Inverse Problems **17** (2001) 985–998 www.iop.org/Journals/ip PII: S0266-5611(01)17312-0

Rational solutions of the KP equation through symmetry reductions: coherent structures and other solutions

Elena Medina and María José Marín

Departamento de Matemáticas, Universidad de Cádiz, Apt. 40, Puerto Real, 11510 Cádiz, Spain

Received 20 September 2000, in final form 13 June 2001

Abstract

Some singular rational travelling wave solutions of the Kadomtsev–Petviashvili (KP) equation are determined by making use of the theory of symmetry reductions. Solutions of KP of the form $u(x - c_1t, y - c_2t)$, c_1, c_2 constants, satisfy the equation $(-c_1\bar{u}_{\bar{x}} - c_2\bar{u}_{\bar{y}} + 6\bar{u}\bar{u}_{\bar{x}} + \bar{u}_{\bar{x}\bar{x}\bar{x}})_{\bar{x}} = \bar{u}_{\bar{y}\bar{y}}$, with $\bar{x} = x - c_1t$, $\bar{y} = y - c_2t$ and $\bar{u}(\bar{x}, \bar{y}) = u(x - c_1t, y - c_2t)$. Some nonclassical symmetries of this equation are determined: then, by considering the corresponding ordinary differential equations, some solutions for the KP equation are obtained. But in the case of the well known one-dimensional soliton, all the solutions we construct in this way are rational functions. Moreover, the KP equation reduces to the previous equation with $c_1 = c_2 = 0$ under a more general symmetry group. Thus, we can use the solutions describing coherent structures to construct large families of (x, y)-rational solutions.

(Some figures in this article are in colour only in the electronic version; see www.iop.org)

It is well known that there are a large number of applications, both in physics and mathematics, of the Kadomtsev–Petviashvili (KP) hierarchy. Several methods in the context of integrable systems have been developed in order to analyse it, describe large families of solutions and study its reductions (see e.g. [1–4]). The first member of this hierarchy, the KP equation

$$(u_t + 6uu_x + u_{xxx})_x = u_{yy} \tag{1}$$

describes the evolution of quasi-one-dimensional shallow water waves when effects of the surface tension and the viscosity are negligible [5, 6].

The first purpose of this paper is to apply the theory of symmetry reductions in partial differential equations (PDEs) to find travelling wave solutions for the KP equation. In order to do that we consider solutions of (1) of the form

$$u(x, y, t) = \bar{u}(\bar{x}, \bar{y}), \qquad \bar{x} = x - c_1t, \quad \bar{y} = y - c_2t$$

that is, a coherent structure in $\mathbb{R}^2$ travelling with constant velocity (c_1, c_2). Introducing this ansatz in (1) we find that the new function $\bar{u}(\bar{x}, \bar{y})$ satisfies the equation

$$(-c_1\bar{u}_{\bar{x}} - c_2\bar{u}_{\bar{y}} + 6\bar{u}\bar{u}_{\bar{x}} + \bar{u}_{\bar{x}\bar{x}\bar{x}})_{\bar{x}} = \bar{u}_{\bar{y}\bar{y}}. \tag{2}$$

From now on, we drop the bar in the variables and we write $u(x, y)$ instead of $\bar{u}(\bar{x}, \bar{y})$.

The classical method for finding symmetry reductions of PDEs is the Lie group method of infinitesimal transformations. The fundamental basis of this technique is that, when a differential equation is invariant under a Lie group of transformations, a reduction transformation exists. To apply the classical method to (2) one looks for fields of the form

$$V = \xi(x, y, u)\frac{\partial}{\partial x} + \eta(x, y, u)\frac{\partial}{\partial y} + \phi(x, y, u)\frac{\partial}{\partial u} \tag{3}$$

that leave invariant the set of solutions of (2). The machinery of Lie group theory provides a systematic method to search for these special group invariant solutions. Most of the required theory and description of the method can be found in [7–9]. As a generalization of the classical Lie group method Blumann and Cole [10] developed the nonclassical method. In order to apply this method to (2) one requires only the subset of solutions of (2) that satisfy the surface condition

$$\xi(x, y, u)u_x + \eta(x, y, u)u_y = \phi(x, y, u)$$

to be invariant under the transformation associated with (3). In this case it is still possible to find a reduction transformation.

In this paper we make use of the nonclassical symmetries of equation (2) to reduce it to some ordinary differential equations. Solutions of these equations lead to invariant solutions of (2) and consequently to travelling wave solutions of the KP equation.

We start by looking for nonclassical symmetries admitted by (2). We distinguish two cases depending on $\eta \not\equiv 0$ or $\eta \equiv 0$.

