F_3(x) = \|x - a_3\|_1$. By theorem 3.1, WE(F_1, F_2) is the thick segment defined by a_1 and a_2 and WE(F_2, F_3) is the dark rectangle with a_2 and a_3 as opposite vertices.

Now, using the previous characterization we are going to obtain a geometrical description of WE(F_1, F_2, F_3). To this end, three technical lemmas are needed. Their proofs can be seen in the appendix.

Lemma 3.1. Whenever the statements

(a) $\bigcap_{i=1}^{3} L_<(F_i, F_i(x)) = \emptyset$,

(b) $I_{ij}^<(x) \neq \emptyset, \forall i \neq j \in \{1, 2, 3\}$,

hold for some $x \in X$, then:

(i) $x + \bigcap_{i=1}^{3} T_{L_{\leqslant}(F_i, F_i(x))}(x) = \{x\}$,

(ii) $\mathrm{WE}(F_i, F_j) \cap (x - (T_{L_{\leqslant}(F_i, F_i(x))}(x) \cap T_{L_{\leqslant}(F_j, F_j(x))}(x))) = \emptyset, \ \forall i \neq j \in \{1, 2, 3\}$.

Lemma 3.2. Whenever

$$\bigcap_{i=1}^{3} L_{<}\big(F_i, F_i(x)\big) \neq \emptyset \quad \text{for some } x \in X \tag{5}$$

then

$$\text{(i)} \quad \{0\} \notin \mathrm{ri}\left(\bigcap_{i=1}^{3} T_{L_{\leqslant}(F_i, F_i(x))}(x)\right),$$

$$\text{(ii)} \quad \mathrm{ri}\left(\bigcap_{i=1}^{3} T_{L_{\leqslant}(F_i, F_i(x))}(x)\right) \neq \emptyset.$$

Lemma 3.3. If $I_{ij}^{<}(x) \neq \emptyset$ for some $x \in X$ then

$$I_{ij}^{<}(x) \cap \mathrm{WE}(F_i, F_j) \neq \emptyset.$$

Corollary 3.1. If $I_{ij}^{<}(x) \neq \emptyset$ for some $x \in X$ then

$$\mathrm{ri}\big(x + T_{L_{\leqslant}(F_i, F_i(x))}(x)\big) \cap \mathrm{ri}\big(x + T_{L_{\leqslant}(F_j, F_j(x))}(x)\big) \cap \mathrm{WE}(F_i, F_j) \neq \emptyset.$$

The next result shows that the 3-criteria solution set is a kind of hull defined by the different bicriteria solution sets.

Definition 3.1 (See figure 2). The curve $z(t)$, $t \in [0, \infty)$, with $z(0) = x$ and $\lim_{t\to\infty} \|z(t)\| = +\infty$ separates the sets A and B, with respect to a convex cone Γ pointed at x, if

(a) $A, B \subset \Gamma$,

(b) there does not exist a continuous curve $y(t) \subset \Gamma$, $\forall t \in [0, 1]$ with $y(0) \in A$, $y(1) \in B$ such that $\{z(t)\colon t \in (0, +\infty)\} \cap \{y(t)\colon t \in [0, 1]\} = \emptyset$.

Remark 3.2. It should be noted that $\|\cdot\|$, used in the definition above, stands for any norm in X. Since, all the norms are equivalent in X we can assume without loss of generality that it is the l_2-norm.

Let

$$\mathrm{WE}(2) := \bigcup_{\substack{i,j \in \{1,2,3\} \\ i \neq j}} \mathrm{WE}(F_i, F_j)$$

be the union of all bicriteria chains for the three considered criteria.

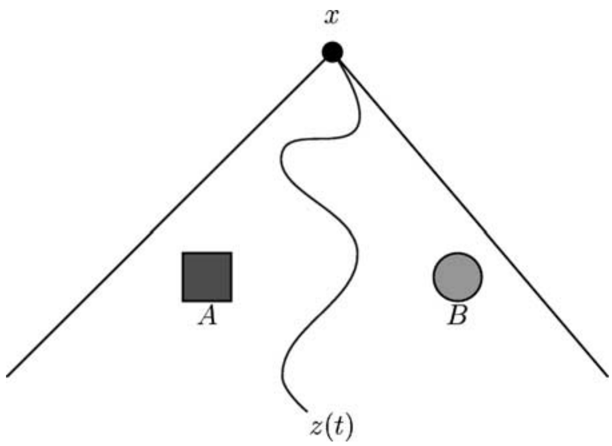

Figure 2. $z(t)$ separates the sets A and B with respect to the pointed cone at x.

Theorem 3.2.

$$\mathrm{WE}(F_1, F_2, F_3) = \mathrm{encl}\big(\mathrm{WE}(2)\big)$$

where encl(WE(2)) is the bounded region encircled by WE(2) and WE(2) itself.

