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# KP THEORY OF EGOROV NETS

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*The theory of multicomponent KP hierarchies is used to characterize explicit examples of Egorov nets. A $\bar\partial$ dressing method for Cauchy propagators is found to be particularly efficient.*

## 1. Introduction

Two decades ago, it was found that the theory of orthogonal nets

$$ds^2 = \sum_{i=1}^{M} H_i^2 (du_i)^2$$

is closely related to the theory of integrable systems of the hydrodynamic type in 1+1 dimensions [1–3]. Moreover, a particular type of orthogonal nets (the $\partial$-invariant Egorov nets) was relevant for classifying topological quantum field theories [4]. In this work, we describe some recent developments regarding the application of the KP theory to the theory of Egorov nets. The relevant underlying system of partial differential equations is [5–7]

$$\frac{\partial \beta_{ij}}{\partial u_k} - \beta_{ik}\beta_{kj} = 0, \quad i,j,k = 1,\dots,N, \text{ with } i,j,k \text{ different},$$
$$\frac{\partial \beta_{ij}}{\partial u_i} + \frac{\partial \beta_{ji}}{\partial u_j} + \sum_{\substack{k=1,\dots,N \\ k\neq i,j}} \beta_{ki}\beta_{kj} = 0, \quad i,j = 1,\dots,N, \ i \neq j,$$

where $\beta_{ij} := H_i^{-1}\partial H_j/\partial u_i$, with the conditions

$$\beta_{ij} = \beta_{ji}, \qquad \partial H_i = 0, \qquad \partial := \sum_j \frac{\partial}{\partial u_j}.$$

In this context, an important mathematical structure, the class of Frobenius manifolds, was proposed [2, 4, 8–10]. Locally, a Frobenius manifold is determined [4] by a flat metric,

$$ds^2 = \sum_{i,j=1}^{N} \eta^{ij}\, dx_i\, dx_j,$$

and a commutative associative algebra structure,

$$\partial_i \cdot \partial_j = \sum_k c_{ij}^k(x)\partial_k, \qquad \partial_i := \frac{\partial}{\partial x^i}, \qquad x^i := \sum_k \eta^{ik} x_k,$$

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with a unity $\partial_1$. The metric $ds^2$ must be invariant with respect to this product, and *the deformed connection*

$$\nabla_i X^j := \partial_i X^j + z \sum_k c^j_{ik}(\mathbf{x}) X^k, \tag{1}$$

where $z$ is a spectral parameter, should have zero curvature.

Much of the structure of Frobenius manifolds is encoded in the systems of *deformed flat coordinates* $\theta_k(z, \mathbf{x})$ [4, 8–10] for connection (1)

$$\nabla_i \nabla_j \theta_k = 0$$

or, equivalently,

$$\partial_i \partial_j \theta_k = z \sum_l c^l_{ij}(\mathbf{x}) \partial_l \theta_k. \tag{2}$$

On the other hand, it follows from the assumptions on $(\eta, c^k_{ij}(\mathbf{x}))$ that there exists a function $F = F(\mathbf{x})$ (the *free energy function*) such that

$$c_{ijk} = \partial_i \partial_j \partial_k F,$$

and because of the associativity property of the algebra, $F$ satisfies the Witten–Dijkgraff–E. Verlinde–H. Verlinde (WDVV) equations [11, 12]

$$\sum_{r,s} \partial_i \partial_j \partial_r F \eta^{rs} \partial_s \partial_m \partial_k F = \sum_{r,s} \partial_i \partial_m \partial_s F \eta^{sr} \partial_r \partial_j \partial_k F. \tag{3}$$

In a system of deformed flat coordinates normalized by

$$\theta_i(0, \mathbf{x}) = x_i, \quad i = 1, \ldots, N,$$

Eq. (2) implies [4] that a free energy function can be derived from

$$\partial_i F(\mathbf{x}) = \frac{\partial \theta_i}{\partial z}(0, \mathbf{x}).$$

Furthermore, the coefficients of the expansions

$$\theta_i(z, \mathbf{x}) = \sum_{p \geq 0} h_{i,p}(\mathbf{x}) z^p \tag{4}$$

determine an infinite family of functionals

$$H_{i,p}[\mathbf{x}] := \int h_{i,p+1}(\mathbf{x}) dt,$$

which are in involution with respect to the Poisson bracket

$$\{x^i(t_1), x^j(t_2)\} := \eta^{ij} \delta'(t_1 - t_2).$$

The corresponding Hamiltonian systems constitute an integrable hierarchy of *systems of the hydrodynamic type*.

