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Potential Symmetries for some Ordinary Differential Equations

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Abstract

Some diffusion equations admitting potential symmetries and the scaling group as a Lie symmetry are considered and some general results are obtained. For all the equations that we have studied, a set of potential symmetries admitted by the diffusion equation is "inherited" by the ODE that emerges as the reduced equation under the scaling group. Using these potential symmetries we find that the order of the ODE can be reduced even if this equation does not admit point symmetries. Moreover, in the case for which the ODE admits a group of point symmetries, we find that the potential symmetries allow us to perform further reductions than its point symmetries.

Key words: differential equation, potential symmetry

1 Introduction

In the last few years we have observed a significant progress in the application of symmetries to the study of nonlinear partial differential equations (PDE's) of physical importance, as well as in finding exact solutions for such equations.

Lie classical symmetries admitted by nonlinear PDE's are useful for finding invariant solutions, as well as to discover whether or not the equation can be linearized by an invertible mapping and to construct an explicit linearization when one exists. Nevertheless an obvious limitation of group-theoretic methods based on local symmetries, is that many PDE's do not have local symmetries. It turns out that PDE's can admit nonlocal symmetries ,whose infinitesimal generators depend on the integrals of the dependent variables in some specific manner. For a given PDE one can find useful nonlocal symmetries by embedding it in an auxiliary "covering" system with auxiliary dependent variables. A point symmetry of the auxiliary system, acting on the space consisting of

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the independent and dependent variables of the given PDE as well as the auxiliary variables, yields a nonlocal symmetry of the given PDE if it does not project onto a point symmetry acting in its space of the independent and dependent variables.

It also happens that if a nonlinear scalar PDE does not admit an infinite-parameter Lie group of contact transformations it is not linearizable by an invertible contact transformation. However most of the interesting linearizations involve non-invertible transformations, such linearizations can be found by embedding given nonlinear PDE's in auxiliary systems of PDE's [1]. In [2] Bluman and Reid derived an algorithm to find new symmetry groups for ordinary differential equations (ODE's). These new symmetries reduce the order of a given ODE in cases where a direct application of Lie's method fails. With respect to the given independent and dependent variables of an ODE these symmetries are in general not of point, contact or Lie-Bäcklund type.

The aim of this paper is to study a family of diffusion equations which admits potential symmetries. We prove that some of the reduced ODE's, which arise by means of invariance of the diffusion equation under the scaling group, "inherit" a set of potential symmetries. These potentials symmetries lead to further order reductions than the classical Lie symmetries of these equations.

It is well known that

$$u_t = (u^{-2}u_x)_x \tag{1}$$

admits, besides Lie-Bäcklund symmetries and recursion operators, potential symmetries. Noteworthy, (1) is the only equation of the form

$$u_t = (u^n u_x)_x \tag{2}$$

which admit classical potential symmetries [1].

In [3] there have been obtained two hierarchies which admit classical potential symmetries. The first of them is

$$u_t = R^m[u](u^{-2}u_x)_x \tag{3}$$

where

$$R[u] = D_x^2 u^{-1} D_x^{-1}. \tag{4}$$

Thus, the first three equations in the hierarchy (3) take the form

$$
\begin{aligned}
u_t &= (u^{-2}u_x)_x \\
u_t &= (u^{-3}u_x)_{xx} \\
u_t &= (u^{-4}u_{xx} - 3u^{-5}{u_x}^2)_{xx}.
\end{aligned}
\tag{5}
$$

These equations admits an infinite set of potential symmetries if we consider the corresponding associated hierarchy of systems

$$v_x = u, \quad v_t = D_x^{-1}R^m[u](u^{-2}u_x)_x. \tag{6}$$

For example, the first three systems in the hierarchy (6) take the form

$$v_x = u, \quad v_t = u^{-2}u_x,$$

$$v_x = u, \quad v_t = (u^{-3}u_x)_x,$$
$$v_x = u, \quad v_t = (u^{-4}u_{xx} - 3u^{-5}u_x^2)_x.$$

The second hierarchy obtained is

$$u_t = R^m[u](u^2u_{xx}) \tag{7}$$

which admits the recursion operator

$$R[u] = u^2D_x^2uD_x^{-1}u^{-2}. \tag{8}$$

The first three equations in the hierarchy (7) are

$$
\begin{aligned}
u_t &= u^2u_{xx} \\
u_t &= (u^3u_{xx})_x \\
u_t &= u^2[uu_x^2 + u^2u_{xx}]_{xx}.
\end{aligned}
\tag{9}
$$

Observe that (9) can be obtained from (5) by means of $u \to \frac{1}{u}$. These equations admit an infinite set of potential symmetries if we consider the corres ponding associated hierarchy of systems

$$v_x = \frac{1}{u}, \quad v_t = -u_x,$$

$$v_x = \frac{1}{u}, \quad v_t = -(uu_x)_x,$$

$$v_x = \frac{1}{u}, \quad v_t = [(-uu_x)_x u]_x.$$

These infinite set of potential symmetries allow us to linearize the corresponding hierarchies of PDE's.