Remark 3.3. It is worth noting that the region encl(WE(2)) is well-defined because the set WE(2) is connected (see [25]). In addition, this region can be equivalently defined, as the set of points such that if $x \in \mathrm{encl}(\mathrm{WE}(2)) \setminus \mathrm{WE}(2)$ there is no continuous curve $c(t)$, $t \in [0, \infty)$, with $c(0) = x$ and $\lim_{t\to\infty} \|c(t)\| = +\infty$, such that $c(t) \notin \mathrm{WE}(2)$, $\forall t \in [0, \infty)$.

Proof. First, we prove that $\mathrm{encl}(\mathrm{WE}(2)) \subseteq \mathrm{WE}(F_1, F_2, F_3)$. Before that, we note that $\mathrm{WE}(F_i, F_j) \subseteq \mathrm{WE}(F_1, F_2, F_3)$ $\forall i, j \in \{1, 2, 3\}$. Thus, we only have to prove that if $x \in \mathrm{encl}(\mathrm{WE}(2)) \setminus \mathrm{WE}(2)$ then $x \in \mathrm{WE}(F_1, F_2, F_3)$. We prove that by contradiction.

If $x \notin \mathrm{WE}(F_1, F_2, F_3)$ we have that $\bigcap_{i=1}^3 L_<(F_i, F_i(x)) \neq \emptyset$. Then, by lemma 3.2, $x - \mathrm{ri}(\bigcap_{i=1}^3 T_{L_\leqslant(F_i,F_i(x))}(x)) \neq \{x\}$. Now, since $x \in \mathrm{encl}(\mathrm{WE}(2)) \setminus \mathrm{WE}(2)$ and $x - \mathrm{ri}(\bigcap_{i=1}^3 T_{L_\leqslant(F_i,F_i(x))}(x))$ is a cone pointed at x then

$$S := \left(x - \mathrm{ri}\left(\bigcap_{i=1}^{3} T_{L_\leqslant(F_i,F_i(x))}(x)\right)\right) \cap \mathrm{WE}(2) \neq \emptyset.$$

Let $y \in S$. Since $y \in x - \mathrm{ri}(\bigcap_{i=1}^3 T_{L_\leqslant(F_i,F_i(x))}(x)) \subseteq \bigcap_{i=1}^3 L_>(F_i, F_i(x))$ then $F_i(x) < F_i(y)$, $i = 1, 2, 3$. Therefore, $y \notin \mathrm{WE}(F_1, F_2, F_3) \supseteq \mathrm{WE}(2)$ which contradicts that $y \in \mathrm{WE}(2)$.

Hence, we have that

$$\mathrm{encl}\big(\mathrm{WE}(2)\big) \subseteq \mathrm{WE}(F_1, F_2, F_3).$$

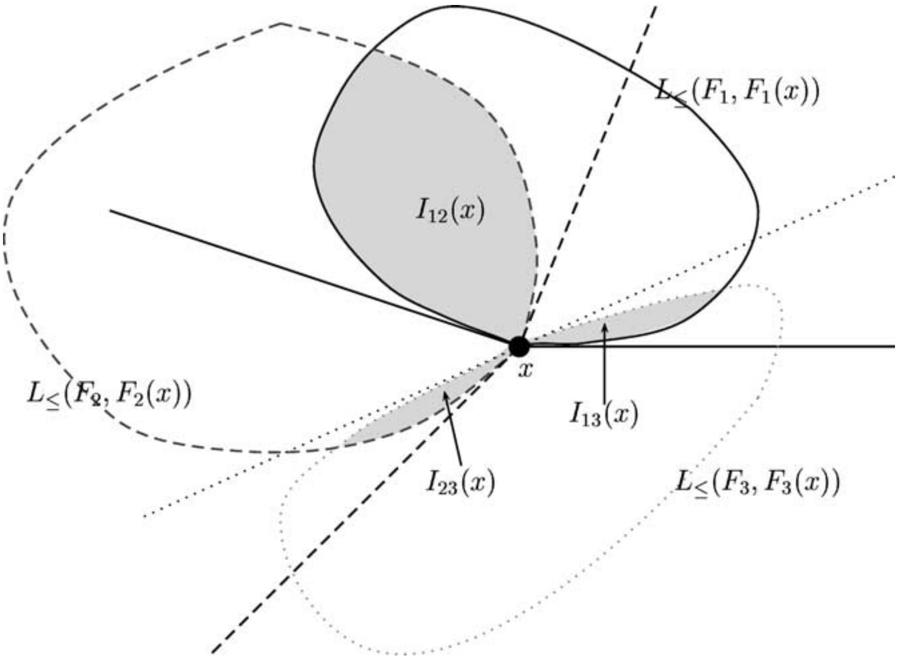

Figure 3. Case $x \in \mathrm{WE}(F_1, F_2, F_3) \setminus \mathrm{WE}(2)$.

Now, we prove the reverse inclusion. Let $x \in \mathrm{WE}(F_1, F_2, F_3)$. We must prove that $x \in \mathrm{encl}(\mathrm{WE}(2))$.

