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C^∞-symmetries and non-solvable symmetry algebras

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If an ordinary differential equation admits the non-solvable Lie algebra $\mathrm{sl}(2, \mathbb{R})$ as symmetry algebra then, when the classical Lie method of reduction is applied, at least one of its generators cannot be used to obtain a second order reduction. In this paper it is proved that these generators can be recovered as C^∞-symmetries of the reduced equations. These C^∞-symmetries can be used to new-order reductions if the order of the last reduced equations is higher than one. As a consequence, a classification of the third-order equations that admit $\mathrm{sl}(2, \mathbb{R})$ as symmetry algebra is given. This step by step method of reduction is applied to the Chazy equation.

Keywords: differential equation; lie groups; symmetry.

1. Introduction

It is well known (Olver, 1986; Ovsiannikov, 1982; Stephani, 1989) that if an nth-order differential equation admits a k-dimensional Lie algebra, $\mathcal{G}$, as symmetry algebra then its general solution can be obtained by means of the general solution of an $(n-k)$th-order reduced equation and the solution of a kth-order auxiliary equation. If $\mathcal{G}$ is solvable, then the general solution of the corresponding auxiliary equation can be obtained by k successive quadratures. Nevertheless, if $\mathcal{G}$ is non-solvable, this step by step method of reduction is no longer applicable. The main reason for this is that, in certain stage of the reduction process, at least one of the generators of $\mathcal{G}$ cannot be used to proceed with the order reductions. In this case we will say, roughly speaking, that the corresponding symmetries have been lost, or that these generators are lost symmetries, for the reduced equations.

In the literature, many recent studies about lost symmetries can be found (Abraham-Shrauner & Guo, 1992, 1993; Abraham-Shrauner, 1993, 1995; Guo & Abraham-Shrauner, 1993; Govinder & Leach, 1995, 1997; Abraham-Shrauner *et al.*, 1995a,b; Leach *et al.*, 1999). These lost symmetries are called type I hidden symmetries and they are difficult to evaluate because there is no general method for determining them. Some indirect methods for hidden symmetries have been introduced (Abraham-Shrauner, 1995) and some non-local symmetries for some specific ordinary differential equations have been studied (Krause, 1994; Govinder & Leach, 1995, 1997). Previously, Olver (1986) pointed out, in two examples, the usefulness of what he called exponential vector fields (although they are not well-defined vector fields), that can be considered the origin of the hidden symmetries theory.

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The importance of the study of hidden symmetries lies in the fact that they can be used to reduce the order of differential equations for which the classical Lie method is not applicable. As an example, González-López (1988) introduced a family of second-order differential equations that can be integrated, but whose point symmetry group is trivial. There are many recent works that try to explain the integrability of these equations (Govinder & Leach, 1995, 1997; Abraham-Shrauner *et al.*, 1995a; Leach *et al.*, 1999). The central principle of these works is to increase the order of the equation by a transformation which produces a Lie symmetry in the higher-order equation. The main difficulty of this method is that the form of the transformations must be assumed *a priori*, and not all of them are useful.

In Muriel & Romero (2001) we have introduced a new class of symmetries, that strictly includes Lie symmetries, for which there exists an algorithm that lets us reduce the order of an ordinary differential equation. They are called C^{∞}-symmetries, and must satisfy a new prolongation formula that provides a procedure to determine them. Many of the known-order reduction processes, that are not consequence of the existence of Lie symmetries, are a consequence of the invariance of the equation under C^{∞}-symmetries. In particular, the integrability of the family of equations that appears in González-López (1988) has been explained by the existence of non-trivial C^{∞}-symmetries. We have also found some ordinary differential equations whose Lie symmetries are trivial and have no obvious order reductions, but can be completely integrated using the new class of symmetries.

Type I hidden symmetries can be recovered as C^{∞}-symmetries of the reduced equations and, in particular, the C^{∞}-symmetries that are consequence of the invariance of the equation under exponential vector fields can be calculated through a well-defined algorithm. While hidden symmetries are manifested as non-local symmetries of the reduced equations, whose coordinate functions depend on integrals of the dependent and independent variables, the C^{∞}-symmetries of an equation are well-defined vector fields on the space of the variables of the equation.

In this paper, we consider nth-order differential equations that are invariant under the non-solvable Lie group $\mathrm{SL}(2, \mathbb{R})$. A base of generators $\{X_1, X_2, X_3\}$ of the Lie algebra $\mathrm{sl}(2, \mathbb{R})$ can always be chosen such that the corresponding Lie brackets are given by

$$\begin{aligned} [X_1, X_2] &= X_1, \\ [X_1, X_3] &= 2X_2, \\ [X_2, X_3] &= X_3. \end{aligned} \tag{1.1}$$

If we use, for instance, the vector field X_1 (resp. X_3) to reduce the order of the equation, the Lie symmetry X_2 is inheritable as Lie symmetry of the reduced equation, but X_3 (resp. X_1) is not inheritable as Lie symmetry. The worst option is to use X_2 first to reduce the order, because both X_1 and X_3, are not inheritable as Lie symmetries of the equation obtained at the first step of the reduction. We prove here that the lost symmetries can be recovered as C^{∞}-symmetries for the reduced equations, and, in consequence, they can be used to reduce successively the order of the equation by three.

In particular, the method provides a complete classification of the third-order differential equations that admit the unimodular group $\mathrm{SL}(2, \mathbb{R})$ as symmetry group, and we have obtained the simplest form of the first-order equations that appear in the last stage of the reduction process.

This step by step method of reduction is applied to the Chazy equation in the final section of this paper. The Chazy equation is a simple example of an ordinary differential equation whose solutions have a moveable natural boundary (Chazy, 1909). In recent years the Chazy equation has assumed added importance because it appears as a reduction of the self-dual Yang–Mills equations(Chakravarty *et al.* 1990). In Clarkson & Olver (1996) and Clarkson (1997) the Chazy equation is studied from a symmetry point of view. The symmetry group associated to the Chazy equation is the most complicated of the three known actions of $SL(2, \mathbb{C})$ on two-dimensional complex spaces, classified by Lie. Clarkson & Olver (1996) describe a connection between these three actions via the standard prolongation process, and use this to interrelate their differential invariants. It allows them to construct fundamental differential invariants of the most complicated action from those of the basic unimodular action. If the original equation is written in terms of these fundamental invariants, its order is reduced by three. This method leads to a formula for the solution of the Chazy equation in terms of solutions of the Lamé equation, that the authors relate to a hypergeometric equation via an elliptic change of variables.

When the step by step method of reduction based on the C^{∞}-symmetries is applied to the Chazy equation, we obtain a Riccati equation, whose general solution can be expressed in terms of the Legendre functions. The general solution of the Chazy equation can be obtained from the general solution of the Riccati equation via two quadratures.

2. Notation and preliminary results

Let us consider an nth-order ordinary differential equation

$$\Delta(x, u^{(n)}) = 0, \tag{2.2}$$

with $(x, u) \in M$, for some open subset $M \subset X \times U \simeq \mathbb{R}^2$. We denote by $M^{(k)}$ the corresponding k-jet space $M^{(k)} \subset X \times U^{(k)}$, for $k \in \mathbb{N}$. Their elements are $(x, u^{(k)}) = (x, u, u_1, \ldots, u_k)$, where, for $i = 1, \ldots, k$, u_i denotes the derivative of order i of u with respect to x. We assume that the implicit function theorem can be applied to (2.2), and, as a consequence, that this equation can locally be written in the explicit form

$$u_n = \Psi(x, u^{(n-1)}). \tag{2.3}$$

The vector field

$$A_{(x,u)} = \frac{\partial}{\partial x} + u_1 \frac{\partial}{\partial u} + \cdots + \Psi(x, u^{(n-1)}) \frac{\partial}{\partial u_{n-1}} \tag{2.4}$$

will be called the vector field associated to (2.3).