If $\eta \not\equiv 0$ we can assume without lack of generality that $\eta \equiv 1$. Then for the infinitesimals $\xi(x, y, u)$, $\phi(x, y, u)$ we get seven determining equations. When trying to solve these equations we get that

$$\xi(x, y, u) = p_1(y)x + p_2(y)$$
$$\phi(x, y, u) = -2p_1(y)u + r(x, y)$$

where p_1 satisfies the ODE

$$\frac{\mathrm{d}^2 p_1}{\mathrm{d}y^2} + 2p_1\frac{\mathrm{d}p_1}{\mathrm{d}y} - 4p_1^3 = 0 \tag{4}$$

and p_2, r can be obtained from p_1. Observe that equation (4) can be reduced to a quadrature. Indeed, by setting $p_1 = \frac{\mathrm{d}p}{\mathrm{d}y}$, multiplying by e^{-2p} and integrating once we have

$$\left(\frac{\mathrm{d}^2 p}{\mathrm{d}y^2} + 2\left(\frac{\mathrm{d}p}{\mathrm{d}y}\right)^2\right)\mathrm{e}^{-2p} = a, \qquad a \text{ arbitrary constant.}$$

Now, the change of variables $p = 2\ln q$ allows a new integration, which transforms the previous equation into

$$\frac{\mathrm{d}q}{\mathrm{d}y} = \sqrt{\frac{a}{6}q^6 + b}, \qquad b \text{ arbitrary constant.}$$

This equation is equivalent to a quadrature. However, solutions in this way are not useful for reducing equation (2). We restrict ourselves to the particular cases $a = 0$ or $b = 0$, which, taking into account that $p_1 = 2\frac{q'}{q}$, lead to

$$\text{If} \quad a = 0, \quad p_1(y) = \frac{2}{y - y_0}$$
$$\text{If} \quad b = 0, \quad p_1(y) = -\frac{1}{y - y_0}.$$

Taking into account that (2) is an autonomous equation, the translation in the previous solution is not relevant. For the sake of simplicity we take $y_0 = 0$. Thus, we consider for (4) the solutions

$$p_1(y) = 0, \qquad p_1(y) = -\frac{1}{y}, \qquad p_1(y) = \frac{2}{y}.$$

Analysing separately these three cases we have:

Symmetry 1.1. $p_1(y) = 0$ leads to

$$\xi(x, y, u) = k_1 + k_2 y, \qquad \eta(x, y, u) = 1 \qquad \phi(x, y, u) = \frac{k_2^2 y}{3} + \frac{k_1 k_2}{3} - \frac{c_2 k_1}{6} \tag{5}$$

being k_1, k_2 arbitrary constants. It is clear that (5) is a classical symmetry of (2).

Symmetry 1.2. $p_1(y) = -\frac{1}{y}$ leads to

$$\begin{aligned} &\xi(x, y, u) = -\frac{x}{y} + c_2 + k_1 y^4, \qquad \eta(x, y, u) = 1 \\ &\phi(x, y, u) = -\frac{x^2}{y^3} + \frac{c_2 x}{y^2} - \frac{2c_1 + c_2^2}{6y} + \frac{2u}{y} + k_1 \left(\frac{xy^2}{3} - \frac{c_2 y^3}{6} \right) + \frac{2k_1^2 y^7}{3} \end{aligned} \tag{6}$$

where again k_1 is an arbitrary constant.

Symmetry 1.3. $p_1(y) = \frac{1}{2y}$ leads to

$$\begin{aligned} &\xi(x, y, u) = \frac{x}{2y} + \frac{c_2}{4} + k_1 y, \qquad \eta(x, y, u) = 1 \\ &\phi(x, y, u) = -\frac{u}{y} + \frac{4c_1 - c_2^2}{24y} + \frac{k_1 x}{3y} + \frac{2k_1^2 y}{3} - \frac{dk_1}{6}. \end{aligned} \tag{7}$$

We point out that, in the particular case that we choose the arbitrary constant k_1 as zero, the symmetry (7) becomes a classical symmetry.

If $\eta \equiv 0$, we have that $\xi \not\equiv 0$, then we can take without lack of generality $\xi \equiv 1$. In this case we obtain three determining equations for ϕ. From these equations one sees that

$$\phi(x, y, u) = h\left(x - \frac{c_2}{2} y\right) u + \varphi(x, y)$$

where h satisfies the equation

$$h'' + 5hh' + 2h^3 = 0. \tag{8}$$

Proceeding in the same way as for equation (4), we can also reduce this equation to a quadrature. However, also in this case, solutions are not manageable, so we just consider the particular solutions $h(s) = 0$, $h(s) = \frac{2}{s}$ and $h(s) = \frac{1}{2s}$. If we choose $h(s) = 0$ we find

$$\varphi(x, y) = \varphi_1(y)x + \varphi_2(y)$$

with

$$\begin{aligned} &\varphi_1'' - 18\varphi_1^2 = 0 \\ &\varphi_2'' - 18\varphi_1\varphi_2 + c_2\varphi_1' = 0. \end{aligned} \tag{9}$$

The general solution of the second equation in (9) involves inverse functions of elliptic integrals, but it is easily seen that $\varphi_1(y) = 0$ and $\varphi_1(y) = \frac{1}{3y^2}$ verify this equation. Thus, solving the third equation in (9) we have:

Symmetry 2.1.

$$\xi(x, y, u) = 1, \qquad \eta(x, y, u) = 0, \qquad \phi(x, y, u) = k_1 y + k_2 \tag{10}$$

with k_1 and k_2 being arbitrary constants. For the case $k_1 = k_2 = 0$ (10) is a classical symmetry.

Symmetry 2.2.

$$\xi(x, y, u) = 1 \qquad \eta(x, y, u) = 0 \qquad \phi(x, y, u) = \frac{x}{3y^2} + k_1 y^3 - \frac{c_2}{6y} \tag{11}$$

where k_1 is a constant.

If we choose $h(s) = \frac{2}{s}$, the equations for φ can be solved and we have

Symmetry 2.3.

$$\xi(x, y, u) = 1 \qquad \eta(x, y, u) = 0 \qquad \phi(x, y, u) = \frac{2u + \frac{c_2^2 - 4c_1}{12}}{x - \frac{c_2}{2}y} + \frac{8}{(x - \frac{c_2}{2}y)^3}. \tag{12}$$

Finally, if we choose $h(s) = \frac{1}{2s}$, equations for φ become incompatible.