First, if there exists a pair $i, j \in \{1, 2, 3\}$ such that $I_{ij}^{<}(x) = \emptyset$ then $x \in \mathrm{WE}(F_i, F_j) \subseteq \mathrm{WE}(2)$.

Second, we consider the case that $I_{ij}^{<}(x) \neq \emptyset$, $\forall i, j \in \{1, 2, 3\}$. Since $x \in \mathrm{WE}(F_1, F_2, F_3)$ then $\bigcap_{i=1}^{3} L_{<}(F_i, F_i(x)) = \emptyset$. Therefore, the conditions of lemmas 3.1 and 3.3 are fulfilled (see figure 3). This implies that

$$C_{ij} := I_{ij}^{<}(x) \cap \mathrm{WE}(F_i, F_j) \neq \emptyset. \tag{6}$$

We must prove that there exists a chain of weakly efficient points for two criteria surrounding the point x. We prove that by contradiction.

Assume that there exists a continuous curve $z(t)$, $t \in [0, \infty)$, under the hypothesis of definition 3.1 such that (see figure 4),

(a) $z(t)$ separates the sets C_{12} and C_{13} with respect to the cone $x + T_{L_{\leqslant}(F_1, F_1(x))}(x)$,

(b) $\{z(t)\colon\ t \in [0, \infty)\} \cap \mathrm{WE}(2) = \emptyset$.

On the other hand, we have the following four assertions:

A1. $\mathcal{X}^*(F_1) \subseteq L_{\leqslant}(F_1, F_1(x)) \subseteq x + T_{L_{\leqslant}(F_1, F_1(x))}(x)$ (by the definition of $\mathcal{X}^*(F_i)$, see (4)).

A2. $T_{L_{\leqslant}(F_1, F_1(x))}(x) \cap T_{L_{\leqslant}(F_i, F_i(x))}(x) = T_{I_{1i}^{\leqslant}(x)}(x)$ for $i = 2, 3$ (by remark 5.3.2 in [13]).

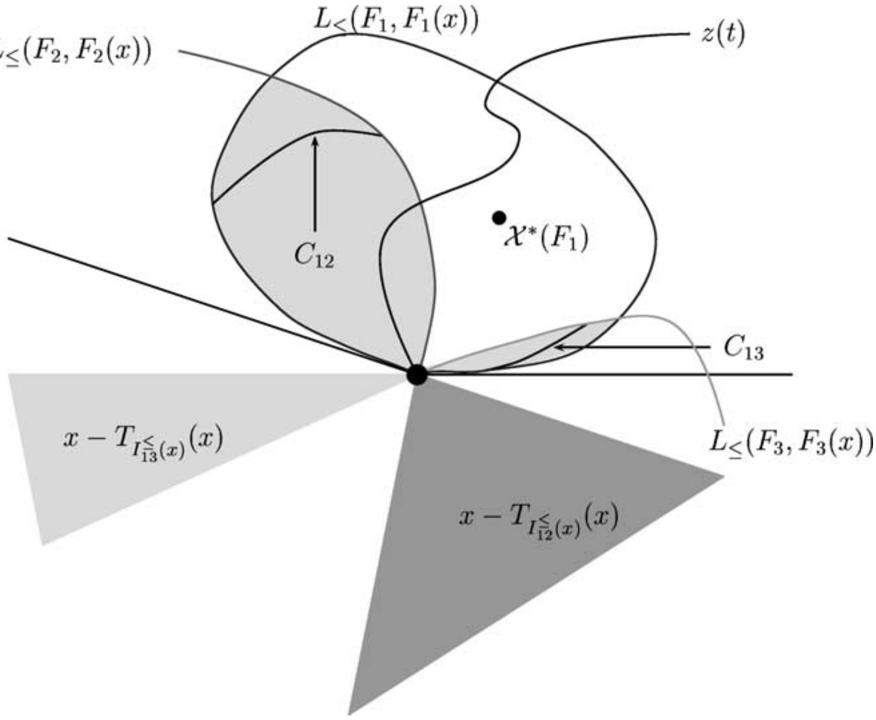

Figure 4. $z(t)$ separates the sets C_{12} and C_{13} with respect to the cone $x + T_{L_{\leqslant}(F_1, F_1(x))}(x)$.

A3. $\mathcal{X}^*(F_1) \cup C_{12} \subseteq \mathrm{WE}(F_1, F_2)$ and $\mathrm{WE}(F_1, F_2) \cap (x - T_{I_{12}^{\leqslant}(x)}(x)) = \emptyset$ (by lemma 3.1).

A4. $\mathcal{X}^*(F_1) \cup C_{13} \subseteq \mathrm{WE}(F_1, F_3)$ and $\mathrm{WE}(F_1, F_3) \cap (x - T_{I_{13}^{\leqslant}(x)}(x)) = \emptyset$ (by lemma 3.1).