2.1 *Lie symmetries and order reductions*

It is well known (Stephani, 1989) that a vector field X on M is a Lie symmetry of the equation (2.3) if and only if there exists a function $\rho \in C^{\infty}(M^{(1)})$ such that

$$[X^{(n-1)}, A_{(x,u)}] = \rho A_{(x,u)}, \tag{2.5}$$

where $X^{(n-1)}$ denotes the usual $(n-1)$th prolongation of the vector field X.

A Lie symmetry X can be used to reduce the order of the equation by one: we introduce variables $\{y = y(x,u), \alpha = \alpha(x,u)\}$ such that the vector field X can be written as $X = \partial/\partial\alpha$, in some open set of variables $\{y, \alpha\}$, that will also be denoted by M. Let us observe that we have used the same letter X for the vector field written in variables $\{y, \alpha\}$. This should not cause misunderstanding: in what follows, the set of variables that must be used will be clear from the context.

Since X is a Lie symmetry of (2.3) it can be written in terms of variables $\{y, \alpha\}$ in the form

$$\alpha_n = \Phi(y, \alpha_1, \alpha_2, \dots, \alpha_{n-1}). \tag{2.6}$$

It can easily be checked that the vector field associated to (2.6) is the vector field

$$A_{(y,\alpha)} = \frac{1}{D_x(y(x,u))} A_{(x,u)}, \tag{2.7}$$

written in the new variables, where D_x denotes the total derivative operator with respect to x.

If we set $w = \alpha_1$ in (2.6) we obtain a reduced equation

$$w_{n-1} = \Phi(y, w, w_1, \dots, w_{n-2}), \tag{2.8}$$

where (y, w) are in some open set $M_1 \subset \mathbb{R}^2$.

Let $\pi_X^{(k)} : M^{(k)} \to M_1^{(k-1)}$ be the projection $(y, \alpha, \alpha_1, \dots, \alpha_k) \mapsto (y, w, \dots, w_{k-1}) = (y, \alpha_1, \dots, \alpha_k)$, for $k \in \mathbb{N}$. A vector field V on $M^{(k)}$ will be called $\pi_X^{(k)}$-projectable if

$$[X^{(k)}, V] = f X^{(k)} \tag{2.9}$$

for some function $f \in C^\infty(M^{(k)})$. This implies that V must take the following form in the variables $\{y, \alpha^{(k)}\}$:

$$V = \xi(y, \alpha_1, \dots, \alpha_k)\frac{\partial}{\partial y} + \eta(y, \alpha, \alpha_1, \dots, \alpha_k)\frac{\partial}{\partial \alpha} + \sum_{i=1}^{k} \eta_i(y, \alpha_1, \dots, \alpha_k)\frac{\partial}{\partial \alpha_i}. \tag{2.10}$$

The $\pi_X^{(k)}$-projection of V on $M_1^{(k-1)}$ is the vector field

$$(\pi_X^{(k)})_*(V) = \xi(y, w, \dots, w_{k-1})\frac{\partial}{\partial y} + \sum_{i=1}^{k} \eta_i(y, w, \dots, w_{k-1})\frac{\partial}{\partial w_{i-1}}. \tag{2.11}$$

With this definition, it can be checked that the vector field $A_{(y,\alpha)}$ is $\pi_X^{(n-1)}$-projectable and its projection is the vector field $A_{(y,w)}$ associated to the reduced equation (2.8).

2.2 C^∞-*symmetries and order reductions*

The concept of Lie symmetry for an ordinary differential equation can be generalized in several ways: conditional symmetries, Lie–Bäcklund symmetries, etc. In Muriel & Romero (2001) we have introduced the concept of C^∞-symmetry. This concept is somewhat similar to the concept of Lie symmetry, but it is based on a different way of prolonging vector fields.

DEFINITION 2.2.1 *New prolongation formula.* Let $X = \xi(x,u)\partial/\partial x + \eta(x,u)\partial/\partial u$ be a vector field defined on M, and let $\lambda \in C^\infty(M^{(1)})$ be an arbitrary function. The *λ-prolongation of order n of X*, denoted by $X^{[\lambda,(n)]}$, is the vector field defined on $M^{(n)}$ by

$$X^{[\lambda,(n)]} = \xi(x,u)\frac{\partial}{\partial x} + \sum_{i=0}^{n} \eta^{[\lambda,(i)]}(x,u^{(i)})\frac{\partial}{\partial u_i}, \tag{2.12}$$

where $\eta^{[\lambda,(0)]}(x,u) = \eta(x,u)$ and

$$\begin{aligned}\eta^{[\lambda,(i)]}(x,u^{(i)}) = {} & D_x(\eta^{[\lambda,(i-1)]}(x,u^{(i-1)})) - D_x(\xi(x,u))u_i \\ & + \lambda(\eta^{[\lambda,(i-1)]}(x,u^{(i-1)}) - \xi(x,u)u_i)\end{aligned} \tag{2.13}$$

for $1 \leqslant i \leqslant n$.

DEFINITION 2.2.2 Let $\Delta(x,u^{(n)}) = 0$ be an nth-order ordinary differential equation. We will say that a vector field X, defined on M, is a $C^\infty(M^{(1)})$-*symmetry* of the equation if there exists a function $\lambda \in C^\infty(M^{(1)})$ such that

$$X^{[\lambda,(n)]}(\Delta(x,u^{(n)})) = 0 \quad \text{when} \quad \Delta(x,u^{(n)}) = 0. \tag{2.14}$$

In this case we will also say that X is a λ-symmetry, or a C^∞-symmetry if there is no confusion.

Recall that if X is a 0-symmetry then X is a classical Lie symmetry.

In Muriel & Romero (2001) it is proved that a vector field X on M is a $C^\infty(M^{(1)})$-symmetry of (2.3) if and only if there exist two functions, $\lambda, \rho \in C^\infty(M^{(1)})$, such that

$$[X^{[\lambda,(n-1)]}, A_{(x,u)}] = \lambda X^{[\lambda,(n-1)]} + \rho A_{(x,u)}. \tag{2.15}$$

We have also proved that if X is a $C^\infty(M^{(1)})$-symmetry then there exists a procedure to reduce the equation to an $(n-1)$th-order equation and a first-order equation.

THEOREM 2.2.1 Let X be a λ-symmetry, with $\lambda \in C^\infty(M^{(1)})$, of the equation $\Delta(x,u^{(n)}) = 0$. Let $y = y(x,u)$ and $w = w(x,u,u_1)$ be two functionally independent invariants of $X^{[\lambda,(n)]}$. The general solution of the equation can be obtained by solving an equation of the form $\Delta_r(y,w^{(n-1)}) = 0$ and an auxiliary equation $w = w(x,u,u_1)$.

The equation $\Delta_r(y,w^{(n-1)}) = 0$ can be constructed as follows: if $y = y(x,u)$ and $w = w(x,u,u_1)$ are two functionally independent invariants of $X^{[\lambda,(n)]}$, then the set

$$y, w, w_1 = \frac{D_x w}{D_x y}, \ldots, w_{n-1} = \frac{D_x w_{n-2}}{D_x y} \tag{2.16}$$

constitutes a complete system of functionally independent invariants of $X^{[\lambda,(n)]}$ and, therefore, the equation can be written in terms of $\{y, w, w_1, \ldots, w_{n-1}\}$.

3. C^∞-symmetries and conservation of symmetries for the Lie algebra $\mathrm{sl}(2, \mathbb{R})$

Let us consider an nth-order differential equation

$$\Delta(x, u^{(n)}) = 0 \tag{3.17}$$

that admits the non-solvable Lie algebra $\mathrm{sl}(2, \mathbb{R})$ as symmetry algebra. A base of generators $\{X_1, X_2, X_3\}$ of $\mathrm{sl}(2, \mathbb{R})$ such that the corresponding Lie brackets are given by (1.1) can always be chosen. Since this Lie algebra is not solvable, the well-established Lie symmetry methods of reduction (see, for instance, Olver, 1986) produce the loss of at least one Lie symmetry at a certain step of the reduction.