The equations (5) and (9) are invariant under the classical scaling group and admit an infinite set of potential symmetries. Under the scaling group the sets of equations (5) and (9) are reduced to two sets of ODE's. These ODE's only admit a one-parameter Lie group of point symmetries but "inherit" potential symmetries that allow us to linearize the ODE's, or to reduce the order by two.

We have also obtained the equations:

$$\begin{aligned} u_t &= (\tfrac{u_x}{u^2+1})_x, \\ u_t &= (\tfrac{u_x}{(u^2+1)^{3/2}})_{xx}, \\ u_t &= \Big((\tfrac{u_x}{(u^2+1)^{3/2}})_x \tfrac{1}{(u^2+1)^{1/2}}\Big)_{xx}, \end{aligned} \tag{10}$$

which are invariant under the classical scaling group and admit potential symmetries. Note that the second and the third of these equations can be generated by applying the operator

$$D_x^2 (u^2+1)^{-\frac{1}{2}} D_x^{-1}$$

to the previous one. Under the scaling group, the set of equations (10) is reduced to a set of ODE's. These ODE's do not admit any classical point symmetry but 'inherit' potential symmetries that allow us to reduce the order by one.

2 The diffusion equation $u_t = (u^{-3}u_x)_{xx}$

We consider the third order diffusion equation

$$u_t = (u^{-3}u_x)_{xx}, \tag{11}$$

which is the second member of (5). This equation admits the four parameter group of infinitesimal generators

$$X_1 = \partial_x, \quad X_2 = \partial_t, \quad X_3 = x\partial_x - u\partial_u, \quad X_4 = 3t\partial_t + u\partial_u.$$

An important class of solutions of (11) arises from its invariance under the scaling group $X = X_3 + X_4$; using this invariance, we have $u = u(z)$ and

$z = xt^{-1/3}$, where u satisfies the ODE

$$zu_z + 3(u^{-3}u_z)_{zz} = 0. \tag{12}$$

For applications it is important to reduce the order of (12). ODE (12) only admits a one-parameter Lie group of point transformations with infinitesimals $\xi = z$ and $\phi = -u$, that allows us to reduce (12) to a second order ODE:

$$\begin{aligned} &-y^6h^5 + y^5h^4 - 3y^2hh'' + 9y^2(h')^2 - 9y^2h^2h' + 6y^2h^4 + \\ &27yhh' - 27yh^3 + 36h^2 = 0. \end{aligned} \tag{13}$$

This equation does not admit any classical Lie symmetry. Nevertheless (12) admits "potential" symmetries derived by considering the associated system

$$\begin{aligned} &v_z - u = 0, \\ &3(u^{-3}u_z)_z + zu - v = 0. \end{aligned} \tag{14}$$

System (14) admits a new group of symmetries with infinitesimals

$$\xi = k_1 z + \eta(v), \quad \phi = -k_1 u - \eta_v(v)u^2, \quad \psi = 0, \tag{15}$$

where $\eta(v)$ satisfies the linear equation

$$\eta_v v + 3\eta_{vvv} - \eta = 0. \tag{16}$$

A solution for (16) is $\eta = k_2 v$. Consequently the integrated equation

$$3(v_z^{-3}v_{zz})_z + zv_z - v = 0 \tag{17}$$

admits a two-parameter Lie group of point transformations with infinitesimals

$$V_1 = z\partial_z, \quad V_2 = v\partial_z. \tag{18}$$

The Lie-bracket is $[V_1, V_2] = -V_2$ which allows us to reduce (17) first by V_2 and then by V_1. The canonical coordinates for V_2 are $w = \frac{z}{v}$, $y = v$, thus, (17) is reduced to the linear second order Bessel ODE

$$3yh'' + 9h' + y^2h = 0, \tag{19}$$

where $h = w_y$ and $'$ stands by $\frac{d}{dy}$. Then, (19) inherits the Lie symmetry V_1, whose projection is $\hat{V}_1 = h\partial_h$ (it is clear that invariance under $\hat{V}_1$ is just a consequence from being (19) a linear equation). This symmetry allows us to reduce (19) to the first order Ricatti ODE,

$$3tg' + 3tg^2 + 9g + t^2 = 0$$

where $t = y$, $g = \frac{h'}{h}$.