Next, we will use symmetries (5)–(7), (10)–(12) to reduce (2) to the corresponding ODEs. In order to perform these reductions we first need to obtain the similarity variables, i.e. the first integrals of the dynamical system

$$\dot{x} = \xi(x, y, u)$$
$$\dot{y} = \eta(x, y, u)$$
$$\dot{u} = \phi(x, y, u).$$

We proceed in this way and we have:

Reduction 1.1. As was said above the similarity variables can be computed as the first integrals of the dynamical system

$$\dot{x} = k_1 + k_2 y$$
$$\dot{y} = 1$$
$$\dot{u} = \frac{k_2^2 y}{3} + \frac{k_1 k_2}{3} - \frac{c_2 k_1}{6}$$

or, equivalently, as the arbitrary constants in the general solutions of the system

$$\frac{dx}{dy} = k_1 + k_2 y,$$
$$\frac{du}{dy} = \frac{k_2^2 y}{3} + \frac{k_1 k_2}{3} - \frac{c_2 k_1}{6}.$$

For this case, both equations in the previous system are nothing but quadratures. Thus we see that the similarity variables are given by

$$z = x - k_1 y - \frac{k_1 y^2}{2}$$
$$u(x, y) = \frac{k_2^2 y^2}{6} + \left(\frac{k_1 k_2}{3} - \frac{c_2 k_2}{6}\right) y + w(z).$$

Consequently (2) becomes (after integrating once)

$$-\frac{k_2^3}{3} z + k_2 w - (c_1 + c_2 k_1 + k_1^2) w' + 6ww' + w''' = k_3 \tag{13}$$

where k_3 is an arbitrary constant.

Reduction 1.2. As in the previous case, we can obtain the similarity variables z and $w(z)$ as the arbitrary constants in the general solution of the system

$$\frac{dx}{dy} = -\frac{x}{y} + c_2 + k_1 y^4$$
$$\frac{du}{dy} = -\frac{x^2}{y^3} + \frac{c_2 x}{y^2} - \frac{2c_1 + c_2^2}{6y} + \frac{2u}{y} + k_1\left(\frac{xy^2}{3} - \frac{c_2 y^3}{6}\right) + \frac{2k_1^2 y^7}{3}.$$

Solving the first equation we have

$$x = \frac{A}{y} + \frac{1}{6}(3c_2 y + k_1 y^5), \qquad A \text{ arbitrary constant}$$

and using it in the second equation we obtain

$$u = \frac{4c_1 - c_2^2}{24} + \frac{25k_1^2 y^8}{216} + \frac{A^2}{6y^4} + By^2 \qquad B \text{ arbitrary constant.}$$

Consequently, the similarity variables z and $w(z)$ are given by

$$z = -\frac{1}{6}y(-6x + 3c_2 y + k_1 y^5)$$
$$u(x, y) = \frac{4c_1 - c_2^2}{24} + \frac{25k_1^2 y^8}{216} + \frac{z^2}{6y^4} + y^2 w(z). \tag{14}$$

Thus the reduced equation is

$$175k_1^2 z - 180k_1 w - 45k_1 z w' - 162ww' - 27w''' = C \tag{15}$$

with C being an arbitrary constant.

Reduction 1.3. Proceeding in the same way as above, we see now that the similarity variables z and $w(z)$ are

$$z = \frac{6x - 3c_2 y - 4k_1 y^2}{6\sqrt{y}}$$
$$u(x, y) = \frac{1}{216}(36c_1 - 9c_2^2 + 64k_1^2 y^2 + 48k_1\sqrt{y}z) + \frac{w(z)}{y} \tag{16}$$

and the reduced equation is

$$w^{(4)} + 6(ww')' = 2w + \frac{7z}{4}w' + \frac{z^2}{4}w''. \tag{17}$$

Reduction 2.1. From (10) it is easily seen that the similarity variables are

$$z = y, \qquad u(x, y) = (k_1 y + k_2)x + w(y) \tag{18}$$

and the reduced equation is,

$$w'' - 6(k_1 y + k_2)^2 + c_2 k_1 = 0. \tag{19}$$

Reduction 2.2. From (11) we have

$$z = y \qquad u(x, y) = \frac{x^2}{6y^2} + \left(k_1 y^3 - \frac{c_2}{6y}\right)x + w(y) \tag{20}$$

where w satisfies the reduced equation

$$w'' - \frac{2w}{y^2} - 6k_1^2 y^6 + 5c_2 k_1 y^2 + \frac{c_1}{3y^2} = 0. \tag{21}$$

Reduction 2.3. The similarity variables are in this case $z = y$ and

$$u(x, y) = \frac{4c_1(2x - c_2 y)^2 - 192 - 4c_2^2 x^2 + 4c_2^3 xy - c_2^4 y^2}{24(2x - c_2 y)^2} + (2x - c_2 y)^2 w(y) \tag{22}$$

and the reduced equation is

$$w'' - 144w^2 = 0. \tag{23}$$

Before proceeding further we remark that, although the infinitesimals in symmetries (6) and (7) depend on both components of the velocity (c_1, c_2), the independent similarity variable z depends only on c_2 and the dependency of $u(x, y)$ in (14) and (16) on c_1 is just through an additive term. This difference between c_1 and c_2 is also clear for symmetries (11) and (12) and the corresponding reductions (20) and (21), and (22) and (23). That is related to the fact that the KP equation describes the evolution of quasi-one-dimensional water waves. Thus the role played in the symmetry reductions for c_1 and c_2 is, as expected, very different.