Next we study how the lost symmetries can be recovered as C^∞-symmetries for the reduced equations, and how they can be used to reduce successively the order of (3.17) by three when $n > 3$. It is sufficient to study the use of sequences that begin with X_1, because the study of the reduction process that begin with X_3 is similar: the corresponding transformation groups associated to X_1 and X_3 are equivalent under a point transformation.

3.1 *Option A: use of the Lie symmetry X_1*

3.1.1 *First step: use of the Lie symmetry X_1.* With the notation introduced in the previous section, if the Lie symmetry X_1 is used to reduce the order of (3.17), the reduced equation

$$\Delta(y, w^{(n-1)}) = 0 \tag{3.18}$$

is obtained, where $(y, w) \in M_1$.

Since $[X_1^{(k)}, X_2^{(k)}] = X_1^{(k)}$, the vector field $X_2^{(k)}$ is $\pi_{X_1}^{(k)}$-projectable, for $k \in \mathbb{N}$. It can be checked that

$$((\pi_{X_1}^{(1)})_*(X_2^{(1)}))^{(k-1)} = (\pi_{X_1}^{(k)})_*(X_2^{(k)}). \tag{3.19}$$

The corresponding projection on M_1, namely $(\pi_{X_1}^{(1)})_*(X_2^{(1)})$ will be denoted by Y_2. The next result holds.

THEOREM 1 The vector field Y_2 is a Lie symmetry of (3.18).

Proof. We must prove that

$$[Y_2^{(n-2)}, A_{(y,w)}] = \rho_2 \cdot A_{(y,w)} \tag{3.20}$$

for some function $\rho_2 \in C^\infty(M_1^{(1)})$. By (3.19), it can be checked that

$$\begin{aligned}
[Y_2^{(n-2)}, A_{(y,w)}] &= [((\pi_{X_1}^{(1)})_*(X_2^{(1)}))^{(n-2)}, (\pi_{X_1}^{(n-1)})_*(A_{(y,\alpha)})] \\
&= [(\pi_{X_1}^{(n-1)})_*(X_2^{(n-1)}), (\pi_{X_1}^{(n-1)})_*(A_{(y,\alpha)})] \\
&= (\pi_{X_1}^{(n-1)})_*([X_2^{(n-1)}, A_{(y,\alpha)}]).
\end{aligned} \tag{3.21}$$

By hypothesis X_2 is a Lie symmetry of the original equation. In variables $\{y, \alpha^{(n-1)}\}$ we can write

$$[X_2^{(n-1)}, A_{(y,\alpha)}] = \widetilde{\rho}_2 \cdot A_{(y,\alpha)}, \qquad \text{where} \quad \widetilde{\rho}_2 = -A_{(y,\alpha)}(X_2(y)) \in C^\infty(M^{(1)}). \tag{3.22}$$

Since $X_1^{(1)}(\widetilde{\rho}_2) = -A_{(y,\alpha)}(X_1^{(1)}(X_2^{(1)}(y))) = -A_{(y,\alpha)}(0) = 0$, the function $\widetilde{\rho}_2 \in C^\infty(M^{(1)})$ is $X_1^{(1)}$-invariant. Therefore, $\widetilde{\rho}_2 = (\pi_{X_1}^{(1)})^*(\rho_2)$ for some $\rho_2 \in C^\infty(M_1)$, and thus

$$\begin{aligned}[Y_2^{(n-2)}, A_{(y,w)}] &= (\pi_{X_1}^{(n-1)})_*(\widetilde{\rho}_2 A_{(y,\alpha)}) \\ &= \rho_2 \cdot (\pi_{X_1}^{(n-1)})_*(A_{(y,\alpha)}) = \rho_2 \cdot A_{(y,w)}.\end{aligned} \tag{3.23}$$

This concludes the proof. □

Let us denote by $\{z = z(y, w), \beta = \beta(y, w)\}$ the change of variables such that Y_2 takes the canonical form $Y_2 = \partial/\partial\beta$. If we set $\mu = \beta_z$, the order of (3.18) is reduced by one:

$$\Delta(z, \mu^{(n-2)}) = 0, \tag{3.24}$$

where $(z, \mu) \in M_2$ for some open set $M_2 \subset \mathbb{R}^2$.

Observe that the vector field X_3 is not π_{X_1}-projectable, hence at the first stage of the reduction the Lie symmetry X_3 has been lost. The symmetry X_3 can be recovered, at the second step of the reduction, as a C^∞-symmetry of (3.24). In order to prove this, we consider the map π such that the following diagram is commutative:

$$\begin{array}{ccc} M^{(2)} & \xrightarrow{\ \pi\ } & M_2 \\ \pi_{X_1}^{(2)} \downarrow & & \uparrow \pi_{Y_2}^{(1)} \\ M^{(2)} & \xleftarrow[\ \varphi\ ]{} & M_2 \end{array} \tag{3.25}$$

where φ stands for the change of variables $(y, w, w_y) \mapsto (z, \beta, \beta_z)$, and $\pi_{Y_2}^{(1)}(z, \beta, \beta_z) = (z, \mu) = (z, \beta_z)$.

3.1.2 *Second step: use of the Lie symmetry Y_2.* In the following theorem it is proved how the vector field X_3 can be recovered as a C^∞-symmetry of the reduced equation (3.24).

THEOREM 2 There exists a function $f_3 \in C^\infty(M^{(1)})$ such that the vector field $f_3 X_3^{(2)}$ is π-projectable and its projection on M_2, $(\pi)_*(f_3 X_3^{(2)})$ is a C^∞-symmetry of (3.24).

Proof. Let $\widetilde{f}_3 \in C^\infty(M_1)$ be such that $Y_2(\widetilde{f}_3) = -\widetilde{f}_3$ and let us write $f_3 = (\pi_{Y_2}^{(1)})^*(\widetilde{f}_3)$. The function f_3 is, by construction, an $X_1^{(1)}$-invariant function. The vector field $f_3 X_3^{(2)}$ is $\pi_{Y_2}^{(2)}$-projectable because

$$[X_2^{(k)}, f_3 X_3^{(k)}] = 0. \tag{3.26}$$

TABLE 1

$[\cdot,\cdot]$	$X_1^{(n-1)}$	$X_2^{(n-1)}$	$f_3X_3^{(n-1)}$	$\varrho A_{(y,\alpha)}$
$X_1^{(n-1)}$	0	$X_2^{(n-1)}$	$2f_3X_2^{(n-1)}$	0
$X_2^{(n-1)}$		0	0	0
$f_3X_3^{(n-1)}$			0	$\widetilde{\lambda}_3(f_3X_3^{(n-1)}) + \widetilde{\rho}_3(\varrho A_{(y,\alpha)})$

We also have

$$[X_1^{(k)}, f_3X_3^{(k)}] = 2f_3X_2^{(k)} + X_1^{(k)}(f_3)X_3^{(k)} = 2f_3X_2^{(k)}. \tag{3.27}$$

Clearly, expressions (3.26) and (3.27) show that $f_3X_3^{(k)}$ is a $\pi^{(k)}$-projectable vector field. Next we prove that its π-projection on M_2, namely $(\pi)_*(f_3X_3^{(2)})$, is a C^∞-symmetry of (3.24). We first observe that the vector field associated to (3.24) is

$$A_{(z,\mu)} = (\pi^{(n-3)})_*(\varrho A_{(y,\alpha)}), \tag{3.28}$$

where $\varrho = 1/D_y z$ (see (2.7)) and ϱ is $X_1^{(2)}$-invariant.