Remark 1 *By making in (12) $u = v_z$, it can be written in the conserved form*

$$\left[3((v_z)^{-3}v_{zz})_z + zv_z - v\right]_z = 0. \tag{20}$$

The integrated equation with $c = 0$ has the form (17). We have found that equation (17) admits two generators V_1 and V_2. On the other hand, it can be easily seen that the equation (20) admits the generator V_1. Besides, the process for the order reduction from (20) to (12) has been done by using the Lie symmetry ∂_v of equation (20). Since $[\partial_v, V_1] = 0$, V_1 is inherited as a Lie symmetry of equation (12). The corresponding inherited symmetry, $z\partial_z - u\partial_u$, is the only Lie symmetry admitted by (12).

The same process can not be done for the other admitted Lie symmetry of (17) because the vector field $V_2 = v\partial_z$ is not a Lie symmetry for equation (20). In fact, it is a conditional symmetry for the equation (20) because it transforms only *a class of solutions of (20) into solutions of (20), that is those solutions that verify equation (17).*

3 The diffusion equation $u_t = (u^3 u_{xx})_x$

Next, we study the third order diffusion equation corresponding to the second member of the hierarchy (7) i.e.

$$u_t = (u^3 u_{xx})_x. \tag{21}$$

This equation admits the four parameter group of infinitesimal generators

$$X_1 = \partial_x, \quad X_2 = \partial_t, \quad X_3 = x\partial_x + u\partial_u, \quad X_4 = 3t\partial_t - u\partial_u.$$

We consider the reduction of (21) under the group $X = -3nX_3 + X_4$, i.e.

$$u = w(z)t^{-\frac{3n+1}{3}} \; z = xt^n$$

where w satisfies the ODE

$$-3nzw_z + (3n+1)w + 3(w^3 w_{zz})_z = 0. \tag{22}$$

As in the previous section, now we try to reduce the order of (22). It is worth noting that in the case $n = -\frac{1}{6}$ this equation is in a conserved form, consequently an order reduction can be done in a trivial way. We can show that equation (22) only admits a one-parameter Lie group with infinitesimals

$$\xi = z, \, \phi = w \tag{23}$$

that allows us to reduce (22) to the second order ODE

$$(-3hh'' + 9(h')^2 - 9h^2h' + 6h^4)y^3 + (9h^3 - 9hh')y^2 + h^5 y - 3nh^4 = 0,$$

that does not admit any Lie symmetry.

We also can show that (22) does not admit potential symmetries by setting $w = v_z$. In this case the ODE (22) becomes

$$\Big(3(v_z^3 v_{zzz} - nzv_z) + (6n+1)v\Big)_z = 0 \tag{24}$$

and the associated system is

$$\begin{aligned} &v_z - w = 0 \\ &3w^3 w_{zz} - 3nzw + (6n+1)v = 0 \end{aligned} \tag{25}$$

that, for $n \neq -\frac{1}{6}$, only admits a one-parameter group of symmetries which does not correspond to a potential symmetry of (22). In fact, it projects onto (23). However we can show that by setting $w = \frac{1}{v_z}$, ODE (22) becomes

$$\Big(-3(v_{zz}v_z^{-3})_z + 3nzv_z + v\Big)_z = 0 \tag{26}$$

and then, it has a new group of "potential" symmetries derived by considering the associated system

$$\begin{aligned} &v_z - w = 0 \\ &3(w_z w^{-3})_z - 3nzw - v = 0 \end{aligned} \tag{27}$$

which has a new group of symmetries with infinitesimals

$$\xi = (\psi_v + k_1)z + \eta(v),\ \phi = w^2(\psi_{vv}z + \eta_v) - k_1 w,\ \psi = \psi(v) \tag{28}$$

where $\eta(v)$ satisfies

$$\eta_v v + 3\eta_{vvv} + 3n\eta = 0 \tag{29}$$

and $\psi(v)$ verifies

$$\begin{aligned} &2\psi_v v + 6\psi_{vvv} + \psi = 0 \\ &2\psi_{vv} v + 3\psi_{vvvv} + 9n\psi_v = 0. \end{aligned} \tag{30}$$