## 2. KP hierarchies and the Grassmannian

The $N$-component KP hierarchy can be introduced as a family of flows in an infinite-dimensional Grassmannian [13, 14]. Let $D(r)$ and $\gamma(r)$ respectively denote the disk $\{z \in \mathbb{C} : |z| \leq r\}$ and its boundary $\{z \in \mathbb{C} : |z| = r\}$, and let $H_{\gamma(r)}$ be the set of Laurent series

$$\sum_{n=-\infty}^{\infty} a_n z^n$$

with the coefficients $a_n \in M_N(\mathbb{C})$ (the ring of $N \times N$ complex matrices), which converge on the circle $\gamma(r)$. Next, two different Grassmannians $\mathrm{Gr}_{\gamma(r)}$ and $\mathrm{Gr}^*_{\gamma(r)}$ are required.

**Definition 1.** The elements of $\mathrm{Gr}_{\gamma(r)}$ are the subsets $W$ of $H_{\gamma(r)}$ such that

1. $W$ is a $M_N(\mathbb{C})$ left-module and
2. the projection operator $P_+ : W \longrightarrow H^+_{\gamma(r)}$ from $W$ into

$$H^+_{\gamma(r)} = \left\{ w \in H_{\gamma(r)} : w = \sum_{n=0}^{\infty} a_n z^n \right\}$$

is a bijective map.

Similarly, $\mathrm{Gr}^*_{\gamma(r)}$ is given by the subsets $V$ of $H_\gamma$ such that

1*. $V$ is a $M_N(\mathbb{C})$ right-module and

2*. the projection operator $P_+ : V \longrightarrow H^+_{\gamma(r)}$ is a bijective map.

There is a map

$$\mathrm{Gr}_{\gamma(r)} \xrightarrow{\ *\ } \mathrm{Gr}^*_{\gamma(r)}, \quad W \mapsto W^*,$$

such that for each given $W \in \mathrm{Gr}_{\gamma(r)}$, the subspace $W^* \in \mathrm{Gr}^*_{\gamma(r)}$ is the set of those $v \in H_{\gamma(r)}$ satisfying

$$\int_{\gamma(r)} w(z) v(z)\, dz = 0 \quad \forall w \in W.$$

Typical elements in the Grassmannians are provided by the $\bar{\partial}$ method. Given an appropriate $N \times N$ matrix distribution $R(z, z')$ with support in $D(r) \times D(r)$, the corresponding $W \in \mathrm{Gr}_{\gamma(r)}$ is the set of restrictions to $\gamma(r)$ of the solutions $w(z)$ of

$$\frac{\partial w}{\partial \bar{z}}(z) = \int_{D(r)} w(z') R(z', z)\, d^2 z'.$$

Then $W^* \in \mathrm{Gr}^*_{\gamma(r)}$ solves

$$\frac{\partial v}{\partial \bar{z}}(z) = -\int_{D(r)} R(z, z') v(z')\, d^2 z'.$$

**Definition 2.** Given $W \in \mathrm{Gr}_{\gamma(r)}$, its *associated KP Baker function* is the unique element $\psi \in W$ that admits a convergent expansion of the form

$$\psi(z, \mathbf{u}) = \chi(z, \mathbf{u}) \psi_0(z, \mathbf{u}), \qquad \chi(z, \mathbf{u}) = I_N + \sum_{n \geq 1} \frac{a_n(\mathbf{u})}{z^n}, \quad \mathbf{u} \in \mathcal{U}(r)^N, \quad z \in \gamma(r).$$

Also, the *adjoint KP Baker function* is the unique element $\psi^* \in W^*$ with the expansion

$$\psi^*(z, \mathbf{u}) = \psi_0(z, \mathbf{u})^{-1} \chi^*(z, \mathbf{u}), \qquad \chi^*(z, \mathbf{u}) = I_N + \sum_{n \geq 1} \frac{a_n^*(\mathbf{u})}{z^n}, \quad \mathbf{u} \in \mathcal{U}(r)^N, \quad z \in \gamma(r).$$

Here, $I_N := \sum_{i=1}^N E_i$ denotes the identity matrix in $M_N(\mathbb{C})$. We note that for all $\mathbf{u} \in \mathcal{U}(r)^N$, both $\chi(z, \mathbf{u})$ and $\chi^*(z, \mathbf{u})$ are analytic functions of $z$ on the domain $\mathbb{C} \setminus D(r)$.