The vector field X_3 is a Lie symmetry of the original equation, and in variables $\{y, \alpha^{(n-1)}\}$ we can write

$$[f_3X_3^{(n-1)}, \varrho A_{(y,\alpha)}] = \widetilde{\lambda}_3(f_3X_3^{(n-1)}) + \widetilde{\rho}_3(\varrho A_{(y,\alpha)}) \tag{3.29}$$

for some functions $\widetilde{\lambda}_3, \widetilde{\rho}_3$.

Thus, we get

$$\begin{aligned}[(\pi^{(n-3)})_*(f_3X_3^{(n-1)}), (\pi^{(n-3)})_*(\varrho A_{(y,\alpha)})] &= (\pi^{(n-3)})_*([f_3X_3^{(n-1)}, \varrho A_{(y,\alpha)}])\\ &= (\pi^{(n-3)})_*(\widetilde{\lambda}_3(f_3X_3^{(n-1)}) + \widetilde{\rho}_3(\varrho A_{(y,\alpha)})).\end{aligned} \tag{3.30}$$

Table 1 shows the corresponding Lie brackets.

Jacobi's identity applied to the families of vector fields $\{X_2^{(n-1)}, f_3X_3^{(n-1)}, \varrho A_{(y,\alpha)}\}$ and $\{X_1^{(n-1)}, f_3X_3^{(n-1)}, \varrho A_{(y,\alpha)}\}$, leads to the following relations:

$$X_2^{(n-1)}(\widetilde{\lambda}_3)\cdot f_3X_3^{(n-1)} + X_2^{(n-1)}(\widetilde{\rho}_3)\cdot \varrho A_{(y,\alpha)} = 0 \tag{3.31}$$

and

$$X_1^{(n-1)}(\widetilde{\lambda}_3)\cdot f_3X_3^{(n-1)} + X_1^{(n-1)}(\widetilde{\rho}_3)\cdot \varrho A_{(y,\alpha)} + 2(\varrho A_{(y,\alpha)}(f_3) + \widetilde{\lambda}_3 f_3)X_2^{(n-1)} = 0. \tag{3.32}$$

Equation (3.31) shows that the functions $\widetilde{\lambda}_3$ and $\widetilde{\rho}_3$ are $X_2^{(n-1)}$-invariant. From (3.32) it follows that the functions $\widetilde{\lambda}_3$ and $\widetilde{\rho}_3$ are $X_1^{(n-1)}$-invariant.

We set $(\pi^{(n-3)})^*(\lambda_3) = \widetilde{\lambda}_3$, $(\pi^{(n-3)})^*(\rho_3) = \widetilde{\rho}_3$, and thus

$$[(\pi^{(n-3)})_*(f_3 X_3^{(n-1)}), A_{(z,\mu)}] = \lambda_3 (\pi^{(n-3)})_*(f_3 X_3^{(n-1)}) + \rho_3 A_{(z,\mu)}. \tag{3.33}$$

Write $Z_3 = (\pi^{(2)})_*(f_3 X_3^{(2)})$. By (2.15) and (3.33), Z_3 is a λ_3-symmetry of (3.24). □

3.1.3 *Third step: use of the C^∞-symmetry Z_3.* The $(n-2)$th-order (3.24) obtained at the second stage of the reduction process admits the vector field Z_3 as a C^∞-symmetry and it can be used to reduce the order of the equation again.

3.2 *Option B: use of the Lie symmetry X_2*

3.2.1 *First step: use of the Lie symmetry X_2.* If the Lie symmetry X_2 is used to reduce the order of (3.17), we obtain an equation in the form

$$\Delta(y, w^{(n-1)}) = 0. \tag{3.34}$$

The Lie symmetries X_1, X_3 are lost for this reduced equation. However, they can be recovered as C^∞-symmetries. This result is proved in the following theorem.

THEOREM 3 There exist two functions $f_1, f_3 \in C^\infty(M)$ such that the vector fields $f_1 X_1^{(1)}$ and $f_3 X_3^{(1)}$ are $\pi_{X_2}^{(1)}$-projectable. Their projections on M_1, $(\pi_{X_2}^{(1)})_*(f_1 X_1^{(1)})$ and $(\pi_{X_2}^{(1)})_*(f_3 X_3^{(1)})$, are C^∞-symmetries of (3.34).

Proof. Let $f_1, f_3 \in C^\infty(M)$ be such that

$$X_2(f_1) = f_1, \qquad X_2(f_3) = -f_3. \tag{3.35}$$

We have

$$[f_1 X_1^{(k)}, X_2^{(k)}] = 0, \qquad [X_2^{(k)}, f_3 X_3^{(k)}] = 0 \tag{3.36}$$

for $k \in \mathbb{N}$; therefore the vector fields $f_1 X_1^{(1)}, f_3 X_3^{(1)}$ are $\pi_{X_2}^{(1)}$-projectable. Let us denote $Y_i = (\pi_{X_2}^{(1)})_*(f_i X_i^{(1)})$ for $i = 1, 3$.

By using the prolongation formula given in Definition 2.2.1, it can be checked that $(f X^{(1)})^{(k)} = f(X)^{[\lambda,(k)]}$, where $\lambda = -D(f)/f$. Hence

$$((\pi_{X_2}^{(1)})_*(f_i X_i^{(1)}))^{(k)} = Y_i^{[\lambda_i,(k)]}, \qquad \lambda_i = \frac{-D_y(f_i)}{f_i} \quad (i = 1, 3). \tag{3.37}$$

Since X_1 and X_3 are Lie symmetries of the original equation, in variables $\{y, \alpha^{(n-1)}\}$ we can write

$$[f_i X_i^{(n-1)}, A_{(y,\alpha)}] = \widetilde{\lambda}_i f_i X_i^{(n-1)} + \widetilde{\rho}_i A_{(y,\alpha)} \quad (i = 1, 3) \tag{3.38}$$

for some functions $\widetilde{\lambda}_i, \widetilde{\rho}_i$. It can be checked, by using Jacobi's identity for the families of vector fields $\{X_2^{(n-1)}, f_1 X_1^{(n-1)}, A_{(y,\alpha)}\}$ and $\{X_2^{(n-1)}, f_3 X_3^{(n-1)}, A_{(y,\alpha)}\}$, that the functions

$\widetilde{\lambda}_i$ and $\widetilde{\rho}_i$ are $X_2^{(1)}$-invariant. We define λ_i and ρ_i by $(\pi_{X_2}^{(n-1)})^*(\lambda_i) = \widetilde{\lambda}_i$ and $(\pi_{X_2}^{(n-1)})^*(\rho_i) = \widetilde{\rho}_i$ for $i = 1, 3$. Therefore,

$$\begin{aligned}[Y_i^{[\lambda_i,(n-1)]}, A_{(y,w)}] &= [(\pi_{X_2}^{(n-1)})_*(f_i X_i^{(n-1)}), (\pi_{X_2}^{(n-1)})_*(A_{(y,\alpha)})] \\ &= (\pi_{X_2}^{(n-1)})_*(\widetilde{\lambda}_i f_i X_i^{(n-1)} + \widetilde{\rho}_i A_{(y,\alpha)}) \\ &= \lambda_i (\pi_{X_2}^{(n-1)})_*(f_i X_i^{(n-1)}) + \rho_i (\pi_{X_2}^{(n-1)})_*(A_{(y,\alpha)}) \\ &= \lambda_i Y_i^{[\lambda_i,(n-1)]} + \rho_i A_{(y,w)}. \end{aligned} \tag{3.39}$$

This concludes the proof. □

As a consequence of the previous theorem, any of the two C^∞-symmetries, Y_1 and Y_3, can be used to reduce the order of (3.34). We only study here the use of the C^∞-symmetry Y_1, because the corresponding study for the vector field Y_3 is similar.