Thus, since $z(t)$ separates C_{12} and C_{13} in $x + T_{L_{\leqslant}(F_1, F_1(x))}(x)$ as well as $\mathcal{X}^*(F_1) \subseteq x + T_{L_{\leqslant}(F_1, F_1(x))}(x)$, one of following three cases must occur:

1. $\mathcal{X}^*(F_1)$ is separated from C_{12} by $z(t)$ in $x + T_{L_{\leqslant}(F_1, F_1(x))}(x)$. Since $\mathrm{WE}(F_1, F_2)$ is a connected set, it can not cross $x - T_{I_{12}^{\leqslant}(x)}(x)$ and it contains $\mathcal{X}^*(F_1) \cup C_{12}$ (assertion A3) we have that $\mathrm{WE}(F_1, F_2) \cap \{z(t)\colon t \in [0, \infty)\} \neq \emptyset$.

2. $\mathcal{X}^*(F_1)$ is separated from C_{13} by $z(t)$ in $x + T_{L_{\leqslant}(F_1, F_1(x))}(x)$. Since $\mathrm{WE}(F_1, F_3)$ is a connected set, it can not cross $x - T_{I_{13}^{\leqslant}(x)}(x)$ and it contains $\mathcal{X}^*(F_1) \cup C_{13}$ (assertions A4) we have that $\mathrm{WE}(F_1, F_3) \cap \{z(t)\colon t \in [0, \infty)\} \neq \emptyset$.

3. $\mathcal{X}^*(F_1) \cap \{z(t)\colon t \in [0, \infty)\} \neq \emptyset$.

Therefore, any of these three cases contradict the initial hypothesis, since $\mathrm{WE}(2) \cap \{z(t)\colon t \in [0, +\infty)\} \neq \emptyset$.

We can use the same arguments with C_{12} and C_{23}, as well as C_{13} and C_{23} to obtain that the point x belongs to the region surrounded by the set of weakly efficient points of the bicriteria problems. □

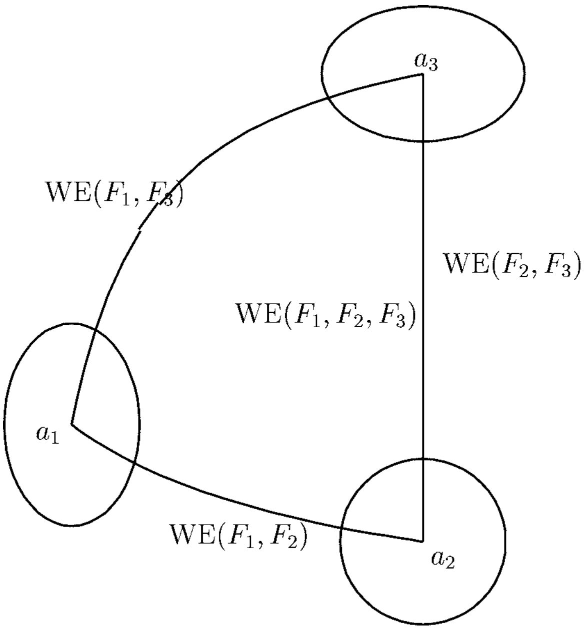

Figure 5. Illustration of example 3.2.

As illustration of the result above we show the following example.

Example 3.2. Let us consider three points $a_1 = (0, 0)$, $a_2 = (3, -1)$ and $a_3 = (3, 3)$ and the functions $F_1(\cdot)$, $F_2(\cdot)$ and $F_3(\cdot)$ such that, for $\rho \geqslant 0$ they have the following level sets:

$$L_{\leqslant}(F_1, \rho) = \left\{(x_1, x_2)\colon\ \frac{x_1^2}{4} + \frac{x_2^2}{9} \leqslant \rho\right\},$$
$$L_{\leqslant}(F_2, \rho) = \left\{(x_1, x_2)\colon\ (x_1 - 3)^2 + (x_2 + 1)^2 \leqslant \rho\right\},$$
$$L_{\leqslant}(F_3, \rho) = \left\{(x_1, x_2)\colon\ \frac{(x_1 - 3)^2}{9} + \frac{(x_2 - 3)^2}{4} \leqslant \rho\right\}.$$

We can see that these three functions are convex functions, therefore by the previous result we obtain the geometrical characterization of the set WE(F_1, F_2, F_3); this set is the shadowed region in figure 5.

Now we are in the right position to show the main result about the geometrical structure of WE($F_1, \dots, F_k$).

Theorem 3.3.

$$\mathrm{WE}(F_1, \dots, F_k) = \bigcup_{i,j,l \in \{1,\dots,k\}} \mathrm{WE}(F_i, F_j, F_l).$$

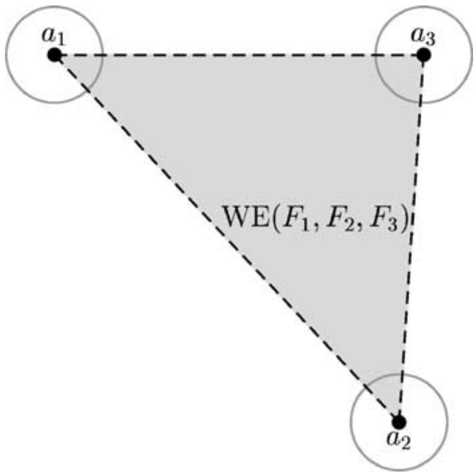

Figure 6. Illustration of example 3.3.