Finally, we want to obtain solutions of (1) by using the reductions we have used until now. By analysing the reduced equations we have:

Equation (13). For the particular case $k_2 = k_3 = 0$, (13) can be integrated once and we have the autonomous equation

$$3w^2 - (c_1 + c_2 k_1 + k_1^2)w + w'' = k_4 \qquad k_4 \text{ arbitrary constant.}$$

Thus, its order can be reduced and we get

$$w' = \sqrt{k_5 + 2k_4 w + dw^2 - 2w^3} \quad d = k_1^2 + c_2 k_1 + c_1 \quad k_5 \text{ arbitrary constant.}$$

In general, the solution of these equations involve inverse functions of elliptic integrals. However, for the case $k_4 = k_5 = 0$ $(d > 0)$ we have

$$w(z) = \frac{d}{2}\operatorname{sech}^2\left[\frac{\sqrt{d}}{2}z + k\right]$$

with k being an arbitrary constant. Thus, the associated solution of (2) is given by

$$u(x, y) = \frac{d}{2}\operatorname{sech}^2\left[\frac{\sqrt{d}}{2}(x + k_1 y) + k\right] \tag{24}$$

and the corresponding solution of KP, $u(x - c_1 t, y - c_2 t)$ is the well known one-dimensional soliton solution for the KP equation.

Equation (15). If we take in (15) $k_1 = 0$ and integrate once, we find

$$w'' + 3w^2 = -\frac{C}{27}z + D \tag{25}$$

with D a new arbitrary constant. Note that equation (25) is, with an appropriate change of variables, the first Painlevé equation ($P'' = P^2 + x$). By choosing $C = 0$, we get an autonomous equation. Consequently, its order can be reduced. Thus we have

$$w' = \sqrt{2w(D - w^2) + E} \tag{26}$$

with E being an arbitrary constant. In this way we have reduced the solution of (25) with $C = 0$ to a quadrature. However, in general, this solution involves inverse functions of elliptic integrals. The simplest case corresponds to $D = E = 0$, for which the solution of (26) is given by

$$w(z) = -\frac{2}{(z + k)^2} \qquad k \text{ arbitrary constant.}$$

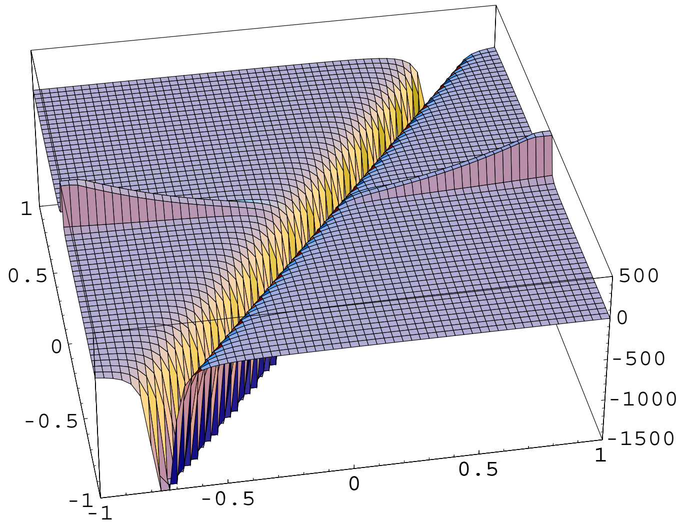


Figure 1. Solution (27) with $k = 0$, $c_1 = 1$, $c_2 = \frac{3}{2}$.

Taking into account (14) we have the solution of (2):

$$u(x, y) = \frac{4c_1 - c_2^2}{24} + \frac{1}{6}\left(\frac{x}{y} - \frac{c_2}{2}\right)^2 - \frac{2y^2}{\left[y\left(x - \frac{c_2 y}{2}\right) + k\right]^2} \tag{27}$$

and the corresponding travelling wave solution of the KP equation is given from (27) as $u(x - c_1 t, y - c_2 t)$. Observe that the solution of (2) we are analysing blows up in

- Two straight lines if $k = 0$:

$$y = 0 \qquad \text{and} \qquad x = \frac{c_2 y}{2}.$$

- A straight line and a hyperbola if $k \neq 0$:

$$y = 0 \qquad \text{and} \qquad y\left(x - \frac{c_2 y}{2}\right) = -k.$$

This fact can be appreciated in figures 1 and 2. Note also that (27) depends on c_1 just through an additive constant while it depends on c_2 in a stronger way.