We choose $\psi = 0$, besides, a particular solution of (29) is easily found for two particular values of n. In fact, we have

for $n = -\frac{1}{3}$ $\eta = k_2 v$ then $\xi = k_1 z + k_2 v,\ \ \phi = k_1 w + k_2 v, \qquad \psi = 0,$

for $n = -\frac{2}{3}$ $\eta = k_2 v^2$ then $\xi = k_1 z + k_2 v^2,\ \phi = k_1 w + 2k_2 v w^2,\ \psi = 0.$

The symmetry groups corresponding to $k_2 \neq 0$ are new symmetry groups of (22) because ξ depends explicitly on v and v can not be expressed in terms of x, u and derivatives of u to some finite order. Next, we consider these two cases separately:

1.- $n = -\frac{1}{3}$: We have found that the equation

$$3(v_{zz}v_z^{-3})_z + zv_z - v = 0 \tag{31}$$

admits a two-parameter Lie group of point transformations with infinitesimals

$$V_1 = z\partial_z,\ V_2 = v\partial_z\,; \tag{32}$$

the Lie-bracket is $[V_1, V_2] = -V_2$ that allow us to reduce the order of (31) first by V_2 to the second order Bessel ODE

$$3yh'' + 9h' - y^2 h = 0 \tag{33}$$

with $y = v$ and $h = \frac{v - zv_z}{v^2 v_z}$. Then, (33) inherits the symmetry group V_1, which in terms of the new variables, has the form $\hat{V_1} = h\partial_h$ and allows us to reduce (33) to the first order ODE,

$$3tg' + 3tg^2 + 9g - t^2 = 0$$

with $t = y$, $g = \frac{h'}{h}$.

2.- $n = -\frac{2}{3}$: For this case we have that the equation

$$3(v_{zz}v_z^{-3})_z + 2zv_z - v = 0 \tag{34}$$

admits a two-parameter Lie group of point transformations with infinitesimals

$$V_1 = z\partial_z,\ V_2 = v^2\partial_z. \tag{35}$$

The Lie-braket is $[V_1, V_2] = -V_2$, thus, we can reduce (34) first by V_2 to the second order Bessel ODE

$$3y^2h'' + 18yh' + 18h + y^3h = 0 \tag{36}$$

with $y = v$, $h = \frac{v - 2zv_z}{v^3 v_z}$; and then (36) inherits the symmetry group V_1 that now takes the form $\hat{V_1} = h\partial_h$ and allows us to reduce (36) to the Ricatti equation,

$$3t^2g' + 3t^2g^2 + 18tg + 18 + t^3 = 0$$

with $t = y$, $g = \frac{h'}{h}$.

4 The fourth order diffusion equation $u_t = [(u^{-2})_{xx}u^{-1}]_{xx}$

Another interesting equation, from the point of view of the symmetry reductions is the fourth order diffusion equation

$$u_t = [(u^{-2})_{xx}u^{-1}]_{xx}, \tag{37}$$

related to the third member of the hierarchy (3) through the change of variables $t' = -2t$. This equation admits the four parameter group of infinitesimal generators

$$X_1 = \partial_x, \quad X_2 = \partial_t, \quad X_3 = x\partial_x - u\partial_u, \quad X_4 = 4t\partial_t + u\partial_u.$$

An important class of solutions of (37) arises from its invariance under the scaling group $X = -4nX_3 + X_4$, i.e.

$$u = w(z)t^{\frac{4n+1}{4}}, \quad z = xt^n$$

where w satisfies the ODE

$$-n(zw_z + w) - \frac{w}{4} + [(w^{-2})_{zz}w^{-1}]_{zz} = 0. \tag{38}$$

We can show that ODE (38) only admits Lie group of point transformations of infinitesimals

$$\xi = k_1 z, \quad \phi = -k_1 w.$$

Nevertheless (38) admits a set of "potential" symmetries derived by considering the associated system

$$\begin{aligned} &v_z - w = 0, \\ &-nzw - \tfrac{v}{4} + [(w^{-2})_{zz}w^{-1}]_z = 0. \end{aligned} \tag{39}$$