Proof. By theorem 3.1, $x \in \mathrm{WE}(F_1, \ldots, F_k)$ if and only if $\bigcap_{1 \leqslant i \leqslant k} L_<(F_i, F_i(x)) = \emptyset$. This intersection is empty if and only if there exist $i, j, l \in \{1, \ldots, k\}$ such that $L_<(F_i, F_i(x)) \cap L_<(F_j, F_j(x)) \cap L_<(F_l, F_l(x)) = \emptyset$ (by Helly's theorem, see [23]) and this is equivalent to $x \in \mathrm{WE}(F_i, F_j, F_l)$. Since in any case we have that

$$\bigcup_{i,j,l \in \{1,\ldots,k\}} \mathrm{WE}(F_i, F_j, F_l) \subset \mathrm{WE}(F_1, \ldots, F_k)$$

the result follows. □

Remark 3.4. This result extends previous characterizations in the literature:

- Taking $F_i(x) = \|x - a_i\|$ with $a_i \in \mathbb{R}^2$ for $i = 1, \ldots, k$ and $\| \cdot \|$ being an strictly convex norm or a norm derived from a scalar product, we get proposition 1.3, theorem 4.3 and corollary 4.1 in [9]. The solution set is the convex hull of the points a_i with $i = 1, \ldots, k$. In example 3.3, we illustrate this result.
- Taking $F_i(x) = \|x - a_i\|$ with $a_i \in \mathbb{R}^2$ for $i = 1, \ldots, k$ and $\| \cdot \|$ being a polyhedral gauge we get theorem 6.1 in [7], where the solution set is the union of elementary convex sets (see [8] for a definition). In example 3.4, we illustrate this result.
- Taking $F_i(x) = \max_{j \in \mathcal{M}} \omega_j^i \|x - a_j\|$ with $a_j \in \mathbb{R}^2$ and $\omega_j^i > 0$ for $i = 1, \ldots, k$ and $j \in \mathcal{M} := \{1, \ldots, m\}$ we get theorem 6.1 in [11], where the solution set is the union of the solution sets for each two functions. In example 3.5, we illustrate the use of this result.

Example 3.3 (See figure 6). Let us consider the points $a_1 = (4, 4)$, $a_2 = (18, -11)$, $a_3 = (19, 4)$ and the functions $F_i(x) = \|x - a_i\|_2$ for $i = 1, 2, 3$. By theorem 3.2, $\mathrm{WE}(F_1, F_2, F_3)$ is the dark region, which in this case is the convex hull of a_1, a_2 and a_3.

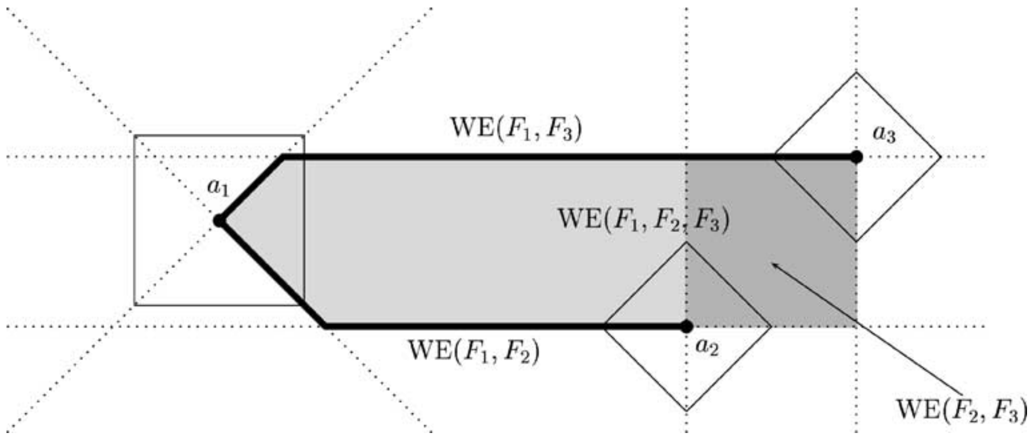

Figure 7. Illustration of example 3.4.

Example 3.4 (See figure 7). Let us consider the points $a_1 = (7, 5)$, $a_2 = (18, 2.5)$, $a_3 = (22, 6.5)$ and the functions $F_1(x) = \|x - a_1\|_\infty$, $F_2(x) = \|x - a_2\|_1$ and $F_3(x) = \|x - a_3\|_1$. By theorem 3.1, WE(F_1, F_2) is the thick path joining a_1 and a_2, WE(F_1, F_3) is the thick path joining a_1 and a_3, and WE(F_2, F_3) is the dark square with a_2 and a_3 as opposite extreme points. Therefore, by theorem 3.2, WE(F_1, F_2, F_3) is the dark region encircled by the union of the three previous sets and the three sets themselves. Notice that this region is the union of two full dimensional elementary convex sets.