3.2.2 *Second step: use of the C^∞-symmetry Y_1.* We choose a system of coordinates $\{z = z(y, w), \beta = \beta(y, w)\}$ such that Y_1 can be written as $\partial/\partial\beta$. Let $\mu = \mu(z, \beta, \beta_z)$ be an invariant of ${Y_1}^{[\lambda_1,(1)]}$ such that it is functionally independent of z. Thus, we obtain an order reduction for (3.34) by using Y_1, which can be written in explicit form as

$$\mu_{n-2} = \Phi(z, \mu^{(n-3)}), \tag{3.40}$$

where $(z, \mu) \in M_2$ for some open set $M_2 \subset \mathbb{R}^2$. Write $\pi_{Y_1}^{[\lambda_1,(k)]} : M^{(k)} \to M_2^{(k-1)}$ for the map $(z, \beta, \mu, \mu_1, \ldots, \mu_{k-1}) \mapsto (z, \mu, \mu_1, \ldots, \mu_{k-1})$. The vector field associated to (3.40) is

$$A_{(z,\mu)} = (\pi_{Y_1}^{[\lambda_1,(n-2)]})_*(A_{(z,\beta,\mu)}), \tag{3.41}$$

where $A_{(z,\beta,\mu)}$ denotes the vector field

$$A_{(z,\beta,\mu)} = \frac{\partial}{\partial z} + \beta_z \frac{\partial}{\partial \beta} + \mu_1 \frac{\partial}{\partial \mu} + \cdots + \Phi(z, \mu^{(n-3)}) \frac{\partial}{\partial \mu_{n-3}}. \tag{3.42}$$

THEOREM 4 There exists a function $g_3 \in C^\infty(M_1)$ such that $g_3 Y_3$ is $\pi_{Y_1}^{[\lambda_1,(1)]}$-projectable and its projection $(\pi_{Y_1}^{[\lambda_1,(1)]})_*(g_3 Y_3)$ is a C^∞-symmetry of (3.40).

Proof. It can be checked that

$$[Y_1^{[\lambda_1,(k)]}, Y_3^{[\lambda_3,(k)]}] = c_1 Y_1^{[\lambda_1,(k)]} + c_3 Y_3^{[\lambda_3,(k)]} \tag{3.43}$$

for some functions $c_1, c_3 \in C^\infty(M_1)$. Let $g_3 \in C^\infty(M_1)$ be a function such that $Y_1(g_3) = -c_3 g_3$. Then

$$[Y_1^{[\lambda_1,(k)]}, g_3 Y_3^{[\lambda_3,(k)]}] = g_3 c_1 Y_1^{[\lambda_1,(k)]} \tag{3.44}$$

and, hence, $g_3 Y_3^{[\lambda_3,(1)]}$ is $\pi_{Y_1}^{[\lambda_1,(k)]}$-projectable.

Since Y_3 is a C^{∞}-symmetry of (3.34), in variables $\{z, \beta, \mu\}$ we can write

$$[g_3 Y_3^{[\lambda_3,(n-2)]}, A_{(z,\beta,\mu)}] = \widehat{\lambda}_3 g_3 Y_3^{[\lambda_3,(n-2)]} + \widehat{\rho}_3 A_{(z,\beta,\mu)}. \tag{3.45}$$

By Jacobi's identity for the vector fields $\{Y_1^{[\lambda_1,(n-2)]}, g_3 Y_3^{[\lambda_3,(n-2)]}, A_{(z,\beta,\mu)}\}$ we get a functional relation of the form

$$0 = F_1 Y_1^{[\lambda_1,(n-2)]} + F_2 g_3 Y_3^{[\lambda_3,(n-2)]} + F_3 A_{(z,\beta,\mu)}, \tag{3.46}$$

where

$$F_1 = Y_1^{[\lambda_1,(n-2)]}(\widehat{\lambda}_3) \tag{3.47}$$

and

$$F_3 = Y_1^{[\lambda_1,(n-2)]}(\widehat{\rho}_3). \tag{3.48}$$

Thus, the functions $\widehat{\lambda}_3$ and $\widehat{\rho}_3$ are $Y_1^{[\lambda_1,(n-2)]}$-invariant. Hence, if λ_3' and ρ_3' are such that $(\pi^{(n-2)})^*(\lambda_3') = \widehat{\lambda}_3$ and $(\pi^{(n-2)})^*(\rho_3') = \widehat{\rho}_3$, we finally obtain

$$[(\pi_{Y_1}^{[\lambda_1,(n-2)]})_*(g_3 Y_3^{[\lambda_3,(n-2)]}), A_{(z,\mu)}] = \lambda_3'(\pi_{Y_1}^{[\lambda_1,(n-2)]})_*(g_3 Y_3^{[\lambda_3,(n-2)]}) + \rho_3' A_{(z,\mu)}. \tag{3.49}$$

Therefore $(\pi_{Y_1}^{[\lambda_1,(1)]})_*(g_3 Y_3^{[\lambda_3,(1)]})$ is a λ_3'-symmetry of (3.40), that will be denoted by Z_3. □

3.2.3 *Third step: use of the C^{∞}-symmetry Z_3.* The $(n-2)$th-order equation obtained at the second stage, once we have used the Lie symmetry X_2 and the C^{∞}-symmetry Y_1, admits the vector field Z_3 as a C^{∞}-symmetry, that can be used to reduce again the order of the equation.

3.3 *Recovery of solutions*

The method of reduction by using C^{∞}-symmetries allows us to reduce the order of any equation admitting sl$(2, \mathbb{R})$ as symmetry algebra by three successive order-one reductions. As a consequence, the general solution of the original equation can be obtained through the reduced one by solving three auxiliary first-order equations. Moreover, if the order reduction process is carried out by following option A, two of these three first-order equations can be solved by integration. From the general solution of (3.24), namely $\mu = H(z)$, we obtain the general solution of (3.18) by integrating with respect to $z : \beta = \int H(z)\,\mathrm{d}z = G(z)$. From $\beta(y, w) = G(z(y, w))$, w can locally be expressed as $w = \widetilde{G}(y)$ and the general solution of (3.17) is obtained by integration with respect to $y : \alpha = \int \widetilde{G}(y)\,\mathrm{d}y = F(y)$, that is, $\alpha(u, x) = F(y(x, u))$ is the general solution of (3.34). The solution of (3.24) is obtained from the solution of the last reduced equation by solving the corresponding first-order auxiliary equation of Theorem 2.2.1.

In the case of option B, the auxiliary equation that allows us to recover the general solution of (3.34) from the solution of (3.40) can be solved by integration with respect to y.

If the order of the original equation is three, after two order reductions, a first-order differential equation is obtained. At this last step of the reduction, the Lie symmetry that has been lost can be recovered as a C^∞-symmetry. In the following section, we show how the method of the C^∞-symmetries is used to classify the first-order equations that appear in the last step of the reduction process.

4. General method to solve a third-order equation admitting the non-solvable symmetry algebra sl(2, $\mathbb{R}$)

Let

$$\Delta(x, u^{(3)}) = 0 \tag{4.50}$$

be an arbitrary third-order equation admitting sl(2, $\mathbb{R}$) as symmetry algebra.

There are four different actions of the group SL(2, $\mathbb{R}$) on a two-dimensional real manifold (González-López *et al.*, 1992). Each one of these actions can be modelled by the transformation group generated by the following vector fields:

$$\textit{Case } 1\colon X_1 = \frac{\partial}{\partial x}, \qquad X_2 = x\frac{\partial}{\partial x}, \qquad X_3 = x^2\frac{\partial}{\partial x}, \tag{4.51}$$

$$\textit{Case } 2\colon X_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \quad X_2 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, \quad X_3 = x^2\frac{\partial}{\partial x} + u^2\frac{\partial}{\partial u}, \tag{4.52}$$

$$\textit{Case } 3\colon X_1 = \frac{\partial}{\partial x}, \qquad X_2 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, \quad X_3 = x^2\frac{\partial}{\partial x} + 2xu\frac{\partial}{\partial u}, \tag{4.53}$$

$$\textit{Case } 4\colon X_1 = \frac{\partial}{\partial x}, \qquad X_2 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, \quad X_3 = (x^2 - u^2)\frac{\partial}{\partial x} + 2xu\frac{\partial}{\partial u}. \tag{4.54}$$

By means of a change of variables in (4.50), we can assume that the symmetry algebra is generated by one of the vector fields $\{X_1, X_2, X_3\}$ given in (4.51)–(4.54). For each of these four cases we have studied the different reduction processes described in the previous section.