For arbitrary k_1 and $C = 0$ two particular solutions of (15) are

$$w(z) = \frac{5k_1}{9} z \qquad \text{and} \qquad w(z) = -\frac{35k_1}{18} z$$

and the corresponding solutions for (2) are given by

$$\begin{aligned} u(x, y) &= \frac{4c_1 - c_2^2}{24} + \frac{25k_1^2 y^8}{216} + \frac{5}{54} k_1 y^3 (6x - 3c_2 y - k_1 y^5) \\ &\quad + \frac{1}{216 y^2}(6x - 3c_2 y - k_1 y^5)^2 \\ u(x, y) &= \frac{4c_1 - c_2^2}{24} + \frac{25k_1^2 y^8}{216} - \frac{35}{108} k_1 y^3 (6x - 3c_2 y - k_1 y^5) \\ &\quad + \frac{1}{216 y^2}(6x - 3c_2 y - k_1 y^5)^2. \end{aligned} \tag{28}$$

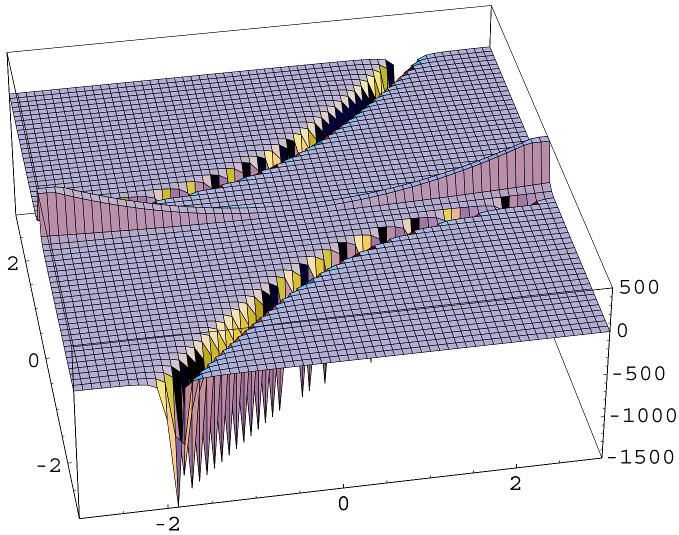


Figure 2. Solution (27) with $k = 1$, $c_1 = 1$, $c_2 = \frac{3}{2}$.

These solutions blow up at the line $y = 0$ and increase as $|y|$ increases for any arbitrary value of x. Only the particular case $k_1 = 0$ (for which both solutions coincide) avoids this increasing for large values of $|y|$.

Equation (17). If we put $w = v'$ in (17) the resulting equation can be trivially integrated twice. Thus we reduce the order and (17) becomes the third order equation:

$$v''' + 3(v')^2 - \frac{z^2}{4}v' - \frac{3z}{4}v = Cz + D.$$

Unfortunately, we have not been able to obtain further order reductions. For simple inspection of (17) one finds the particular solution

$$w(z) = -\frac{2}{z^2}$$

and consequently the solution of (2):

$$u(x, y) = \frac{1}{216}(36c_1 - 9c_2^2 + 48k_1x - 24k_1c_2y + 32k_1^2y^2) - \frac{72}{(6x - 3c_2y - 4k_1y^2)^2}. \qquad (29)$$

It is clear that this solution blows up at

- a straight line if $k_1 = 0$: $2x - c_2y = 0$,
- a parabola if $k_1 \neq 0$: $6x = 3c_2y + 4k_1y^2$,

and in the case $k_1 \neq 0$ (29) increases for large values of $|y|$, although its increase is much slower than for (28). We plot this solution for $k_1 = 0$ and $k_1 \neq 0$ in figures 3 and 4 respectively.

Equation (19). Equation (19) can be trivially integrated, and using (18) we obtain the solution of (2)

$$u(x, y) = \frac{k_1^2}{2}y^4 + 2k_1k_2y^3 - \frac{1}{2}(c_2k_1 - 6k_2^2)y^2 + k_3y + k_4 + (k_1y + k_2)x \qquad (30)$$

where k_3 and k_4 are arbitrary constants.

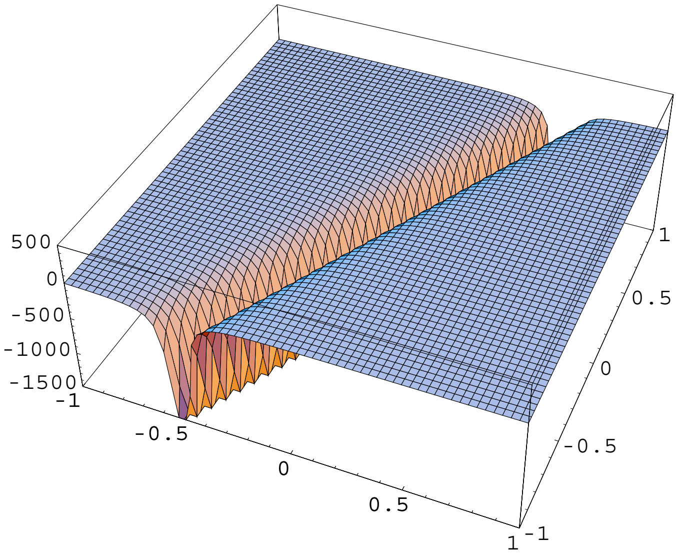


Figure 3. Solution (29) with $k_1 = 0$, $c_1 = 1$, $c_2 = 1$.