Example 3.5 (See figure 8). Let us consider the points $a_1 = (4, 16)$, $a_2 = (10, 5)$, $a_3 = (25, 12)$ and the functions $F_i(x) = \|x - a_i\|_\infty$ for $i = 1, 2, 3$. By theorem 3.1, WE(F_1, F_2) $= R_1$, WE(F_1, F_3) $= R_2 \cup R_4$, WE(F_2, F_3) $= R_3 \cup R_4$. By theorem 3.2, WE(F_1, F_2, F_3) $= R_1 \cup R_2 \cup R_3 \cup R_4$. Notice that for this example it holds that WE(F_1, F_2, F_3) = WE(F_1, F_2) $\cup$ WE(F_1, F_3) $\cup$ WE(F_2, F_3).

As a direct consequence of the results of this section we get the following pseudoalgorithm.

Input: $F_1(\cdot), \ldots, F_k(\cdot)$: inf-compact, convex functions.
Output: WE($F_1, F_2, \ldots, F_k$).
Steps:

1. Compute the sets WE(F_i, F_j) $\forall i, j \in \{1, \ldots, k\}$.
2. Compute WE(F_i, F_j, F_l) for all $i, j, l \in \{1, \ldots, k\}$.
3. Compute WE($F_1, F_2, \ldots, F_k$) $= \bigcup_{i,j,l \in \{1,\ldots,k\}}$ WE(F_i, F_j, F_l).
4. END.

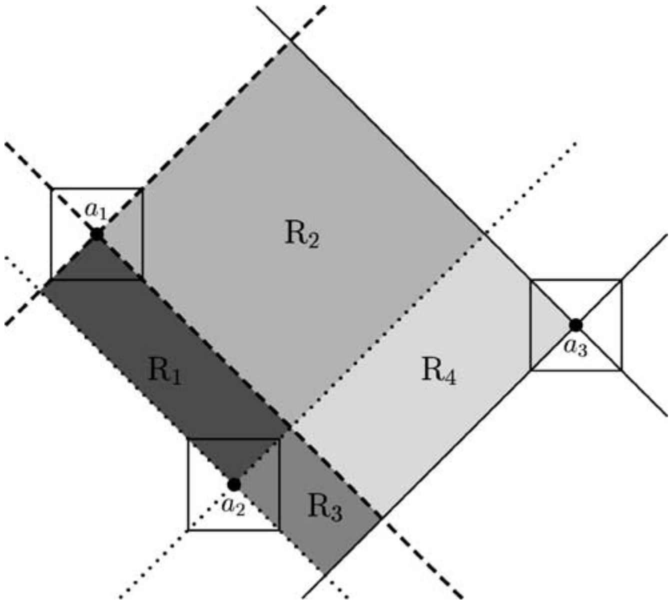

Figure 8. Illustration of example 3.5.

4. Concluding remarks

In this paper we have developed a geometrical characterization of the solution set for a general multicriteria problem in two-dimension spaces.

It should be noted, that in the literature this problem had been solved for particular cases, as for instance in the papers by Durier and Michelot [9], Durier [7], Carrizosa et al. [4], Hamacher and Nickel [11], Ndiaye and Michelot [16] and Ohsawa [17]. The model studied in this paper includes a large number of classic models of multicriteria problems as well as many other whose solution were not previously known.

Appendix

Proof of lemma 3.1. The first assertion is equivalent to prove that $\bigcap_{i=1}^{3} T_{L_{\leqslant}(F_i,F_i(x))}(x) = \{0\}$. We prove this fact by contradiction. Assume that there exists $y \neq 0$ such that $y \in \bigcap_{i=1}^{3} T_{L_{\leqslant}(F_i,F_i(x))}(x)$, then four cases may occur:

1. $y \in \mathrm{ri}(T_{L_{\leqslant}(F_i,F_i(x))}(x))$, $i = 1, 2, 3$ (see figure 9).

Since $y \in \bigcap_{i=1}^{3} \mathrm{ri}(T_{L_{\leqslant}(F_i,F_i(x))}(x))$, there exists $\lambda_i > 0$ such that $x + \lambda_i y \in L_{<}(F_i, F_i(x))$ for $i = 1, 2, 3$. We define $\lambda := \min\{\lambda_1, \lambda_2, \lambda_3\} > 0$. Using $x \in \bigcap_{i=1}^{3} L_{\leqslant}(F_i, F_i(x))$ and the convexity of $\bigcap_{i=1}^{3} L_{<}(F_i, F_i(x))$ we have that $x + \lambda y \in \bigcap_{i=1}^{3} L_{<}(F_i, F_i(x))$, and this contradicts (a).