The Baker function satisfies the so-called $N$-component KP hierarchy. This hierarchy is an infinite system of linear equations

$$\frac{\partial \psi}{\partial u_{i,n}} = P_{i,n}(\mathbf{u}, \partial)\psi, \quad i = 1, \dots, N, \quad n \geq 1, \quad \partial := \partial_1 + \cdots + \partial_N, \tag{5}$$

where $P_{i,n}(\mathbf{u}, \partial)$ is a family of linear differential operators in $\partial$.

The first few members of hierarchy (5) are

$$\frac{\partial \psi}{\partial u_{i,1}} = E_i \partial \psi + [a_1, E_i]\psi, \quad i = 1, \dots, N,$$

which can be rewritten as

$$\frac{\partial \boldsymbol{\psi}_i}{\partial u_k} = \beta_{ik} \boldsymbol{\psi}_k, \quad i \neq k, \tag{6}$$

with

$$\boldsymbol{\psi}_i := (\psi_{i1}, \dots, \psi_{iN}), \quad u_k := u_{k,1}, \quad \beta = a_1.$$

Analogously, the adjoint Baker function satisfies the linear system

$$\frac{\partial \boldsymbol{\psi}_j^*}{\partial u_k} = \boldsymbol{\psi}_k^* \beta_{kj}, \quad j \neq k, \tag{7}$$

where

$$\boldsymbol{\psi}_i^* := \begin{pmatrix} \psi_{1i}^* \\ \vdots \\ \psi_{Ni}^* \end{pmatrix}.$$

The compatibility of either (6) or (7) implies the Darboux system of equations for a conjugate net. Moreover, (6) and (7) show that for a given set of rotation coefficients $\beta_{ij}$, there is an associated family of conjugate nets with tangent vectors and Lamé coefficients given by $(\mathbf{X}_i)_j := \mathbf{X}_{ij}$ and $H_i = \mathbf{H}_{li}$, $l = 1, \dots, N$, where

$$\mathbf{X}(\mathbf{u}) := \int_{\mathbb{C}} \psi(z, \mathbf{u}) N(z) \, d^2 z, \qquad \mathbf{H}(\mathbf{u}) := \int_{\mathbb{C}} M(z) \psi^*(z, \mathbf{u}) \, d^2 z.$$

Here, $N(z)$ and $M(z)$ are appropriate $N \times N$ matrix distributions.

## 3. The Cauchy propagator

**Definition 3.** Given $W \in \mathrm{Gr}_{\gamma(r)}$, its *associated Cauchy propagator* is the Green's function $\Psi = \Psi(z, z', \mathbf{u})$ of the $\bar{\partial}$ operator,

$$\frac{\partial \Psi}{\partial \bar{z}}(z, z', \mathbf{u}) = \pi \delta(z - z'), \quad z, z' \in \mathbb{C} \setminus D(r), \quad \mathbf{u} \in \mathcal{U}(\infty)^N,$$

satisfying the boundary conditions that

1. for every fixed $\mathbf{u} \in \mathcal{U}(\infty)^N$ and $z' \in \mathbb{C} \setminus D(r)$, the restriction of $\Psi$ to $\gamma(r)$, as a function of $z$, is an element of $W$ and
2. as $z \longrightarrow \infty$,

$$\Psi(z, z', \mathbf{u}) = \mathcal{O}\left(\frac{1}{z}\right)\psi_0(z, \mathbf{u}).$$

The next theorem [2] relates the Cauchy propagator to the Baker function. The notation

$$[z] = ([z]_1, \ldots, [z]_N), \qquad [z]_i = \left(\frac{1}{z}, \ldots, \frac{1}{nz^n}, \ldots\right)$$

is used.