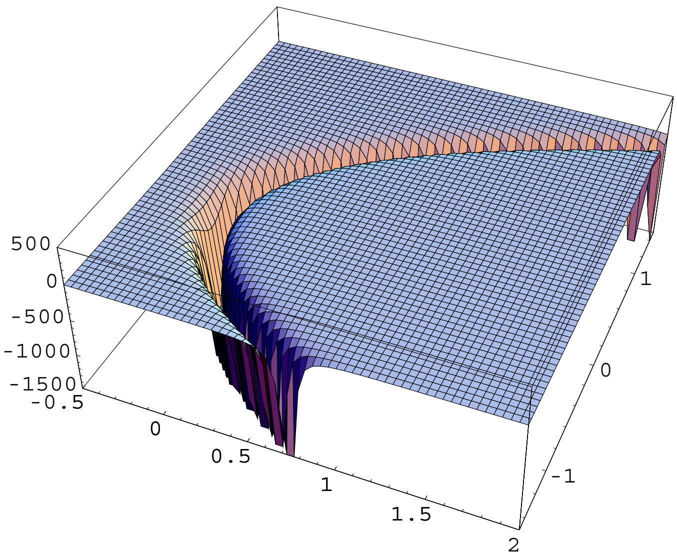


Figure 4. Solution (29) with $k_1 = 1$, $c_1 = 1$, $c_2 = 1$.

Equation (21). Equation (21) can be integrated by standard means. Then using (20) we obtain the solution of (2)

$$u(x, y) = \frac{x^2}{6y^2} + \left(k_1 y^3 - \frac{c_2}{6y}\right) x + \frac{c_1}{6} + \frac{k_2}{y} + k_3 y^2 - \frac{c_2 k_1}{2} y^4 + \frac{k_1^2}{9} y^8. \tag{31}$$

Equation (23). As in some previous cases, the general solution of (23) involves inverse functions of elliptic integrals. However, a solution is given by $w(y) = \frac{1}{24y^2}$. Using (22) we have

$$u(x, y) = \frac{4c_1(2x - c_2 y)^2 - 192 - 4c_2^2 x^2 + 4c_2^3 xy - c_2^4 y^2}{24(2x - c_2 y)^2} + \frac{(2x - c_2 y)^2}{24y^2}. \tag{32}$$

This solution blows up in two lines: $y = 0$ and $x = \frac{c_2}{2} y$.

Finally, note that, as the equation (2) is invariant under translation, we can generate new solutions by putting $x \to x - x_0$, $y \to y - y_0$ in any of the solutions we have obtained.

Summarizing, by making use of the nonclassical symmetries of a reduced equation of the KP equation, we have been able to find several rational solutions of KP that describe coherent structures, i.e. solutions whose temporal evolution consists in a movement with constant velocity (c_1, c_2). Indeed, expressions of the form

$$u(x - c_1 t, y - c_2 t),$$

with u given by (24), (27)–(31) or (32) are solutions of the KP equation.

Moreover, we can construct large families of (x, y)-rational solutions of the KP equation, starting with our solutions (27)–(32) of (2) in the following way: in [11], Clarkson and Winternitz apply the *direct method* (see, e.g., [12] for a description of this method) in order to reduce the KP equation to PDEs with two independent variables. In particular, they find that making the ansatz

$$\begin{aligned} z_1(x, y, t) &= x\, f(t) + \frac{h(t)}{2} + \frac{y^2\, f'(t)}{2} + \frac{y\, g'(t)}{2\, f(t)} \\ z_2(x, y, t) &= y\, f(t)^2 + g(t) \\ u(x, y, t) &= f(t)^2\, w(z_1(x, y, t), z_2(x, y, t)) - \frac{x\, f'(t)}{6\, f(t)} \\ &\quad - \frac{-g'(t)^2 + 2\, f(t)^3\, h'(t)}{24\, f(t)^4} - \frac{y^2\,(-2\, f'(t)^2 + f(t)\, f''(t))}{12\, f(t)^2} \\ &\quad - \frac{y\,(-3\, f'(t)\, g'(t) + f(t)\, g''(t))}{12\, f(t)^3} \end{aligned} \tag{33}$$

where f, g, h are arbitrary functions of t, the new dependent variable w satisfies the equation

$$w_{z_2 z_2} = (6 w_{z_1} w + w_{z_1 z_1 z_1})_{z_1}. \tag{34}$$

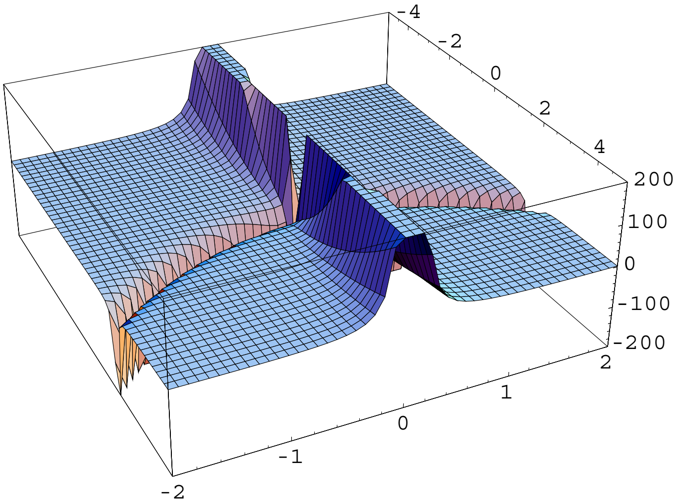
It is clear that (34) coincides with our equation (2) if we put $c_1 = c_2 = 0$. Thus, by introducing our solutions (27)–(32) as the w function in (33), we get families of solutions of KP which are rational functions in the spatial variables (x, y) and depend on t through three arbitrary functions. For example, substituting (27), (29) and (32) in (33) we obtain