2. $y \in \mathrm{ri}(T_{L_{\leqslant}(F_i,F_i(x))}(x))$, $i = 1, 2$, and $y \notin \mathrm{ri}(T_{L_{\leqslant}(F_3,F_3(x))}(x))$.

Then, one of the facets of $T_{L_{\leqslant}(F_3,F_3(x))}(x)$ belongs to $\bigcap_{i=1}^{2} \mathrm{ri}(T_{L_{\leqslant}(F_i,F_i(x))}(x))$. Hence, we have that $\bigcap_{i=1}^{3} \mathrm{ri}(T_{L_{\leqslant}(F_i,F_i(x))}(x)) \neq \emptyset$ and we are in case 1.

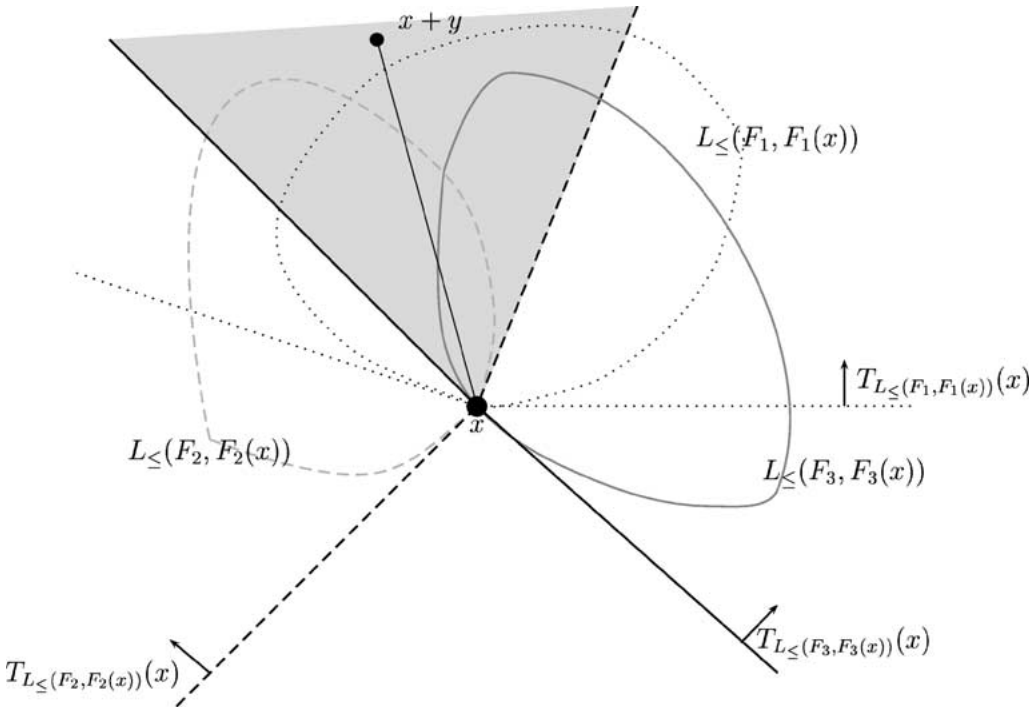

Figure 9. Case $\bigcap_{i=1}^{3} \mathrm{ri}(T_{L_{\leqslant}(F_i,F_i(x))}(x)) \neq \emptyset$.

3. $y \in \mathrm{ri}(T_{L_{\leqslant}(F_1,F_1(x))}(x))$ and $y \notin \mathrm{ri}(T_{L_{\leqslant}(F_i,F_i(x))}(x))$, $i = 2, 3$.
Then, one of the facets of $\bigcap_{i=2}^{3} T_{L_{\leqslant}(F_i,F_i(x))}(x)$ belongs to $\mathrm{ri}(T_{L_{\leqslant}(F_1,F_1(x))}(x))$.
Moreover, since $I_{23}^{<}(x) \neq \emptyset$ then

$$\mathrm{ri}\left(\bigcap_{i=2}^{3} T_{L_{\leqslant}(F_i,F_i(x))}(x)\right) = \bigcap_{i=2}^{3} \mathrm{ri}\big(T_{L_{\leqslant}(F_i,F_i(x))}(x)\big) \neq \emptyset.$$

This implies that

$$\bigcap_{i=2}^{3} \mathrm{ri}\big(T_{L_{\leqslant}(F_i,F_i(x))}(x)\big) \cap \mathrm{ri}\big(T_{L_{\leqslant}(F_1,F_1(x))}(x)\big) \neq \emptyset,$$

and we are again in case 1.

4. $y \notin \mathrm{ri}(T_{L_{\leqslant}(F_i,F_i(x))}(x))$, $i = 1, 2, 3$.