In fact, (39) has a group of symmetries with infinitesimals

$$\xi = k_1 z + \eta(v), \quad \phi = -k_1 w - \eta_v(v)w^2, \quad \psi = 0, \tag{40}$$

where $\eta(v)$ satisfies the linear equation

$$\eta_v v - 8\eta_{vvvv} + 4n\eta = 0. \tag{41}$$

A solution for (41) is $\eta = k_2 v^{-4n}$ where $n = -\frac{1}{4}, -\frac{1}{2}, -\frac{3}{4}$. Consequently, the integrated equation

$$-nzv_z - \frac{v}{4} + [(v_z^{-2})_{zz}v_z^{-1}]_z = 0 \tag{42}$$

admits the two parameter Lie group of infinitesimals

$$\begin{aligned} &V_1 = z\partial_z, \ \ V_2 = v\partial_z \ \text{ if } n = -\tfrac{1}{4}, \\ &V_1 = z\partial_z, \ \ V_2 = v^2\partial_z \text{ if } n = -\tfrac{1}{2}, \\ &V_1 = z\partial_z, \ \ V_2 = v^3\partial_z \text{ if } n = -\tfrac{3}{4}. \end{aligned}$$

In the three cases we have $[V_1, V_2] = -V_2$, then we can reduce (42) first by V_2 and then by V_1. For example, for $n = -\frac{1}{4}$ we reduce (42) by V_2 to the linear third order ODE

$$8yh''' + 32h'' + y^2h = 0 \tag{43}$$

with $y = v$, $h = \frac{d}{dy}(\frac{z}{v})$, and then (43) inherits the symmetry group $\hat{V}_1 = h\partial_h$ that allows us to reduce (43) to the second order ODE,

$$8tg'' + 24tgg' - t^2 + 8tg^3 + 32g' + 32g^2 = 0$$

with $t = y$, $g = \frac{h'}{h}$.

5 The diffusion equation $u_t = \left(\frac{1}{(u^2+1)^{3/2}}u_x\right)_{xx}$

Next, we study the third order diffusion equation corresponding to the second member of (10) i.e.

$$u_t = \left(\frac{1}{(u^2+1)^{3/2}}u_x\right)_{xx}. \tag{44}$$

The first member of (10) has been considered in [2]. An important class of solutions of (44) arises from its invariance under the scaling group $X = x\partial_x + 3t\partial_t$. Using this invariance we have

$$u = u(z), \quad z = xt^{-1/3},$$

where u satisfies the ODE

$$zu_z + 3(\frac{1}{(u^2+1)^{3/2}}u_z)_{zz} = 0. \tag{45}$$

For applications it is important to reduce the order of (45). This equation does not admit any point symmetry but the order can be reduced by using potential symmetries. In fact, the associated system

$$\begin{aligned} &v_z - u = 0 \\ &3(\tfrac{1}{(u^2+1)^{3/2}}u_z)_z + zu - v = 0, \end{aligned} \tag{46}$$

admits a group of transformations with infinitesimals

$$\xi = v, \quad \phi = -(u^2+1), \quad \psi = -z. \tag{47}$$

This symmetry group is a new symmetry group for (45) because ξ depends explicitly on v and v can not be expressed in terms of x, u and derivatives of u to some finite order. This new group also allows us to reduce the order of

(45). The canonical coordinates $r = (z^2 + v^2), w = \arctan\left(\frac{v}{z}\right)$, corresponding to (47), lead to express the ODE

$$3\left(\frac{v_{zz}}{(v_z^2+1)^{3/2}}\right)_z + zv_z - v = 0$$

in the form

$$-3r(r^2y^2+1)y_{rr}+9r^3y(y_r)^2+9r^2y^2y_r-9y_r-r^2y(r^2y^2+1)^{5/2}+3r^3y^5+12ry^3 = 0,$$

where $y = w_r$.

6 Conclusions

In this paper we have considered a family of diffusion PDE's which admit potential symmetries as well as point symmetries. An important class of solutions of these equations arises from its invariance under the scaling group, using this invariance we get a family of ODE's. Knowing the importance of reducing the order of these ODE's, we have derived *potential* symmetries for them. We have found:

- ODE's, such as (45), which do not admit point symmetries, but whose order can be reduced by using a potential symmetry.
- ODE's, such as (12),(22) and (38), which just admit a one-parameter Lie group of point symmetries and the corresponding reduced equations do not admit Lie symmetries. Nevertheless we have used potential symmetries to reduce the order by two as well as to linearize them.

Similar results, that will appear in a separated paper [4], have been obtained for a generalized form of equation (11).

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