$$
\begin{aligned}
u(x,y,t) = {} & -\frac{x\,f'(t)}{6\,f(t)} + f(t)^2\,\frac{\left(x\,f(t)+\frac{h(t)}{2}+\frac{y^2\,f'(t)}{2}+\frac{y\,g'(t)}{2\,f(t)}\right)^2}{6\,(y\,f(t)^2+g(t))^2} \\
& -f(t)^2\,\frac{2\,(y\,f(t)^2+g(t))^2}{\left[k+(y\,f(t)^2+g(t))\left(x\,f(t)+\frac{h(t)}{2}+\frac{y^2\,f'(t)}{2}+\frac{y\,g'(t)}{2\,f(t)}\right)\right]^2} \\
& -\frac{-g'(t)^2+2\,f(t)^3\,h'(t)}{24\,f(t)^4} - \frac{y^2\,(-2\,f'(t)^2+f(t)\,f''(t))}{12\,f(t)^2} \\
& -\frac{y\,(-3\,f'(t)\,g'(t)+f(t)\,g''(t))}{12\,f(t)^3} \\
u(x,y,t) = {} & -\frac{x\,f'(t)}{6\,f(t)} \\
& -\frac{72 f(t)^2}{\left[-4\,k_1\,(y\,f(t)^2+g(t))^2+6x\,f(t)+3h(t)+3y^2\,f'(t)+3\frac{y\,g'(t)}{2\,f(t)}\right]^2} \\
& +\frac{4}{27}\,f(t)^2\,{k_1}^2\,(y\,f(t)^2+g(t))^2 \\
& +\frac{2}{9}\,f(t)^2\,k_1\left(x\,f(t)+\frac{h(t)}{2}+\frac{y^2\,f'(t)}{2}+\frac{y\,g'(t)}{2\,f(t)}\right) \\
& -\frac{-g'(t)^2+2\,f(t)^3\,h'(t)}{24\,f(t)^4} - \frac{y^2\,(-2\,f'(t)^2+f(t)\,f''(t))}{12\,f(t)^2} \\
& -\frac{y\,(-3\,f'(t)\,g'(t)+f(t)\,g''(t))}{12\,f(t)^3} \\
u(x,y,t) = {} & -\frac{x\,f'(t)}{6\,f(t)} - \frac{2 f(t)^2}{\left(x\,f(t)+\frac{h(t)}{2}+\frac{y^2\,f'(t)}{2}+\frac{y\,g'(t)}{2\,f(t)}\right)^2} \\
& +\frac{\left(x\,f(t)+\frac{h(t)}{2}+\frac{y^2\,f'(t)}{2}+\frac{y\,g'(t)}{2\,f(t)}\right)^2}{6\,(y\,f(t)^2+g(t))^2} - \frac{-g'(t)^2+2\,f(t)^3\,h'(t)}{24\,f(t)^4} \\
& -\frac{y^2\,(-2\,f'(t)^2+f(t)\,f''(t))}{12\,f(t)^2} - \frac{y\,(-3\,f'(t)\,g'(t)+f(t)\,g''(t))}{12\,f(t)^3}.
\end{aligned}
\tag{35}
$$

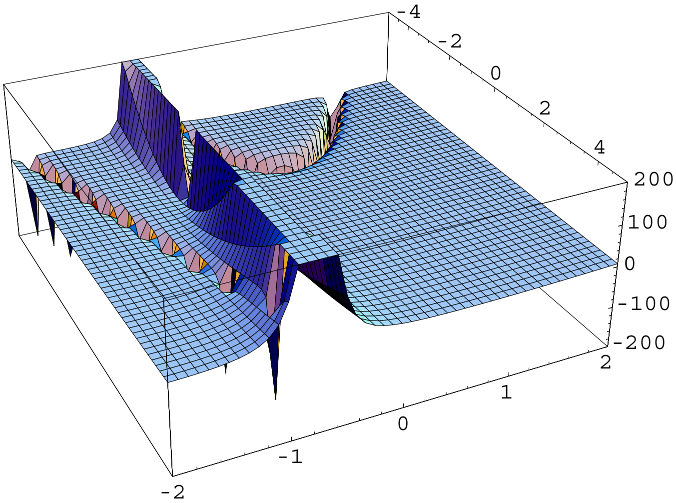
We plot the solution (35) in figure 5 for the choice $k = 1$, $f(t) = t^3 \sin(t)$, $g(t) = t^2 \cos(t)$, $h(t) = t \sin(t)$ and $t = -2$ (figures 5(*a*)), $t = 1$ (figure 5(*b*)) and $t = 4$ (figure 5(*c*)).

Note that all our solutions are invariant solutions under an infinite-dimensional subgroup of the Lie symmetry group of the KP equation. Using other subgroups (corresponding to Lie symmetries or nonclassical symmetries), new reductions can be performed, and invariant solutions under these subgroups can be obtained.

Remark. We have obtained, using the symmetry reduction procedure, some families of (x, y)-rational solutions of the KP equation. It is well known that this equation is integrable. Consequently, a lot of methods in the theory of integrable systems are available to find exact solutions of this equation: Hirota tau functions, methods in quantum field theory, vertex operators, Lax pairs, Grassmannian approach, etc. In particular, rational solutions have been constructed using these techniques. For example, in [13] Kac uses the description of the KP hierarchy through the bilinear identity and the vertex operators in order to show that any Schur polynomial is a τ-function for the KP hierarchy. Jimbo and Miwa [14] use fermionic fields to characterize polynomial τ functions for the KP hierarchy. More recently, Wilson [15] shows that the rational solutions of the KP hierarchy vanishing as $|x| \to \infty$ are parametrized by a



(*a*)



(*b*)

Figure 5. (*a*) Solution (35) with $k = 1$ $f(t) = t^3 \sin t$, $g(t) = t^2 \cos t$, $h(t) = t \sin t$ and $t = -2$. (*b*) Solution (35) with $k = 1$ $f(t) = t^3 \sin t$, $g(t) = t^2 \cos t$, $h(t) = t \sin t$ and $t = 1$. (*c*) Solution (35) with $k = 1$ $f(t) = t^3 \sin t$, $g(t) = t^2 \cos t$, $h(t) = t \sin t$ and $t = 4$.