**Theorem 1.** *The Cauchy propagator associated with an element $W$ of $\mathrm{Gr}_{\gamma(r)}$ can be written in terms of the KP wave functions $\psi$ and $\psi^*$ as*

$$\Psi(z, z', \mathbf{u}) = \begin{cases} -\dfrac{1}{z'}\psi^*(z', \mathbf{u})\psi(z, \mathbf{u} + [z']) & \text{for } |z| \leq |z'|, \\ \dfrac{1}{z}\psi^*(z', \mathbf{u} - [z])\psi(z, \mathbf{u}) & \text{for } |z'| \leq |z|. \end{cases}$$

*The entries of $\Psi$ satisfy the differential equation*

$$\frac{\partial \Psi_{jk}}{\partial u_i}(z, z', \mathbf{u}) = \psi^*_{ji}(z', \mathbf{u})\psi_{ik}(z, \mathbf{u}). \tag{8}$$

As a consequence of (8), the net function of the conjugate net with tangent vectors and Lamé coefficients respectively given by $(\mathbf{X}_i)_j := \mathbf{X}_{ij}$ and $H_i = \mathbf{H}_{li}$, $l = 1, \ldots, N$, is given by the $l$th row of the matrix function

$$\mathbf{x} := \int_{\mathbb{C}\times\mathbb{C}} M(z')\Psi(z, z')N(z)\, d^2z\, d^2z' + \mathbf{x}_0,$$

where $\mathbf{x}_0$ is an arbitrary constant matrix.

## 4. Egorov reduction

**Definition 4.** An element $W \in \mathrm{Gr}_{\gamma(r)}$ satisfies *the Egorov reduction* if

1. for every $w \in W$, the function $\tilde{w}(z) := zw(z)$ is also in $W$ and
2. for every $v \in W^*$, the function $\tilde{v}(z) := v(-z)^{\mathrm{t}}$ is in $W$.

The next theorem was proved in [15].

**Theorem 2.** *If $W \in \mathrm{Gr}_{\gamma(r)}$ satisfies the Egorov reduction, then for any nonsingular matrix $\mathcal{N}$, the functions*

$$\theta_i(z, \mathbf{u}) := \left(\mathcal{N}^{\mathrm{t}}\left(\Psi(z, 0, \mathbf{u}) - \frac{1}{z}\right)\mathcal{N}\right)_{1i}, \quad i = 1, \ldots, N, \tag{9}$$

*are a system of normalized deformed flat coordinates for a Frobenius manifold determined by*

1. *the $\partial$-invariant Egorov metric*

$$ds^2 = \sum_{i=1}^{N} H_i^2 (du_i)^2, \quad H_i(\mathbf{u}) := \left(\psi(0, \mathbf{u})\mathcal{N}\right)_{i1},$$

2. *the system of flat coordinates*

$$x_i := \theta_i(0, \mathbf{u}), \quad i = 1, \dots, N,$$

$$ds^2 = \sum_{i,j=1}^{N} \eta^{ij}\, dx_i\, dx_j, \quad \eta = (\mathcal{N}^{\mathrm{t}}\mathcal{N})^{-1},$$

3. *the structure constants*

$$c_{ij}^{l} = \sum_{k=1}^{N} \frac{\partial u_k}{\partial x^i} \frac{\partial u_k}{\partial x^j} \frac{\partial x^l}{\partial u_k}.$$

We note that as a consequence of (9) and (4), every $W \in \mathrm{Gr}_{\gamma(r)}$ that satisfies the Egorov reduction determines a hierarchy of systems of the hydrodynamic type with Hamiltonian densities given by

$$h_{i,p}(\mathbf{x}) = \frac{1}{(p+1)!} \frac{\partial^{p+1}}{\partial z^{p+1}} \big(\mathcal{N}^{\mathrm{t}} z \Psi(z, 0, \mathbf{u}) \mathcal{N}\big)_{1i} \Big|_{z=0}.$$

## 5. Dressing conjugate nets

We now consider the dressing method for conjugate nets [16]. Let $D(r)$ and $D(\tilde{r})$ be two disks centered at the origin with $r < \tilde{r}$. Let $\gamma(r)$ and $\gamma(\tilde{r})$ denote their respective boundaries and $A$ denote the annulus $D(\tilde{r}) - D(r)$.

**Definition 5.** A matrix distribution $R = R(z, z')$ with support in $A \times A$ determines a *dressing transformation*

$$T_R : \mathrm{Gr}_{\gamma(r)} \mapsto \mathrm{Gr}_{\gamma(\tilde{r})}, \quad W \mapsto \widetilde{W},$$

where for every $W \in \mathrm{Gr}_{\gamma(r)}$, the corresponding $\widetilde{W} \in \mathrm{Gr}_{\gamma(\tilde{r})}$ is the set of boundary values on $\gamma(\tilde{r})$ of matrix functions $w = w(z)$ satisfying the $\bar{\partial}$ equation

$$\frac{\partial w}{\partial \bar{z}}(z) = \int_A w(z') R(z', z)\, d^2 z', \quad z \in A,$$

and such that the restriction of $w$ to $\gamma(r)$ is an element of $W$.