The results are summarized in Tables 2–5. The variables x, u are used for the original equation; the variables y, α are the coordinates such that the vector field used in the first reduction can be written in the canonical form $\partial/\partial\alpha$. We set $\alpha_y = w$ and thus y, w are the independent and dependent variables of the equation obtained at the first stage of the reduction. In the second stage of the reduction, we use variables z, β, μ such that the corresponding vector field is of the form $\partial/\partial\beta$. Remember that μ coincides with β_z only in the case when the vector field is a Lie symmetry, otherwise μ is a first-order invariant. In the boxes corresponding to inherited symmetries we show the symmetries, and the corresponding function λ for which they are λ-symmetries. If $\lambda = 0$, the symmetries are Lie symmetries. Every symmetry has been obtained by the method described in the previous theorems. We have included an Appendix where the option B of case 1 is worked out in full detail.

TABLE 2 *Case* 1: $X_1 = \partial/\partial x$, $X_2 = x\partial/\partial x$, $X_3 = x^2\partial/\partial x$

	Use of X_1	Use of X_2
I	$y = u, \alpha = x,$ $Y_2 = w\partial/\partial w$ $\lambda_2 = 0$	$y = u, \alpha = \ln(x),$ $Y_1 = -w\partial/\partial w$ $\lambda_1 = -w$
	Use of Y_2	Use of Y_1
II	$z = y, \beta = \ln(w),$ $\mu = \beta_z$ $Z_3 = 2\partial/\partial\mu,$ $\lambda_3 = \mu$	$z = y, \beta = -\ln(w),$ $\mu = \beta_z - e^{-\beta}$ $Z_3 = -2\partial/\partial\mu$ $\lambda_3 = -\mu$
	Use of Z_3	Use of Z_3
III	$s = z, r = \frac{1}{2}\mu$	$s = z, r = -\frac{1}{2}\mu$
	Reduced equation $r_s = r^2 + C(s)$	

We have omitted the symmetries corresponding to the vector field X_3 because they can be deduced from the tables: by using the following changes of variables, X_1 is transformed into X_3, X_3 into X_1 and X_2 into $-X_2$:

$$\begin{aligned} &\textit{Case } 1: \bar{x} = -\frac{1}{x}, \quad \bar{u} = u; \qquad &&\textit{Case } 2: \bar{x} = -\frac{1}{x}, \quad \bar{u} = -\frac{1}{u}; \\ &\textit{Case } 3: \bar{x} = -\frac{1}{x}, \quad \bar{u} = -\frac{u}{x^2}; \qquad &&\textit{Case } 4: \bar{x} = -\frac{x}{x^2+u^2}, \quad \bar{u} = \frac{u}{x^2+u^2}. \end{aligned} \tag{4.55}$$

In every one of cases 1–4, $C(s)$ is a function that depends on the original equation. We emphasize that the reduction process has been performed step by step and, as a consequence, the general solution of the original equation can be obtained from the reduced one by solving two first-order differential equations. In the case of option A, these two equations can be solved directly by integration, as was stated in Section 3.3.

In the case of option B, one of these equations can be solved by integration. If $\mu = H(z)$ denotes the general solution of the last reduced equation, it can be checked that the other corresponding auxiliary first-order equations are the following:

Case 1: $\beta_z - e^{-\beta} = H(z)$,
Case 2: $(1 - e^{\beta} + e^{\beta}\beta_z)H(z) = e^{\beta}\beta_z$,
Cases 3, 4: $\beta_z = H(z)(\beta - z)$.

The auxiliary equations obtained for Cases 3 and 4 are linear, and the equations corresponding to Cases 1 and 2 can easily be transformed into linear first-order equations. In conclusion, the step by step process of reduction allows us to recover the general solution of the original equations by integration or by solving first-order equations that are linear.

TABLE 3 *Case* 2: $X_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial u}, X_2 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, X_3 = x^2\frac{\partial}{\partial x} + u^2\frac{\partial}{\partial u}$

	Use of X_1	Use of X_2
I	(1.A) $y = x - u, \alpha = x,$ $Y_2 = y\frac{\partial}{\partial y}$ $\lambda_2 = 0$	(1.B) $y = \frac{x}{u}, \alpha = \ln(u),$ $Y_1 = (1-y)\frac{\partial}{\partial y} - w^2(-1+y)\frac{\partial}{\partial w}$ $\lambda_1 = -w$
	Use of Y_2	Use of Y_1
II	(2.A) $z = w, \beta = \ln(y),$ $\mu = \beta_z$ $Z_3 = -2\,(-1+z)\,z\frac{\partial}{\partial z} + \mu\,(-3-2\,(-3+\mu)\,z + 2\,\mu\,z^2)\frac{\partial}{\partial \mu},$ $\lambda_3 = \mu$	(2.B) $z = y + \frac{1}{w}, \beta = -\ln(1-y),$ $\mu = \dfrac{e^{\beta}\beta_z}{1 - e^{\beta} + e^{\beta} z}$ $Z_3 = -2z\frac{\partial}{\partial z} - \mu(-3+2\mu(-1+z)z)\frac{\partial}{\partial \mu}$ $\lambda_3 = -\mu$
	Use of Z_3	Use of Z_3
III	(3.A) $s = \frac{1}{\mu}(1 + 2\,\mu\,z - 6\,\mu\,z^2 + 4\,\mu\,z^3)((-1+z)z)^{-3/2}, r = (\frac{z}{-1+z})^{1/2}$	(3.B) $s = \frac{z^{-3/2}}{\mu}(1 + 2z\mu(1+z)), r = z^{-1/2}$
	Reduced equation $r_s = (2 + 2r^2 - rs)C(s)$	

TABLE 4 *Case* 3: $X_1 = \frac{\partial}{\partial x}$, $X_2 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}$, $X_3 = x^2\frac{\partial}{\partial x} + 2xu\frac{\partial}{\partial u}$

	Use of X_1	Use of X_2
I	$y = u, \alpha = x,$	$y = \frac{x}{u}, \alpha = \ln(u),$
	$Y_2 = y\frac{\partial}{\partial y}$	$Y_1 = \frac{\partial}{\partial y} + w^2\frac{\partial}{\partial w}$
	$\lambda_2 = 0$	$\lambda_1 = -w$
	Use of Y_2	Use of Y_1
II	$z = w, \beta = \ln(y),$	$z = y + \frac{1}{w}, \beta = y,$
	$\mu = \beta_z$	$\mu = \dfrac{\beta_z}{\beta - z}$
	$Z_3 = -2\,z^2\dfrac{\partial}{\partial z} + 2\,\mu\,z\,(3 + \mu\,z)\dfrac{\partial}{\partial \mu}$	$Z_3 = -2z^2\frac{\partial}{\partial z} - 2z\mu(-3 + z\mu]\frac{\partial}{\partial \mu}$
	$\lambda_3 = \mu$	$\lambda_3 = -\mu$
	Use of Z_3	Use of Z_3
III	$s = \dfrac{2}{\mu\,z^3} + \dfrac{1}{z^2}, r = \dfrac{1}{2\,z},$	$s = \dfrac{1}{z^2} - \dfrac{2}{z^3\mu}, r = \dfrac{1}{2z},$
	Reduced equation	
	$r_s = (4r^2 - s)C(s)$	

4.1 *Classification of equations*

As a consequence of the step by step method of reduction, a complete classification of the third order ordinary differential equations that admit the non-solvable Lie algebra $\mathrm{sl}(2, \mathbb{R})$ as symmetry algebra can be carried out. If we write each of the first-order reduced equations, obtained in Tables 2–5, in terms of the original system of coordinates, we obtain the third-order differential equations that admit $\mathrm{sl}(2, \mathbb{R})$ as symmetry algebra. They are shown in Table 6, together with the associated reduced equations.