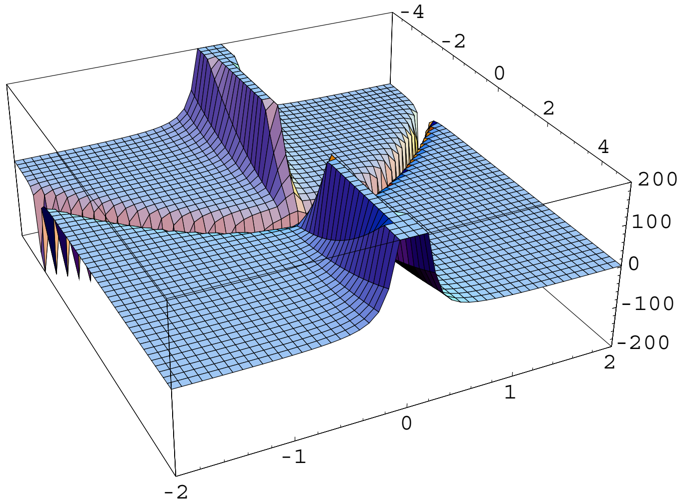
certain Grassmannian, and he uses this description to construct rational solutions.

Thus all the solutions constructed in the above works vanish as $|x| \to \infty$. These solutions are related to the Calogero–Moser system [16], a classical particle system with inverse square potential.

The solutions in the present work do not vanish as $|x| \to \infty$. In fact, they are of the form

$$u(x, y, t) = a(y, t)x^2 + b(y, t)x + c(y, t) + \frac{k}{(x - d(y, t))^2}$$

where $k = -2$ or 0. Note that some particular solutions of this type have been obtained in [17] by using a pair Lax approach and imposing the condition that L^4 and L_+^6 are commuting



(*c*)

Figure 5. (Continued.)

differential operators (L being the pseudo-differential Lax operator). It can be easily checked that the coefficients a, b, c and d satisfy the system

$$
\begin{aligned}
&a_{yy} = 36a^2\\
&b_{yy} = a_t + 36ab\\
&c_{yy} = b_t + 6b^2 + 12ac\\
&d_{yy} = 12ad + 6b\\
&d_t + d_y^2 - 6ad^2 - 6bd - 6c = 0
\end{aligned}
$$

if $k = -2$ and the system consisting of the first three previous equations if $k = 0$. Moreover, acting with the symmetry group of the KP equation on the rational solutions in [13, 14] or [15], we can construct solutions of the form

$$u(x, y, t) = a(y, t)x^2 + b(y, t)x + c(y, t) - 2\sum_{i=1}^{n} \frac{1}{(x - x_i(y, t))^2}.$$

For these solutions, x_i, $i = 1, \ldots, n, a, b, c$ satisfy a generalized Calogero–Moser system.

Acknowledgments

We would like to thank the referees and a member of the Editorial Board for their suggestions. We also thank Professor M L Gandarias for her advice. EM was partially supported by CICYT proyecto PB98-0578.

References

[1] Zakharov V E, Manakov S V, Novikov S P and Pitaevsky L P 1984 *Theory of Solitons* (Moskow: Nauka)
[2] Ablowitz M J and Segur H 1981 *Solitons and Inverse Scattering Transform* (Philadelphia: SIAM)

[3] Ablowitz M J and Clarkson P A 1991 *Solitons, Nonlinear Evolution Equation and Inverse Scattering* (Cambridge: Cambridge University Press)
[4] Konopelchenko B G 1993 *Solitons in Multidimensions* (Singapore: World Scientific)
[5] Kadomtsev B B and Petviashvili V I 1970 *Sov. Phys.–Dokl.* **15** 539–41
[6] Whitham G B 1974 *Linear and Nonlinear Waves* (New York: Wiley)
[7] Olver P J 1986 *Applications of Lie Groups to Differential Equations* (Berlin: Springer)
[8] Ovsiannikov L V 1982 *Group Analysis of Differential Equations* (New York: Academic)
[9] Bluman G W and Kumei S 1989 *Symmetries and Differential Equations* (Berlin: Springer)
[10] Bluman G W and Cole J D 1969 *J. Math. Mech.* **18** 1025–42
[11] Clarkson P A and Winternitz P 1991 *Physica* D **49** 257–72
[12] Clarkson P A and Kruskal M D 1989 *J. Math. Phys.* **30** 2201–13
[13] Kac V G 1990 *Infinite Dimensional Lie Algebras* (Cambridge: Cambridge University Press)
[14] Jimbo M and Miwa T 1983 *Publ. RIMS* Kyoto University **19** 943–1001
[15] Wilson G 1998 *Invent. Math.* **133** 1–41
[16] Krichever I M 1978 *Funkt. Anal. Appl.* **12.1** 59–61
[17] Grünbaum F A 1989 *Phys. Lett.* A **139** 146–50