We have that $y \in \bigcap_{i=1}^{3} T_{L_{\leqslant}(F_i,F_i(x))}(x)$ then $y \in \mathrm{rbd}(T_{L_{\leqslant}(F_i,F_i(x))}(x))$, $i = 1, 2, 3$. Hence, there exists a common facet for the three cones. Since $T_{L_{\leqslant}(F_i,F_i(x))}(x)$ and $T_{L_{\leqslant}(F_j,F_j(x))}(x)$ are convex and

$$\mathrm{ri}\big(T_{L_{\leqslant}(F_i,F_i(x))}(x)\big) \cap \mathrm{ri}\big(T_{L_{\leqslant}(F_j,F_j(x))}(x)\big) \neq \emptyset$$

for all $i, j \in \{1, 2, 3\}$, the cones $T_{L_{\leqslant}(F_i, F_i(x))}(x)$ and $T_{L_{\leqslant}(F_j, F_j(x))}(x)$ lie in the same halfspace generated by the common facet of the three cones. Therefore, $\bigcap_{i=1}^{3} \mathrm{ri}(T_{L_{\leqslant}(F_i, F_i(x))}(x))$ is not empty and we are again in case 1.

Now, we prove the second assertion. Let $y \in T_{I_{ij}^{\leqslant}(x)}(x)$, then $x - y \in L_{\geqslant}(F_i, F_i(x)) \cap L_{\geqslant}(F_j, F_j(x))$ because $x - T_{I_{ij}^{\leqslant}(x)}(x) \subseteq L_{\geqslant}(F_i, F_i(x)) \cap L_{\geqslant}(F_j, F_j(x))$. Thus, we have that $F_k(x) \leqslant F_k(x - y)$ $k = i, j$. On the other hand, from (b) and theorem 3.1, we obtain that $x \notin \mathrm{WE}(F_i, F_j)$. Hence $x - y \notin \mathrm{WE}(F_i, F_j)$ and therefore $\mathrm{WE}(F_i, F_j) \cap (x - T_{I_{ij}^{\leqslant}(x)}(x)) = \emptyset$.

Since

$$\begin{aligned} \emptyset \neq I_{ij}^{<}(x) &= L_{<}\big(F_i, F_i(x)\big) \cap L_{<}\big(F_j, F_j(x)\big) \\ &\subseteq \mathrm{ri}\big(L_{\leqslant}\big(F_j, F_j(x)\big)\big) \cap \mathrm{ri}\big(L_{\leqslant}\big(F_j, F_j(x)\big)\big) \end{aligned}$$

we have that (see remark 5.3.2 in [13]) $T_{L_{\leqslant}(F_i, F_i(x))}(x) \cap T_{L_{\leqslant}(F_j, F_j(x))}(x) = T_{I_{ij}^{\leqslant}(x)}(x)$ and the result follows. □

Proof of lemma 3.2. First, since $\bigcap_{i=1}^{3} T_{L_{\leqslant}(F_i, F_i(x))}(x)$ is a pointed cone at 0 then its relative interior does not contain 0 and (i) is proved.

By (5) we have that $\bigcap_{i=1}^{3} \mathrm{ri}(L_{\leqslant}(F_i, F_i(x))) \neq \emptyset$ then $\bigcap_{i=1}^{3} T_{L_{\leqslant}(F_i, F_i(x))}(x) = T_{\bigcap_{i=1}^{3} L_{\leqslant}(F_i, F_i(x))}(x)$ (see [13]).

On the other hand, since $\bigcap_{i=1}^{3} L_{\leqslant}(F_i, F_i(x)) \subseteq x + T_{\bigcap_{i=1}^{3} L_{\leqslant}(F_i, F_i(x))}(x)$ then

$$\begin{aligned} \emptyset \neq \bigcap_{i=1}^{3} L_{<}\big(F_i, F_i(x)\big) &\subseteq \mathrm{ri}\left(\bigcap_{i=1}^{3} L_{\leqslant}\big(F_i, F_i(x)\big)\right) \\ &\subseteq \mathrm{ri}\big(x + T_{\bigcap_{i=1}^{3} L_{\leqslant}(F_i, F_i(x))}(x)\big) = x + \mathrm{ri}\big(T_{\bigcap_{i=1}^{3} L_{\leqslant}(F_i, F_i(x))}(x)\big). \end{aligned}$$

Thus, since $x \notin \bigcap_{i=1}^{3} L_{<}(F_i, F_i(x))$ and

$$\bigcap_{i=1}^{3} L_{<}\big(F_i, F_i(x)\big) \subseteq x + \mathrm{ri}\big(T_{\bigcap_{i=1}^{3} L_{<}(F_i, F_i(x))}(x)\big)$$

we conclude that $\mathrm{ri}(T_{\bigcap_{i=1}^{3} L_{\leqslant}(F_i, F_i(x))}(x)) \neq \emptyset$ and the result follows. □

Proof of lemma 3.3. The set $I_{ij}^{<}(x)$ is the set of points strictly dominating x. That means that any $y \in I_{ij}^{<}(x)$ verifies $F_l(y) < F_l(x)$, $l = i, j$. Therefore, $\mathrm{WE}(F_i, F_j) \cap I_{ij}^{<}(x) \neq \emptyset$. □

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