For the case of a separable kernel

$$R(z, z') = \pi \sum_{k=1}^{m} \sum_{l=1}^{n} C_{k\ell} f_k(z) g_\ell(z'),$$

the dressing of the Cauchy propagator can be explicitly performed. Here, $C_{k\ell}$ are constant complex $N \times N$ matrices, and $f_k$ and $g_\ell$ are scalar distributions. In order to determine the corresponding transformation, it is useful to introduce the notation

$$\mu_k(z) := \int_A \Psi(z', z) f_k(z')\, d^2 z', \quad k = 1, \dots, m,$$

$$\nu_\ell(z) := \int_A \Psi(z, z') g_\ell(z')\, d^2 z', \quad \ell = 1, \dots, n,$$

$$\omega_{\ell k} := \int_{A \times A} \Psi(z', z'') f_k(z') g_\ell(z'')\, d^2 z'\, d^2 z'', \quad k = 1, \dots, m, \quad \ell = 1, \dots, n.$$

We also define the matrices

$$\boldsymbol{\mu} = (\mu_1, \dots, \mu_m) : A \to M_{N\times mN}(\mathbb{C}), \qquad \boldsymbol{\nu} = \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_n \end{pmatrix} : A \to M_{nN\times N}(\mathbb{C}),$$

$$\boldsymbol{C} = (C_{kl}) \in M_{mN\times nN}(\mathbb{C}), \qquad \boldsymbol{\omega} = (\omega_{\ell k}) \in M_{nN\times mN}(\mathbb{C}).$$

It then follows that [16]

$$\widetilde{\Psi}(z, z') = \Psi(z, z') + \boldsymbol{\mu}(z')\boldsymbol{C}(1 - \boldsymbol{\omega}\boldsymbol{C})^{-1}\boldsymbol{\nu}(z).$$

## 6. Dressing Egorov nets

The Egorov reduction is preserved under the dressing if

$$v \in \widetilde{W}^* \Rightarrow v^{\mathrm{t}}(-z) \in \widetilde{W}.$$

For kernels satisfying

$$zR(z, z') = z'R(-z', -z)^{\mathrm{t}},$$

$$R(z, z') = R(-z', -z)^{\mathrm{t}},$$

the corresponding dressing transformations preserve this reduction. Furthermore, these conditions imply

$$R(z, z') = R_0(z)\delta(z - z'), \qquad R_0(z) = R_0(-z)^{\mathrm{t}}.$$

Separable kernels of this type are

$$R_0(z) = \pi \sum_{k=1}^{n} [C_k\delta(z - p_k) + C_k^{\mathrm{t}}\delta(z + p_k)],$$

where $C_k$ are complex $N\times N$ matrices and $p_k \in \mathbb{C}$, i.e.,

$$R(z, z') = \pi \sum_{k=1}^{n} [C_k\delta(z - p_k)\delta(z' - p_k) + C_k^{\mathrm{t}}\delta(z + p_k)\delta(z' + p_k)].$$

The corresponding dressing transformation, which in principle may suffer from singularity problems, becomes well-defined provided

$$C_k^2 = 0, \quad k = 1, \dots, n.$$

Explicit examples of Egorov nets and their corresponding Frobenius manifolds can thus be characterized by dressing the vacuum solution [15]. For example, the free energy function

$$\begin{aligned} F(x_1, \dots, x_N) = {} & \frac{1}{6}x_1^3 + \frac{1}{p}(x_2^2 + \dots + x_N^2) + \frac{x_1}{2}(x_2^2 + \dots + x_N^2) + \\ & + \frac{1}{6p^3}[1 + 2p^2(x_2^2 + \dots + x_N^2)]\sqrt{1 - p^2(x_2^2 + \dots + x_N^2)} - \\ & - \sum_{j=2}^{N} \frac{x_j^2}{2p} \log\Big[ -\frac{2x_j}{c_j} \frac{1 + \sqrt{1 - p^2(x_2^2 + \dots + x_N^2)}}{x_2^2 + \dots + x_N^2} \Big] \end{aligned}$$

satisfies WDVV associativity equations (3).

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