5. The Chazy equation

In this section we apply the results obtained in the previous sections to the particular case of the Chazy equation. The simplest of the equations introduced by Chazy (1909, 1910, 1911) takes the form

$$v_{xxx} = 2vv_{xx} - 3v_x^2. \tag{5.56}$$

The Chazy equation is important since it is the simplest example of an ordinary differential equation whose solutions have a moveable natural boundary. In recent years the importance of the Chazy equation has increased because it appears as a reduction of the self-dual Yang–Mills equations (Chakravarty *et al.*, 1990).

TABLE 5 *Case* 4: $X_1 = \frac{\partial}{\partial x}, X_2 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, X_3 = (x^2 - u^2)\frac{\partial}{\partial x} + 2xu\frac{\partial}{\partial u}$

	Use of X_1	Use of X_2
I	$y = u, \alpha = x,$ $Y_2 = y\dfrac{\partial}{\partial y}$ $\lambda_2 = 0$	$y = \dfrac{x}{u}, \alpha = \ln(u),$ $Y_1 = \dfrac{\partial}{\partial y} + w^2\dfrac{\partial}{\partial w}$ $\lambda_1 = -w$
	Use of Y_2	Use of Y_1
II	$z = w, \beta = \ln(y),$ $\mu = \beta_z$ $Z_3 = -2(1+z^2)\dfrac{\partial}{\partial z} + 2\mu(3z + \mu z^2 + \mu)\dfrac{\partial}{\partial \mu}$ $\lambda_3 = \mu$	$z = y + \dfrac{1}{w}, \beta = y,$ $\mu = \dfrac{\beta_z}{\beta - z}$ $Z_3 = -2(1+z^2)\dfrac{\partial}{\partial z} - 2\mu(-3z + \mu z^2 + \mu)\dfrac{\partial}{\partial \mu}$ $\lambda_3 = -\mu$
	Use of Z_3	Use of Z_3
III	$s = \dfrac{1}{\mu}(-1 + \mu z + \mu z^3)(1+z^2)^{-3/2}, r = -\frac{1}{2}\arctan(z)$	$s = \dfrac{1}{\mu}(1 + \mu z + \mu z^3)(1+z^2)^{-3/2}, r = -\frac{1}{2}\arctan(z)$
	Reduced equation $r_s = (s + \sin(2r))C(s)$	

TABLE 6

	Original equations	Reduced equations
Case 1	$u_3 = \dfrac{3\,u_2^2}{2\,u_1} - 2\,u_1^3\,C(u)$	$r_s = r^2 + C(s)$
Case 2	$u_3 = \dfrac{3\,u_2^2}{2\,u_1} + \dfrac{u_1^2}{2\,(u-x)^2 C((2\,u_1 + 2\,u_1^2 + u_2\,(-u+x))u_1^{-3/2})}$	$r_s = (2 + 2r^2 - rs)C(s)$
Case 3	$u_3 = \dfrac{-1}{8\,u^2 C(u_1^2 - 2\,u\,u_2)}$	$r_s = (4r^2 - s)C(s)$
Case 4	$u_3 = \dfrac{3\,u_1\,u_2^2}{1+u_1^2} + \dfrac{(1+u_1^2)^2}{2\,u^2 C((1+u_1^2+u\,u_2)(1+u_1^2)^{-3/2})}$	$r_s = (s + \sin(2r))C(s)$

It is well known that the Chazy equation admits a three-dimensional symmetry group with infinitesimal generators:

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = x\frac{\partial}{\partial x} - v\frac{\partial}{\partial v}, \qquad X_3 = x^2\frac{\partial}{\partial x} - (2xv+6)\frac{\partial}{\partial v}. \tag{5.57}$$

The corresponding Lie brackets are given by (1.1). By means of the map $u = x + 6/v$, the Lie algebra (5.57) is mapped to the Lie algebra (4.52):

$$X_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \qquad X_2 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, \qquad X_3 = x^2\frac{\partial}{\partial x} + u^2\frac{\partial}{\partial u}, \tag{5.58}$$

and equation (5.56) becomes

$$(u-x)^2 u_{xxx} = 6\,(-u_x + 2\,{u_x}^2 - {u_x}^3 + u\,u_{xx} + u\,u_x\,u_{xx} - u_{xx}\,x - u_x\,u_{xx}\,x). \tag{5.59}$$

5.1 *Order reductions and recovery of solutions*

The general solutions of the Riccati equations of the form

$$r_s = (2 + 2r^2 - rs)C(s) \tag{5.60}$$

that arise as consequences of the order reduction processes corresponding to symmetry algebras of type (4.53) can be obtained by standard methods. They can be transformed into different well-known types of ordinary differential equations. By means of the map

$$r = -\frac{u_s}{2uC(s)}, \tag{5.61}$$

they take the form of the linear second-order equations (Kamke, 1971)

$$C(s)u_{ss} + (sC(s)^2 - C'(s))u_s + 4uC(s)^3 = 0. \tag{5.62}$$

When the changes of variables (1.A)–(3.A) in Table 3 are used to reduce the order of the Chazy equation following the sequence $X_1 \rightarrowtail Y_2 \rightarrowtail Z_3$, we get the following Riccati equation:

$$2 + 2\,r^2 - r\,s + 3\,r_s\,(-16 + s^2) = 0. \tag{5.63}$$

The equation obtained through the changes of variables (1.B)–(3.B), that is, following the sequence $X_2 \rightarrowtail Y_1 \rightarrowtail Z_3$, is

$$2 + 2r^2 - rs - 3r_s(-16 + s^2) = 0 \tag{5.64}$$

which can be transformed into (5.63) by changing r into $1/r$. The reduction processes that follow the sequences $X_3 \rightarrowtail Y_2 \rightarrowtail Z_3$ and $X_2 \rightarrowtail Y_3 \rightarrowtail Z_1$, and use the changes of variables of Case 2 given in (4.55), lead to the same equation (5.63).

For (5.63) where $C(s) = -1/(3(-16 + s^2))$, (5.62) takes the form

$$9(-16 + s^2)^2 u_{ss} + 15s(-16 + s^2)u_s + 4u = 0. \tag{5.65}$$

We prove now that the general solution of (5.65) can be expressed in terms of the Legendre functions. Denote by $P_\nu(t)$ and $Q_\nu(t)$ the Legendre functions of the first and second kind, respectively, that are solutions of the Legendre equation

$$(1 - t^2)\rho_{tt} - 2\rho_t t + \nu(\nu + 1)\rho = 0. \tag{5.66}$$

It can be checked that the general solution of (5.63) can be expressed as follows:

$$r(s) = \frac{k_2 Q_{-1/6}\left(\frac{1}{4}\right) + k_1 P_{-1/6}\left(\frac{1}{4}\right)}{k_2 Q_{-5/6}\left(\frac{1}{4}\right) + k_1 P_{-5/6}\left(\frac{1}{4}\right)}. \tag{5.67}$$

From the general solution of (5.63), and through the change of variables (3.A), which appears in Table 3, the step by step method of reduction allows us to obtain the general solution of the Chazy equation by two successive integrations (see the changes of variables (2.A) and (1.A) in Table 3).

The general solution of the reduced equation obtained by following option B is $\widetilde{r}(s) = 1/r(s)$, where $r(s)$ is given by (5.67). In this case, in order to obtain the general solution of the Chazy equation, we must solve the auxiliary first-order differential equation that appears in Section 4.1:

$$(1 - e^\beta + e^\beta \beta_z)H(z) = e^\beta \beta_z. \tag{5.68}$$

This equation can easily be transformed into the linear first-order equation

$$\bar{\beta}_z = (1 + \bar{\beta}(z - 1))H(z), \tag{5.69}$$

where $\bar{\beta} = e^\beta$.

6. Conclusions

When the classical Lie method is used to reduce the order of any ordinary differential equation admitting the three-dimensional non-solvable Lie algebra sl(2, $\mathbb{R}$) as symmetry algebra, then at least one of its generators is lost in the reduction process.

Nevertheless, in this paper we have proved that the method of reduction by using C^∞-symmetries can be applied to carry out three successive order-one reductions, if the order of the original equation is $n > 3$.

If $n = 3$, after two order reductions a first-order differential equation is obtained. At this last step of the reduction, the Lie symmetry that has been lost can be recovered as a C^∞-symmetry. This fact allows us to give a complete classification of the third-order equations that admit $\mathrm{sl}(2, \mathbb{R})$, as well as the corresponding reduced equations (first-order equations).

The main consequence of this step by step method of reduction is that the general solution of the original equation can be obtained from the reduced one by solving two first-order differential equations. In the case of option A, these two equations can be solved directly by integration. In the case of option B, one of these equations can be solved by integration, and the other one can easily be transformed into a linear first-order equation.

Finally, in order to illustrate the main results, we have applied our method to the Chazy equation. The step by step method of reduction leads to a Riccati equation, whose general solution can be expressed in terms of the Legendre functions.

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Appendix

In this appendix we work out in more detail one of the cases of Section 4. Since option A is the traditional method of reduction (except that Z_3 is not a Lie symmetry), we study option B of Case 1 to show how the method works in practice.

First step, use of X_2. The vector field $X_2 = x\partial/\partial x$ can be written in the canonical form $X_2 = \partial/\partial\alpha$ in coordinates $y = u$, $\alpha = \ln(x)$. The first prolongations of the vector fields in variables $\{y, \alpha, \alpha_y\}$ are given by

$$X_1^{(1)} = \mathrm{e}^{-\alpha}\frac{\partial}{\partial\alpha} - \mathrm{e}^{-\alpha}\alpha_y\frac{\partial}{\partial\alpha_y}, \qquad X_2^{(1)} = \frac{\partial}{\partial\alpha}, \qquad X_3^{(1)} = \mathrm{e}^{\alpha}\frac{\partial}{\partial\alpha} + \mathrm{e}^{\alpha}\alpha_y\frac{\partial}{\partial\alpha_y}. \quad \text{(A.1)}$$

A pair of functions f_1 and f_3 of Theorem 3 such that $X_2(f_1) = f_1$ and $X_2(f_3) = -f_3$ is given, in coordinates $\{y, \alpha\}$, by $f_1 = \mathrm{e}^{\alpha}$ and $f_3 = \mathrm{e}^{-\alpha}$. The corresponding vector fields

$$f_1X_1^{(1)} = \frac{\partial}{\partial\alpha} - \alpha_y\frac{\partial}{\partial\alpha_y}, \qquad f_3X_3^{(1)} = \frac{\partial}{\partial\alpha} + \alpha_y\frac{\partial}{\partial\alpha_y} \quad \text{(A.2)}$$

are $\pi^{(1)}_{X_2}$-projectable (Theorem 3) and the projections on M_1 are given, in variables $\{y, w = \alpha_y\}$, by $Y_1 = -w\partial/\partial w$ and $Y_3 = w\partial/\partial w$. The vector fields Y_1 and Y_3 are C^{∞}-symmetries for the functions

$$\lambda_1 = -\frac{D_y(f_1)}{f_1} = -\frac{e^{\alpha}\alpha_y}{e^{\alpha}} = -\alpha_y = -w, \qquad \lambda_3 = -\frac{D_y(f_3)}{f_3} = -\frac{e^{-\alpha}\alpha_y}{e^{-\alpha}} = \alpha_y = w. \tag{A.3}$$

The corresponding first prolongations, in variables $\{y, w, w_y\}$, are given by

$$Y_1^{[\lambda_1,(1)]} = -w\frac{\partial}{\partial w} + (w^2 - w_y)\frac{\partial}{\partial w_y}, \qquad Y_3^{[\lambda_3,(1)]} = w\frac{\partial}{\partial w} + (w^2 + w_y)\frac{\partial}{\partial w_y}. \tag{A.4}$$

Second step, use of Y_1. The vector field Y_1 can be written in the canonical form $Y_1 = \partial/\partial\beta$, in coordinates $\{z = y, \beta = -\ln(w)\}$. The first prolongations (A.4), in variables $\{z, \beta, \beta_z\}$, are given by

$$Y_1^{[\lambda_1,(1)]} = \frac{\partial}{\partial\beta} - e^{-\beta}\frac{\partial}{\partial\beta_z}, \qquad Y_3^{[\lambda_3,(1)]} = -\frac{\partial}{\partial\beta} - e^{-\beta}\frac{\partial}{\partial\beta_z}. \tag{A.5}$$

A first-order invariant of $Y_1^{[\lambda_1,(1)]}$ is given by $\mu = \beta_z - e^{-\beta}$. In coordinates $\{z, \beta, \mu\}$:

$$Y_1^{[\lambda_1,(1)]} = \frac{\partial}{\partial\beta}, \qquad Y_3^{[\lambda_3,(1)]} = -\frac{\partial}{\partial\beta} - 2e^{-\beta}\frac{\partial}{\partial\mu}. \tag{A.6}$$

According to the proof of Theorem 4,

$$[Y_1^{[\lambda_1,(1)]}, Y_3^{[\lambda_3,(1)]}] = c_1 Y_1^{[\lambda_1,(1)]} + c_3 Y_3^{[\lambda_3,(1)]}, \tag{A.7}$$

where $c_1, c_3 \in C^{\infty}(M_1)$. In this case it can easily be checked that $c_1 = c_3 = -1$. A function g_3 as in Theorem 4, that satisfies $Y_1(g_3) = -c_3 g_3 = g_3$, is given, in coordinates $\{z, \beta\}$, by $g_3 = e^{\beta}$. The vector field $g_3 Y_3^{[\lambda_3,(1)]}$ is $\pi_{Y_1}^{[\lambda_1,(1)]}$-projectable and its projection, in variables $\{z, \mu\}$, is $Z_3 = -2\partial/\partial\mu$. The vector field Z_3 is a C^{∞}-symmetry and the corresponding function λ_3' of Theorem 4 can be calculated from relation (3.49). It can be checked that, in this case, $\lambda_3' = -\mu$. In variables $\{z, \mu, \mu_z\}$ the first prolongation is given by

$$Z_3^{[\lambda_3',(1)]} = -2\frac{\partial}{\partial\mu} + 2\mu\frac{\partial}{\partial\mu_z}. \tag{A.8}$$

Third step, use of Z_3. The vector field Z_3 can be written in the canonical form $Z_3 = \partial/\partial r$ in coordinates $\{s = z, r = -\frac{1}{2}\mu\}$. The first prolongation in variables $\{s, r, r_s\}$ is given by

$$Z_3^{[\lambda_3',(1)]} = \frac{\partial}{\partial r} + 2r\frac{\partial}{\partial r_s}. \tag{A.9}$$

A first-order invariant of $Z_3^{[\lambda_3',(1)]}$ is $\gamma = r_s - r^2$. In terms of $\{s, \gamma^{(n-3)}\}$, where n is the order of the original equation, we get an $(n-3)$th-order reduced equation, that can be expressed in the form

$$\gamma_{n-3} = C(s, \gamma^{(n-4)}). \tag{A.10}$$

When $n = 3$, instead of (A.10) we obtain an equation of the form $\gamma = C(s)$, that is, we have obtained the general form of the first-order equations that appears after two order reductions:

$$r_s - r^2 = C(s). \tag{A.11}$$